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ORTHOGONAL POLYNOMIALS RELATED TO THE OSCILLATORY-CHEBYSHEV WEIGHT FUNCTION[1]

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A b s t r a c t. *In this paper we discuss the existence question for polynomials orthogonal with respect to the moment functional*

$$L(p)=\int_{-1}^{1} p(x)x(1-x^2)^{-1/2}e^{\mathrm{i}\zeta x}dx,\quad \zeta\in\mathbb{R}.$$

Since the weight function alternates in sign in the interval of orthogonality, the existence of orthogonal polynomials is not assured. A nonconstructive proof of the existence is given. The three-term recurrence relation for such polynomials is investigated and the asymptotic formulae for recursion coefficients are derived.

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1. *Introduction*

Let $\mathcal{P}$ be the space of all algebraic polynomials and $\mathcal{P}_n$ be the linear space of all algebraic polynomials of degree at most n.

Let a linear functional L be given on the linear space of all algebraic polynomials, i.e., let the functional L satisfy following equality, for each $P, Q \in \mathcal{P}$,

$$L(\alpha P + \beta Q) = \alpha L(P) + \beta L(Q), \quad \alpha, \beta \in \mathbb{C}.$$

The value of the linear functional L at every polynomial is known if the values of L are known at the set of all monomials, due to linearity. The corresponding values of the linear functional L at the set of monomials are called the moments and we denote them by μ_k, $k \in \mathbb{N}_0$,

$$L(x^k) = \mu_k, \quad k \in \mathbb{N}_0.$$

In [3, p. 7], the following definition can be found.

Definition 1 A sequence of polynomials $\{P_n(x)\}_{n=0}^{+\infty}$ is called the polynomial sequence orthogonal with respect to the moment functional L, provided for all nonnegative integers m and n,

- $P_n(x)$ is polynomial of degree n,
- $L(P_n(x)P_m(x)) = 0$, if $m \neq n$,
- $L(P_n^2(x)) \neq 0$.

If the sequence of orthogonal polynomials exists for a given linear functional L, then L is called quasi-definite or regular linear functional. Under the condition $L(P_n^2(x)) > 0$, the functional L is called positive definite (see [3]).

Using only linear algebraic tools the following theorem can be stated (see [3, p. 11]).

Theorem 1. *The necessary and sufficient conditions for the existence of a sequence of orthogonal polynomials with respect to the linear functional L are that for each $n \in \mathbb{N}$ the Hankel determinants*

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \mu_n \\ \mu_2 & \mu_3 & \mu_4 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \mu_{n+1} & \cdots & \mu_{2n-2} \end{vmatrix} \neq 0. \tag{1}$$

In this paper we consider the linear functional L given by

$$L(p) = \int_{-1}^{1} p(x)x(1-x^2)^{-1/2}e^{\mathrm{i}\zeta x}dx, \tag{2}$$

where $\zeta \in \mathbb{R}$. In order to prove the existence of the corresponding orthogonal polynomials, we need to compute the sequence of moments μ_k, $k \in \mathbb{N}_0$, and then to prove that all Hankel determinants are different from zero. The case of orthogonality with respect to the linear functional $L(p) = \int_{-1}^{1} p(x)xe^{\mathrm{i}m\pi x}dx$, $m \in \mathbb{N}$, was investigated in [8].

Denote the sequence of moments by $\mu_k(\zeta)$, $k \in \mathbb{N}_0$. Then, for each $k \in \mathbb{N}_0$, we can easily verify that

$$\mu_k(\zeta) = \int_{-1}^{1} x^{k+1}(1-x^2)^{-1/2}e^{\mathrm{i}\zeta x}dx = \overline{\int_{-1}^{1} x^{k+1}(1-x^2)^{-1/2}e^{-\mathrm{i}\zeta x}dx} = \overline{\mu_k(-\zeta)}.$$

This means that we need only to discuss the case $\zeta > 0$, since the corresponding results for $\zeta < 0$ can be obtained by a simple conjugation. We exclude the case $\zeta = 0$ since for this case $\mu_0 = \Delta_0 = 0$, so that the linear functional L is not regular. In what follows we assume $\zeta > 0$. Let J_ν be the Bessel function of the order ν defined by (cf. [14, p. 40])

$$J_\nu(z) = \sum_{m=0}^{+\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}. \tag{3}$$

Theorem 2. *The sequence of moments satisfy the following recurrence relation*

$$\mu_{k+2} = -\frac{k+2}{\mathrm{i}\zeta}\mu_{k+1} + \mu_k + \frac{k+1}{\mathrm{i}\zeta}\mu_{k-1}, \quad k \in \mathbb{N}, \tag{4}$$

with the initial conditions

$$\mu_0 = \mathrm{i}\pi J_1(\zeta),\ \mu_1 = \frac{\pi}{\zeta}(\zeta J_0(\zeta) - J_1(\zeta)),\ \mu_2 = \frac{\mathrm{i}\pi}{\zeta^2}(\zeta J_0(\zeta) + (\zeta^2-2)J_1(\zeta)). \tag{5}$$

P r o o f: We start with the following simple equality

$$\int_{-1}^{1} x^{k+1}(1-x^2)(1-x^2)^{-1/2}e^{\mathrm{i}\zeta x}dx = \mu_k - \mu_{k+2}, \quad k \in \mathbb{N}_0.$$

If we apply integration by parts to the integral which appears in the previous equality, we get

$$\int_{-1}^{1} x^{k+1}\sqrt{1-x^2}e^{\mathrm{i}\zeta x}dx \;=\; \frac{x^{k+1}}{\mathrm{i}\zeta}\sqrt{1-x^2}e^{\mathrm{i}\zeta x}\Big|_{-1}^{1} + \frac{1}{\mathrm{i}\zeta}\int_{-1}^{1} \frac{x^{k+2}}{\sqrt{1-x^2}}e^{\mathrm{i}\zeta x}dx$$

$$
\begin{aligned}
& -\frac{k+1}{\mathrm{i}\zeta}\int_{-1}^{1}\frac{x^k-x^{k+2}}{\sqrt{1-x^2}}\,e^{\mathrm{i}\zeta x}dx \\
= \; & \frac{1}{\mathrm{i}\zeta}\mu_{k+1}-\frac{k+1}{\mathrm{i}\zeta}(\mu_{k-1}-\mu_{k+1}), \quad k\in\mathbb{N},
\end{aligned}
$$

so that we have

$$
\mu_{k+2}=-\frac{k+2}{\mathrm{i}\zeta}\mu_{k+1}+\mu_k+\frac{k+1}{\mathrm{i}\zeta}\mu_{k-1}, \quad k\in\mathbb{N}.
$$

In order to start the recursion we need to compute the moments μ_0, μ_1 and μ_2. Because of the symmetry argument, we see that

$$
\begin{aligned}
\mu_0 &= \mathrm{i}\int_{-1}^{1}x(1-x^2)^{-1/2}\sin\zeta x\,dx,\\
\mu_1 &= \int_{-1}^{1}x^2(1-x^2)^{-1/2}\cos\zeta x\,dx,\\
\mu_2 &= \mathrm{i}\int_{-1}^{1}x^3(1-x^2)^{-1/2}\sin\zeta x\,dx,
\end{aligned}
$$

The series expansion for the function $e^{\mathrm{i}\zeta x}$ is valid for $|\zeta x|<+\infty$, and it converges uniformly everywhere in the complex plane (see [6]). This gives an opportunity to integrate term by term (see [13]), so that we have

$$
\begin{aligned}
\mu_0 &= \sum_{k=0}^{+\infty}\frac{(\mathrm{i}\zeta)^k}{k!}\int_{-1}^{1}\frac{x^{k+1}}{\sqrt{1-x^2}}\,dx=\sqrt{\pi}\sum_{k=0}^{+\infty}\frac{(\mathrm{i}\zeta)^k}{k!}\frac{1+(-1)^{k+1}}{2}\frac{\Gamma((k+2)/2)}{\Gamma((k+3)/2)}\\
&= \mathrm{i}\pi\sum_{k=0}^{+\infty}(-1)^k\frac{\zeta^{2k+1}}{(2k+1)!}\frac{(2k+1)!!}{(2k+2)!!}=\pi\sum_{k=0}^{+\infty}(-1)^k\frac{(\zeta/2)^{2k+1}}{k!(k+1)!}=\mathrm{i}\pi J_1(\zeta),
\end{aligned}
$$

where $J_1(\zeta)$ is the Bessel function of the order one. Here, we used the known expressions for the moments of the Chebyshev weight of the first kind given in [7] and the series expansion for the Bessel function given in (3). Also, we used the simple fact that

$$
\sqrt{\pi}\frac{1+(-1)^{k+1}}{2}\frac{\Gamma((k+2)/2)}{\Gamma((k+3)/2)}=\pi\frac{(2n+1)!!}{(2n+2)!!}, \quad k=2n+1,
$$

which can be verified by the induction argument. Using the same method we derive the corresponding expression for the moments μ_1 i μ_2. $\square$

To explore further the moment sequence we adopt the following notation

$$\mu_k = \frac{\mathrm{i}\pi}{(\mathrm{i}\zeta)^k}(P_k J_1 + \zeta Q_k J_0), \quad k \in \mathbb{N}_0. \tag{6}$$

We have the following statement.

Theorem 3. *The polynomials P_k and Q_k in ζ^2 with integer coefficients of degrees $2[k/2]$ and $2[(k-1)/2]$, respectively, satisfy the following recurrence relation*

$$y_{k+2} = -(k+2)y_{k+1} - \zeta^2 y_k - (k+1)\zeta^2 y_{k-1},$$

with initial conditions

$$\begin{aligned} P_0 &= 1, \quad P_1 = -1, \quad P_2 = 2 - \zeta^2, \\ Q_0 &= 0, \quad Q_1 = 1, \quad\;\;\, Q_2 = -1. \end{aligned}$$

The term with ζ^0 in P_k is equal to $(-1)^k k!$, $k \in \mathbb{N}_0$.

P r o o f: Putting (6) into the recurrence relation for the moments (4), we get

$$\begin{aligned} \frac{\mathrm{i}\pi}{(\mathrm{i}\zeta)^{k+2}}(P_{k+2}J_1 + \zeta Q_{k+2}J_0) &= -\frac{\mathrm{i}\pi(k+2)}{(\mathrm{i}\zeta)^{k+2}}(P_{k+1}J_1 + \zeta Q_{k+1}J_0) \\ &\quad -\frac{\mathrm{i}\pi\zeta^2}{(\mathrm{i}\zeta)^{k+2}}(P_k J_1 + \zeta Q_k J_0) - \frac{(k+1)\mathrm{i}\pi\zeta^2}{(\mathrm{i}\zeta)^{k+2}}(P_{k-1}J_1 + \zeta Q_{k-1}J_0). \end{aligned}$$

Since the equation is valid for all real $\zeta \neq 0$ and the functions J_0 and J_1 are linearly independent, it is just enough to read term with the functions J_0 and J_1 to obtain the recurrence relation stated.

The initial conditions for the moments give $P_0 = 1$, $P_1 = -1$, $P_2 = 2-\zeta^2$ and $Q_0 = 0$, $Q_1 = 1$, $Q_2 = -1$, for polynomials P_k and Q_k, respectively.

Now, obviously P_0, P_1 and P_2 are real polynomials with integer coefficients. According to the recurrence, P_3 is also real polynomial in ζ and has degree 2, i.e.,

$$P_3 = -(1+2)P_2 - \zeta^2 P_1 - \zeta^2 P_0 = -6 + 2\zeta^2.$$

Suppose that P_{k-1}, P_k and P_{k+1} are real polynomials in ζ, with degrees $2[(k-1)/2]$, $2[k/2]$ and $2[(k+1)/2]$, respectively. Then, using the recurrence relation

$$P_{k+2} = -(k+2)P_{k+1} - \zeta^2 P_k - (k+1)\zeta^2 P_{k-1},$$

we deduce the following recurrence

$$A_{2n+2} = -A_{2n},\ A_{2n+3} = -(2n+3)A_{2n+2} - A_{2n+1} - (2n+2)A_{2n}, \quad n \in \mathbb{N}_0,$$

for the leading coefficient in P_k, $k \in \mathbb{N}_0$, with initial conditions $A_0 = 1$, $A_1 = -1$, $A_2 = -1$. It can be checked by a direct calculation that we have the solution $A_{2n} = (-1)^n$, $A_{2n+1} = (-1)^{n+1}(n+1)$, $n \in \mathbb{N}_0$.

For the Q-sequence we have

$$Q_0 = 0, \quad Q_1 = 1, \quad Q_2 = -1, \quad Q_3 = 3 - \zeta^2.$$

We can easily verify the degrees of these initial polynomials. Suppose that Q_{k-1}, Q_k and Q_{k+1} are polynomials with degrees $2[(k-2)/2]$, $2[(k-1)/2]$ and $2[k/2]$, respectively. Using the recurrence for these polynomials, we can obtain the corresponding recurrence for their leading coefficients. Namely, we have

$$A_{2n+3} = -A_{2n+1},\ A_{2n+2} = -(2n+2)A_{2n+1} - A_{2n} - (2n+1)A_{2n-1}, \quad n \in \mathbb{N}_0,$$

with initial conditions $A_0 = 0$, $A_1 = 1$, $A_2 = -1$. It can be checked directly that the solution is $A_{2n+1} = (-1)^n$ and $A_{2n} = (-1)^{n+1}n$, $n \in \mathbb{N}_0$.

To prove the statement on the coefficient A_k with ζ^0 in P_k, $k \in \mathbb{N}_0$, we use the initial conditions $A_0 = 1$, $A_1 = -1$, and $A_2 = 2$. Then, by the recurrence relation, we obtain

$$A_{k+2} = -(k+2)A_{k+1}, \quad k \in \mathbb{N},$$

so that we conclude easily $A_k = (-1)^k k!$, $k \in \mathbb{N}_0$. $\square$

Can we say anything about the existence of orthogonal polynomials? To illustrate the problem, we can calculate the Hankel determinant Δ_2 in the form

$$\Delta_2 = \frac{\pi^2 J_1^2(\zeta)}{\zeta^2}\left(-\zeta^2 \frac{J_0^2(\zeta)}{J_1^2(\zeta)} + \zeta \frac{J_0(\zeta)}{J_1(\zeta)} + 1 - \zeta^2\right).$$

It is easy to conclude that $\Delta_2 = 0$, provided

$$\frac{J_0(\zeta)}{J_1(\zeta)} = \frac{1 \pm \sqrt{5 - 4\zeta^2}}{2\zeta},$$

but any solution must be real so it must be $|\zeta| < \sqrt{5}/2$. A careful numerical inspection shows that we cannot find solution for this equation, and it seems that it does not exist in the set of real numbers.

However, this is not the case with Δ_3. Using some computer algebra, it can be checked easily that

$$\Delta_3 = \frac{\mathrm{i}\pi^3 J_1^3}{\zeta^6}\left(7\zeta^3\frac{J_0^3}{J_1^3} + (2\zeta^2 - 21)\zeta^2\frac{J_0^2}{J_1^2} + \zeta(5\zeta^2 + 12)\frac{J_0}{J_1} + 2\zeta^4 - 15\zeta^2 + 4\right).$$

The smallest positive solution for the equation $\Delta_3 = 0$ is given by

$$\zeta = 6.459008151994783455531721397032502543805710669120882\ldots,$$

so that for this particular number, the sequence of orthogonal polynomials does not exist.

However, there is the way to ensure the existence of orthogonal polynomials. Choose ζ to be any positive zero of the Bessel function $J_0(\zeta)$. Then our sequence of moments becomes

$$\mu_k = \frac{\mathrm{i}\pi}{(\mathrm{i}\zeta)^k}P_k J_1(\zeta).$$

Because of the interlacing property of the positive zeros of the Bessel functions (see [14, p. 479]), we know that $J_1(\zeta) \neq 0$.

Theorem 4. *Suppose ζ is a positive zero of the Bessel function J_0. Then the sequence of polynomials orthogonal with respect to the functional L, given by* (2)*, exists.*

P r o o f: We give the proof of this statement using the fact that all zeros of the Bessel function J_0 are transcendental numbers [10] (see also [11], [4], [12]). Our sequence of moments is given by

$$\mu_k = \frac{\mathrm{i}\pi}{(\mathrm{i}\zeta)^k}P_k J_1(\zeta), \quad k \in \mathbb{N}_0,$$

where we know the basic properties of the polynomials P_k, $k \in \mathbb{N}_0$, stated in Theorem 3.

Consider the Hankel determinants Δ_k, $k \in \mathbb{N}$, given in (1). We have to prove that the determinants $\Delta_k \neq 0$, $k \in \mathbb{N}_0$. Then, according to Theorem 1, the sequence of orthogonal polynomials exists. Consider determinant Δ_k and extract from every of its rows the factor $\mathrm{i}\pi J_1(\zeta)$. Denoting the obtained determinant with Δ_k', we have

$$\Delta_k = (\mathrm{i}\pi J_1(\zeta))^{k+1}\Delta_k'.$$

Now, we can consider the determinant Δ'_k as a Hankel determinant for the sequence of moments $\mu'_\nu = (\mathrm{i}\zeta)^{-\nu} P_\nu$, $\nu \in \mathbb{N}_0$. If we now extract from every ν-th row the factor $1/(\mathrm{i}\zeta)^{\nu-1}$ and after that from j-th column the factor $1/(\mathrm{i}\zeta)^{j-1}$, we obtain new determinant Δ''_k and equality

$$\Delta_k = (\mathrm{i}\pi J_1(\zeta))^{k+1} \Delta'_k = \frac{(\mathrm{i}\pi J_1(\zeta))^{k+1}}{(\mathrm{i}\zeta)^{k(k+1)}} \Delta''_k.$$

The determinant Δ''_k is the Hankel determinant for the sequence of moments $\mu''_\nu = P_\nu$, $\nu \in \mathbb{N}_0$. Hence, the value of Δ''_k is certain polynomial in ζ^2, since all its elements are polynomials in ζ^2. Since ζ is transcendental number, the polynomial with integer coefficients Δ''_k cannot be zero at ζ, because all its zeros must be algebraic numbers. There is only one possibility for Δ''_k to have ζ as its zero, if the polynomial Δ''_k is identically zero.

Thus, we have to prove that Δ''_k is not identically zero. Since Δ''_k is a polynomial in ζ and all its coefficients are polynomials in ζ, the term with ζ^0 of the polynomial Δ''_k equals the Hankel determinant which elements are terms with ζ^0 in the polynomials P_ν, $\nu \in \mathbb{N}_0$. According to Theorem 3, we know that $P_\nu(0) = (-1)^\nu \nu!$, $\nu \in \mathbb{N}_0$, so that the term with ζ^0 in the polynomial Δ''_k equals the Hankel determinant $\hat{\Delta}_k$ for the sequence of moments $\hat{\mu}_\nu = (-1)^\nu \nu!$, $\nu \in \mathbb{N}_0$.

If we extract -1 from the rows $2\nu + 1$, $\nu = 0, 1, \ldots, 2[k/2]$, and from the columns $2j + 1$, $j = 0, 1, \ldots, 2[k/2]$, we get the Hankel determinant $\tilde{\Delta}_k$ for the sequence of moments $\tilde{\mu}_\nu = \nu!$, $\nu \in \mathbb{N}_0$, where the following equation holds

$$\hat{\Delta}_k = \tilde{\Delta}_k.$$

Now, it is easy to recognize the sequence of moments $\tilde{\mu}_\nu = \nu!$, as the sequence of moments for the Laguerre measure (see [7]). But, then it is easy to compute $\widetilde{\Delta}_k$,

$$\widetilde{\Delta}_k = \hat{\Delta}_k = \prod_{\nu=0}^{k} (\nu!)^2.$$

This means that Δ''_k is not a polynomial which is identically equal to zero. Hence $\Delta''_k \neq 0$, which implies that $\Delta_k \neq 0$. The previous discussion is valid for any $k \in \mathbb{N}$, which means that $\Delta_k \neq 0$, $k \in \mathbb{N}$. $\square$

In the rest of this paper we denote by p_n, $n \in \mathbb{N}_0$, the orthonormal sequence of polynomials with respect to the linear functional L, and by π_n, $n \in \mathbb{N}_0$, we denote the monic version of this polynomial sequence.

2. *Asymptotic formulae*

First we prove one auxiliary result, explaining the asymptotic properties of polynomials Q_n orthogonal with respect to the weight function w, defined by $w(x) = \chi_{[-1,1]}(x)(1-x^2)^{-1/2}e^{\mathrm{i}\zeta x}$.

Since this weight function is in Magnus class of the complex weight functions (see [5]), the polynomials Q_n, $n \in \mathbb{N}_0$, exist asymptotically. Also, their three-term recurrence coefficients have the asymptotic behavior of the class $M(0,1)$ introduced and studied in [9]. Denote the three-term recurrence coefficients for the corresponding orthonormal polynomials by α_n^Q and β_n^Q. Then we have

$$\lim_{n\to+\infty} \alpha_n^Q = 0, \qquad \lim_{n\to+\infty} \beta_n^Q = \frac{1}{2}.$$

Actually, using some recent results, we know even more.

Lemma 1. *For the monic polynomial Q_n, $n \in \mathbb{N}_0$, orthogonal with respect to the weight function $\omega(x) = \chi_{[-1,1]}(x)(1-x^2)^{-1/2}\exp(\mathrm{i}\zeta x)$, we have the following asymptotic formula*

$$\frac{Q_{n+1}(0)}{Q_n(0)} = \frac{1}{2}\frac{\cosh\dfrac{\zeta - \mathrm{i}(n+1)\pi}{2}}{\cosh\dfrac{\zeta - \mathrm{i}n\pi}{2}} + O(q^n),$$

where $0 < q < 1$.

P r o o f. We use the following result proved in [1]. Suppose h is a complex function being analytic in some neighborhood of the interval $[-1,1]$, which is different from zero on the interval $[-1,1]$. Then, the monic orthogonal polynomials Q_n with respect to $h(x)\chi_{[-1,1]}(x)(1-x^2)^{-1/2}$ exist asymptotically, and

$$\gamma_n Q_n(x) = \varphi_+(x) + \varphi_-(x) + O(q^n), \quad x \in [-1,1], \quad 0 < q < 1,$$

where

$$\gamma_n^{-1} = 2^{-n}\exp\left(\frac{1}{2\pi}\int_{-1}^{1}\frac{\log h(x)}{(1-x^2)^{-1/2}}\mathrm{d}x\right)$$

and

$$\varphi(z) = (z + (z^2-1)^{1/2})^n \exp\left(-\frac{1}{2\pi}(z^2-1)^{1/2}\int_{-1}^{1}\frac{\log h(x)}{z-x}\frac{\mathrm{d}x}{(1-x^2)^{1/2}}\right).$$

Here, $\varphi_+(x)$ is the limit of $\varphi(z)$ as z approaches $x \in [-1,1]$ over the upper half-plane of the complex z plane and $\varphi_-(x)$ is the limit of $\varphi(z)$ as z approaches $x \in [-1,1]$ from the lower half-plane of the complex z plane. The square root is chosen such that it has cut along the interval $[-1,1]$ and it behaves as z as z approaches ∞.

We use $x = \cos\theta \in [-1,1]$, $\theta \in [0,\pi]$. First, we calculate the integral which appears in the φ function for $\mathrm{Im}(z) > 0$. So we have

$$\begin{aligned}
\int_{-1}^{1} \frac{\mathrm{i}\zeta x}{z-x}\frac{\mathrm{d}x}{(1-x^2)^{1/2}} &= -\mathrm{i}\pi\zeta + \mathrm{i}\zeta z\int_{-1}^{1}\frac{1}{z-x}\frac{\mathrm{d}x}{(1-x^2)^{1/2}} \\
&= -\mathrm{i}\pi\zeta + \mathrm{i}\zeta\int_{-1}^{1}\frac{\mathrm{d}x}{(1-x^2)^{1/2}}\sum_{k=0}^{+\infty}\frac{x^k}{z^k} \\
&= -\mathrm{i}\pi\zeta + \mathrm{i}\zeta\sum_{k=0}^{+\infty}\frac{m_k^C}{z^k} \\
&= -\mathrm{i}\pi\zeta\left(1-\frac{1}{(1-z^{-2})^{1/2}}\right),
\end{aligned}$$

where m_k^C are the moments for Chebyshev weight of the first kind. This gives

$$\varphi_+(x) = e^{\mathrm{i}n\theta}\exp\left(-\frac{\zeta}{2}(\sin\theta + \mathrm{i}\cos\theta)\right)$$

and

$$\varphi_-(x) = e^{-\mathrm{i}n\theta}\exp\left(\frac{\zeta}{2}(\sin\theta - \mathrm{i}\cos\theta)\right).$$

Using the mentioned result, we can calculate directly

$$\gamma_n Q_n(x) = e^{\mathrm{i}n\theta}\exp\Big(-\frac{\zeta}{2}(\sin\theta+\mathrm{i}x)\Big) + e^{-\mathrm{i}n\theta}\exp\Big(\frac{\zeta}{2}(\sin\theta-\mathrm{i}x)\Big) + O(q^n)$$

and also, for $\theta = \pi/2$, we have

$$\frac{Q_{n+1}(0)}{Q_n(0)} = \frac{1}{2}\frac{\cosh\dfrac{\zeta-\mathrm{i}(n+1)\pi}{2}}{\cosh\dfrac{\zeta-\mathrm{i}n\pi}{2}} + O(q^n). \quad \square$$

Using the polynomials Q_n, $n \in \mathbb{N}_0$, we can express the polynomials π_n, $n \in \mathbb{N}_0$, and the corresponding three-term recurrence coefficients. Thus, we have the following statement:

Theorem 5. *Suppose the sequence of orthogonal polynomials Q_n exists for $n > N_Q$. Then, for polynomials orthogonal with respect to the functional L given by* (2), *we have*

$$\pi_n(x) = \frac{1}{x}\left(Q_{n+1}(x) - \gamma_n Q_n(x)\right), \quad n > N_Q, \tag{7}$$

where

$$\gamma_n = \frac{Q_{n+1}(0)}{Q_n(0)}, \quad n > N_Q.$$

The recursion coefficients for the sequence π_n, $n \in \mathbb{N}_0$, in the recurrence relation can be expressed in the following form

$$\alpha_n = -\gamma_n - \frac{(\beta_{n+1}^Q)^2}{\gamma_n}, \quad \beta_{n+1}^2 = \frac{\gamma_{n+1}}{\gamma_n}(\beta_{n+1}^Q)^2, \quad n > N_Q.$$

P r o o f. For the (monic) orthogonal polynomials Q_k, $z \in \mathbb{C}$, $Q_k(z) \neq 0$, $k \in \mathbb{N}$, the polynomials

$$\pi_n(x; z) = \frac{1}{x - z}\left(Q_{n+1}(x) - \frac{Q_{n+1}(z)}{Q_n(z)}\,Q_n(x)\right)$$

are known as the kernel polynomials (cf. [3]). Several results are known in the case when the point z is not in the supporting set of the measure of orthogonality and provided the sequence Q_n exists. In our case $z = 0$ and, therefore, we give the proof here.

Thus, supposing that the sequence of polynomials is given by (7), we have

$$\int_{-1}^{1} x^\nu \pi_n(x) \frac{x e^{\mathrm{i}\zeta x}}{\sqrt{1-x^2}}\,\mathrm{d}x = \int_{-1}^{1} x^\nu (Q_{n+1}(x) - \gamma_n Q_n(x)) \frac{e^{\mathrm{i}\zeta x}}{\sqrt{1-x^2}}\,\mathrm{d}x = 0,$$

provided $n > \nu$. According to the uniqueness property, up to a multiplicative constant, the polynomials (7) are orthogonal with respect to the weight function $x(1-x^2)^{-1/2}\exp(\mathrm{i}\zeta x)\chi_{[-1,1]}$. We note that $\gamma_n = Q_{n+1}(0)/Q_n(0)$.

If we assume that polynomials Q_n, $n > N_Q$, satisfy the following three-term recurrence relation

$$Q_{n+1}(x) = (x - \alpha_n^Q)Q_n(x) - (\beta_n^Q)^2 Q_{n-1}(x), \quad n > N_Q + 1,$$

it is clear that

$$\gamma_{n+1} = -\alpha_{n+1}^Q - \frac{(\beta_{n+1}^Q)^2}{\gamma_n}, \quad n > N_Q.$$

Also, for polynomials π_n, $n > N_Q$, we have

$$\begin{aligned}
x\pi_{n+1} &= Q_{n+2} - \gamma_{n+1}Q_{n+1} \\
&= (x - \alpha_{n+1}^Q)Q_{n+1} - (\beta_{n+1}^Q)^2 Q_n - \left(-\alpha_{n+1}^Q - \frac{(\beta_{n+1}^Q)^2}{\gamma_n}\right)Q_{n+1} \\
&= \left(x + \gamma_n + \frac{(\beta_{n+1}^Q)^2}{\gamma_n}\right)Q_{n+1} - \gamma_n Q_{n+1} - (\beta_{n+1}^Q)^2 Q_n \\
&= (x - \alpha_n)x\pi_n + \gamma_n(\beta_n^Q)^2 Q_{n-1} \\
&\quad + \left(-\gamma_n(x - \alpha_n^Q) - (\beta_{n+1}^Q)^2 + \gamma_n\left(x + \gamma_n + \frac{(\beta_{n+1}^Q)^2}{\gamma_n}\right)\right)Q_n \\
&= (x - \alpha_n)x\pi_n + \gamma_n(\beta_n^Q)^2 Q_{n-1} + \gamma_n\left(\alpha_n^Q + \gamma_n\right)Q_n \\
&= (x - \alpha_n)x\pi_n + \gamma_n(\beta_n^Q)^2 Q_{n-1} - \frac{\gamma_n}{\gamma_{n-1}}(\beta_n^Q)^2 Q_n \\
&= (x - \alpha_n)x\pi_n - \frac{\gamma_n}{\gamma_{n-1}}(\beta_n^Q)^2 x\pi_{n-1},
\end{aligned}$$

wherefrom we read directly the corresponding expressions for the three-term recurrence coefficients. □

Theorem 6. *For the three-term recurrence coefficients of the polynomial sequence p_n, $n \in \mathbb{N}_0$, we have the following asymptotic formulae*

$$\alpha_{2n+k} \to \frac{\mathrm{i}(-1)^k}{2}\left(\tanh\frac{\zeta}{2} - \coth\frac{\zeta}{2}\right), \quad n \to +\infty, \quad k = 0, 1,$$

and

$$\beta_{2n+k}^2 \to \frac{1}{4}\left(\tanh^2\frac{\zeta}{2}\right)^{(-1)^k}, \quad n \to +\infty, \quad k = 0, 1.$$

P r o o f. This theorem is a direct consequence of Lemma 1 and Theorem 5. □

Theorem 7. *Let J be the associated Jacobi operator created using three-term recurrence coefficients α_k and β_k, $k \in \mathbb{N}_0$. Then, we have*

$$\sigma_{\mathrm{ess}}(J) = [-1, 1].$$

P r o o f. According to Theorem presented in [2], we know that the essential spectrum of the Jacobi matrix, with periodic three-term recurrence coefficients of the basic period m, can be obtained as an inverse image of the interval $[-2,2]$ of the mapping

$$h(x) = \frac{p_{2m-1}}{p_{m-1}}.$$

In our case we have $m = 2$. Suppose that we give the sequences of three-term recurrence coefficients $\alpha_{2n+k} = a_k$, $\beta_{2n+k} = b_k$, $k = 0, 1$. Then, we can express $h(x)$ as

$$b_0 b_1 h(x) = x^2 - (a_0 + a_1)x + a_0 a_1 - b_0^2 - b_1^2.$$

For our case

$$a_0 = \frac{\mathrm{i}}{2}\left(\tanh\frac{\zeta}{2} - \coth\frac{\zeta}{2}\right) = -a_1 \quad \text{and} \quad b_0 = \frac{1}{2}\tanh\frac{\zeta}{2} = \frac{1}{4b_1}.$$

We get easily $h(x) = 4(x^2 - 1/2)$. The inverse mappings of h are the mappings

$$h_1^{-1} = \frac{(h+2)^{1/2}}{2}, \quad h_2^{-1} = -\frac{(h+2)^{1/2}}{2},$$

respectively.

If we let h change in $[-2,2]$ we get as a result exactly $[-1,1]$. $\square$

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