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SUBCONVEXITY FOR THE RIEMANN ZETA-FUNCTION AND THE DIVISOR PROBLEM[1]

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A b s t r a c t. A simple proof of the classical subconvexity bound $\zeta(\frac{1}{2} + it) \ll_\varepsilon t^{1/6+\varepsilon}$ *for the Riemann zeta-function is given, and estimation by more refined techniques is discussed. The connections between the Dirichlet divisor problem and the mean square of* $|\zeta(\frac{1}{2} + it)|$ *are analysed.*

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1. *Convexity for the Riemann zeta-function*

Let as usual

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \qquad (\Re\mathrm{e}\, s > 1) \tag{1.1}$$

denote the Riemann zeta-function (see the monographs [22], [23], and [41] for an extensive account), defined by analytic continuation for values of $s = \sigma + it$ for which $\sigma \le 1$. The zeta-function occupies a fundamental place

[1]Dedicated to Professor Matti Jutila on the occasion of his retirement.

in analytic number theory, its connection with prime numbers being evident from (1.1). The problem of the estimation of $\zeta(\frac{1}{2}+it)$ and the evaluation of its mean square is a central one in zeta-function theory. The functional equation for $\zeta(s)$ is

$$\zeta(s) = \chi(s)\zeta(1-s),\ \chi(s) = 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s) \quad (\forall s \in \mathbb{C}), \tag{1.2}$$

and it provides a weak form (a stronger form is with logarithms instead of the "ε" factors) of the so-called "convexity" bound in the so-called "critical strip" $0 \le \sigma \le 1$. Namely the defining series (1.1) for $\zeta(s)$ is convergent for $\sigma = 1+\varepsilon$, and by (1.2) it follows that

$$\zeta(-\varepsilon+it) \ll_\varepsilon |\chi(-\varepsilon+it)| \ll_\varepsilon |t|^{1/2+\varepsilon}, \tag{1.3}$$

since by Stirling's formula for the gamma-function one has, for fixed σ,

$$\chi(s) = (2\pi/t)^{\sigma+it-1/2}\ \mathrm{e}^{i(t+\pi/4)}\Big(1+O(t^{-1})\Big) \qquad (t \ge t_0 > 0). \tag{1.4}$$

As usual ε (> 0) denotes arbitrarily small constants, not necessarily the same ones at each occurrence, while $a \ll_\varepsilon b$ means that the implied $\ll$–constant depends (only) on ε. From $\zeta(1+\varepsilon+it) \ll_\varepsilon 1$, (1.3) and convexity, namely the Phragmén–Lindelöf principle (e.g., see the Appendix to [23]), one obtains the desired convexity bound

$$\zeta(\sigma+it) \ll_\varepsilon t^{\frac{1}{2}(1-\sigma)+\varepsilon} \qquad (0 \le \sigma \le 1,\ t \ge 2). \tag{1.5}$$

As mentioned, one can replace "ε" in (1.5) by a log-power by using the easily established bounds

$$\zeta(1+it) \ll \log t, \qquad \zeta(it) \ll \sqrt{t}\log t \qquad (t \ge 2).$$

Naturally, one wishes to obtains sharper bounds than (1.5), namely bounds where the constant in front of $1-\sigma$ in (1.5) is less than $1/2$ for $\sigma \ge 1/2$. The first such constant was $1/3$ (for $\frac{1}{2} \le \sigma \le 1$), obtained by Hardy and Littlewood in 1921 (see Notes to [23, Chapter 7] for a historic discussion of bounds for $\zeta(\frac{1}{2}+it)$), and the best possible bound, up to "ε", is $\zeta(\sigma+it) \ll_\varepsilon t^\varepsilon$ for $\sigma \ge 1/2$, which is known as the Lindelöf hypothesis. The best known result, that $\zeta(\frac{1}{2}+it) \ll_\varepsilon t^{32/205+\varepsilon}$, $32/205 = 0.15609\ldots$ is due to M.N. Huxley [14]. This is the result of almost a century of continued research, obtained by varied and refined techniques, and the current exponent $32/205$

is obviously very far from the exponent zero suggested by the Lindelöf hypothesis. This shows the great difficulty of proving the Lindelöf hypothesis, and indirectly indicates the enormous difficulty of the Riemann hypothesis (that all complex zeros of $\zeta(s)$ have real part 1/2). Namely it is not very difficult to show (see [36], [23, Chapter 1]) that the Riemann hypothesis, considered by many to be the greatest open problem in Mathematics, implies the Lindelöf hypothesis (it is not known whether the converse implication is true). In fact if the Riemann hypothesis is true, then (see E. C. Titchmarsh's monograph [41]) one has the bound

$$\zeta(\tfrac12 + it) \ll \exp\left(A\frac{\log t}{\log\log t}\right) \qquad (A > 0, t \ge 2),$$

which is stronger than the Lindelöf hypothesis, namely the bound $\zeta(\frac12 + it) \ll_\varepsilon t^\varepsilon$.

There is a natural, very important aspect of this subject which concerns "convexity" bounds for a large class of Dirichlet series. This is the so-called Selberg class $\mathcal{S}$, consisting of Dirichlet series $F(s) = \sum_{n=1}^\infty f(n)n^{-s}$ $(\sigma > 1)$ which possess several properties analogous to $\zeta(s)$, the most important ones being the Euler product over primes (see (1.1)) and the functional equation (see (1.2)) involving gamma-factors. For a comprehensive survey of $\mathcal{S}$, the reader is referred to the paper of J. Kaczorowski and A. Perelli [34]. For each function $F \in \mathcal{S}$ there exists a convexity bound analogous to (1.5). In this general context 'subconvexity' means a bound which improves on the appropriate analogue of (1.5), especially when $\sigma = \frac12$, which is the so-called "critical line" in the theory of functions in $\mathcal{S}$.

The aim of this paper is twofold. Firstly, we shall provide a simple, self-contained subconvexity bound for $\zeta(s)$, the prototype of all functions from $\mathcal{S}$. We shall also give a brief outline of the Bombieri–Iwaniec method at the end of the paper, which furnished the hitherto best bounds for $\zeta(\frac12 + it)$. Secondly, we shall analyse the connection between the mean square of $|\zeta(\frac12 + it)|$ and the classical Dirichlet divisor problem, namely the estimation of the function

$$\Delta(x) = \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1), \tag{1.6}$$

where $d(n)$ is the number of divisors of n and $\gamma = -\Gamma'(1) = 0.57721\ldots$ is Euler's constant. This connection is a natural one, since from (1.1) it follows that

$$\zeta^2(s) \;=\; \sum_{n=1}^{\infty} d(n)n^{-s} \qquad (\sigma > 1),$$

so that $d(n)$ is generated by $\zeta^2(s)$. It is not yet clear whether an analogue of such a connection exists in general for elements of $\mathcal{S}$, or represents an intrinsic property of a subclass of $\mathcal{S}$ containing $\zeta(s)$.

2. *Subconvexity for the Riemann zeta-function*

We start from [22, Theorem 1.2] or [23, Lemma 7.1], which bounds $\zeta(\frac12 + iT)$ by its mean square over a short interval

$$\begin{aligned} |\zeta(\tfrac12 + iT)|^2 \; &\ll \log T \left(1 + \int_{T-\log^2 T}^{T+\log^2 T} |\zeta(\tfrac12 + it)|^2 \,\mathrm{d}t\right) \\ &\ll \log T \int_{T-2G}^{T+2G} f(t)|\zeta(\tfrac12 + it)|^2 \,\mathrm{d}t. \end{aligned} \tag{2.1}$$

Here $T^\varepsilon \le G \le T^{1-\varepsilon}$, and $f(t)$ is a smooth, non-negative function supported in $[T - 2G, T + 2G]$ such that $f(t) = 1$ for $T - G \le t \le T + G$. The second bound in (2.1) is trivial, while the first one rests on contour integration of $\zeta^2(s + z)\Gamma(z)$ over z and the use of the functional equation (1.2) for $\zeta(s)$, and thus may be considered to be elementary. Although (2.1) appears to be wasteful, it turns out that it is an effective starting point for the estimation of $\zeta(\frac12 + it)$.

By the approximate functional equation for $\zeta(s)$ (see e.g., Theorem 4.4 of [23]) the second integral in (2.1) is majorized by $O(\log T)$ integrals of the type

$$\begin{aligned} I : \; &= \int_{T-2G}^{T+2G} f(t)\Big|\sum_{N<n\le N_1} n^{-\frac12 - it}\Big|^2 \,\mathrm{d}t \\ &= O(G) + \sum_{N<m\ne n\le N_1} (mn)^{-\frac12} \int_{T-2G}^{T+2G} f(t)(m/n)^{it} \,\mathrm{d}t, \end{aligned} \tag{2.2}$$

where $1 \ll N < N_1 \le 2N \ll T^{1/2}$. We may assume $G \ll N$, for otherwise the contribution of I is $\ll G$ by the well-known mean value theorem for Dirichlet polynomials (see e.g., Theorem 5.2 of [23]). By symmetry it may be assumed that $m > n$, thus $m = n + r$, $r \ge 1$. Note that

$$\int_{T-2G}^{T+2G} f(t)(m/n)^{it} \,\mathrm{d}t = \frac{i}{\log \frac{m}{n}} \int_{T-2G}^{T+2G} f'(t)(m/n)^{it} \,\mathrm{d}t,$$

so that integrating the last integral in (2.2) sufficiently many times by parts and using $f^{(j)}(t) \ll_j G^{-j}$, $1/\log(m/n) \ll N/r$, it follows that the contribution of $r > N^{1+\varepsilon}/G$ is $\ll 1$. Therefore we have

$$I \ll_\varepsilon G + \left| \sum_{1\le r\le N^{1+\varepsilon}/G} \int_{T-2G}^{T+2G} f(t) \sum_{N<n\le N_1} n^{-\frac12}(n+r)^{-\frac12}(1+r/n)^{it}\,\mathrm{d}t \right|. \tag{2.3}$$

The sum over n in (2.3) is written as

$$\sum_{N-G<n\le N_1+G} \varphi(n) n^{-1/2}(n+r)^{-1/2}(1+r/n)^{it} + O(G/N), \tag{2.4}$$

where $\varphi(t)\,(\ge 0)$ is a smooth function supported in $[N-G, N_1+G]$ such that $\varphi(t) = 1$ for $N \le t \le N_1$, and thus $\varphi^{(j)}(t) \ll_j G^{-j}$. The sum in (2.4) is treated by the Poisson summation formula ($F(x)$ is real valued, smooth and compactly supported in $[0,\infty)$)

$$\sum_{n=1}^\infty F(n) = \int_0^\infty F(x)\,\mathrm{d}x + 2\sum_{k=1}^\infty \int_0^\infty F(x)\cos(2\pi kx)\,\mathrm{d}x,$$

applied with $F(x)$ equal to the real (respectively imaginary) part of

$$\varphi(x)x^{-1/2}(x+r)^{-1/2}(1+r/x)^{it}.$$

Performing again sufficiently many integrations by parts it is seen that the contribution of $k > T^{1+\varepsilon}N^{-2}r$ will be $\ll 1$. We are left with the sums

$$\sum_{k\le T^{1+\varepsilon}N^{-2}r} \int_{N-G}^{N_1+G} \varphi(x)x^{-1/2}(x+r)^{-1/2}(1+r/x)^{it}\exp(\pm 2i\pi kx)\,\mathrm{d}x.$$

The exponential factor here is of the form

$$\exp(ig(x)),\quad g(x) := t\log\Bigl(1+\frac{r}{x}\Bigr) \pm 2k\pi x,\quad g''(x) = t\frac{2rx+r^2}{(x^2+rx)^2} \ll \frac{rT}{N^3}.$$

Thus applying the second derivative test (Lemma 2.2 of [23]) to the integral we deduce that it is $\ll N^{-1}(N^3/(rT))^{1/2}$. By the first derivative test ([23] Lemma 2.1) the total contribution of $\int_0^\infty F(x)\,\mathrm{d}x$ is clearly $\ll_\varepsilon GT^\varepsilon$. Hence from the preceding estimates we have

$$\begin{aligned} I &\ll_\varepsilon T^\varepsilon \left(G + G\sum_{r\le N^{1+\varepsilon}/G} N^{1/2}(rT)^{-1/2} \sum_{k\le T^{1+\varepsilon}N^{-2}r} 1 \right) \\ &\ll_\varepsilon T^\varepsilon(G + T^{1/2}G^{-1/2}) \ll_\varepsilon T^\varepsilon G \end{aligned}$$

for $G \ge T^{1/3}$. Hence, with $G = T^{1/3}$, (2.1) yields the bound $\zeta(\frac{1}{2}+it) \ll_\varepsilon t^{1/6+\varepsilon}$, which is the desired subconvexity estimate. As is to be expected, more refined estimates lead to exponents $< 1/6$, which will be shown in Section 4.

There are other ways to obtain the desired subconvexity estimate. For example, one can use [23, Theorem 1.8]. For the estimation of the zeta-sum

$$\sum_{N<n\le N_1} n^{-1/2-it},$$

one then uses the exponent pair $(\frac{1}{6}, \frac{2}{3})$ ([23] Chapter 2 or M.N. Huxley [8]), which immediately leads to the classical estimate of Hardy–Littlewood (see e.g., E.C. Titchmarsh's monograph [41, Chapter 5])

$$\zeta(\tfrac{1}{2}+it) \;\ll\; t^{1/6}\log t. \tag{2.5}$$

The fact is that one can show in an elementary way that $(\frac{1}{6}, \frac{2}{3})$ is an exponent pair, by using [23, Lemma 2.5] (Weyl's inequality) and then Poisson summation (in the form of, say, [23, Lemma 7]). This is probably the quickest way to attain the subconvexity bound (2.5), but perhaps not self-contained as the previous approach of ours. For two related approaches the reader is referred to Chapter 5 of Titchmarsh's book [41].

One can also use the approach of D.R. Heath-Brown [6], by using P. Gallagher's inequality (e.g., see H.L. Montgomery [38, Lemma 1.10])

$$\int_{-G}^{G}\Bigl|\sum_{n=1}^{\infty} a_n n^{-it}\Bigr|^2 \,\mathrm{d}t \;\ll\; G^2\int_0^\infty\Bigl|\sum_{y}^{y\mathrm{e}^{1/G}} a_n\Bigr|^2\frac{\mathrm{d}y}{y} \tag{2.6}$$

applied to $a_n = n^{-iT}$ if $N < n \le N_1 \le 2N$ and $a_n = 0$ otherwise. This easily leads to

$$\int_{T-G}^{T+G}|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t \ll G\log T \qquad (T^{1/3} \ll G \ll T),$$

giving again (2.5) in view of (2.1). This is quick indeed, but requires the knowledge of (2.6).

3. *The Dirichlet divisor problem*

It is interesting that one can also establish the analogy of the mean square of $|\zeta(\frac{1}{2}+it)|$ with the classical Dirichlet divisor problem, namely the estimation of the error-term function $\Delta(x)$, defined by (1.6). By using the elementary formula (see [23, eq. (14.41)])

$$\sum_{n\le z} n^{-1} \;=\; \log z+\gamma-\psi(z)z^{-1}+O(z^{-2}) \quad (z\ge 2), \quad \psi(x)=x-[x]-\tfrac{1}{2},$$

and writing

$$\sum_{n\le x} d(n)=\sum_{mn\le x}1=2\sum_{n\le\sqrt{x}}[x/n]-[\sqrt{x}\,]^2,$$

one easily arrives at

$$\Delta(x) \;=\; -2\sum_{n\le\sqrt{x}}\psi(x/n)+O(1). \tag{3.1}$$

But, for $T^\varepsilon\le G\le T^{1/2}$ and suitable $C>0$, we trivially have

$$C\Delta(T)-\frac{1}{G}\int_{T-2G}^{T+2G} f(t)\Delta(t)\,\mathrm{d}t=\frac{1}{G}\int_{T-2G}^{T+2G} f(t)(\Delta(T)-\Delta(t))\,\mathrm{d}t\ll_\varepsilon GT^\varepsilon,$$

since $d(n)\ll_\varepsilon n^\varepsilon$, where $f(t)$ is as in (2.1). Using the Fourier expansion

$$\psi(x)=-\frac{1}{\pi}\sum_{m=1}^\infty\frac{\sin(2\pi mx)}{m} \qquad (x\notin\mathbb{Z})$$

and the fact that the above series is boundedly convergent, it follows from (3.1) that

$$\Delta(T)=\frac{2}{\pi CG}\sum_{n\le\sqrt{T}}\sum_{m=1}^\infty\int_{T-2G}^{T+2G} f(t)\frac{\sin(2\pi mt/n)}{m}\,\mathrm{d}t+O_\varepsilon(GT^\varepsilon).$$

If we perform sufficiently many integrations by parts, using the fact that $f^{(\ell)}(t)\ll_\ell G^{-\ell}$ for $\ell=0,1,2,\dots$, it transpires that the contribution of m satisfying $m>nx^\varepsilon G^{-1}$ is negligible, that is, it is $O(1)$. Hence we are left with the estimate

$$\Delta(T)\ll\frac{\log^2 T}{G}\sup_{M\le N^{1+\varepsilon}/G,N\ll\sqrt{x}}\int\limits_{T-2G}^{T+2G} f(t)\Bigl|\frac{1}{M}\sum_{M<m\le M'}\sum_{N<n\le N'}\exp\Bigl(\frac{2\pi imt}{n}\Bigr)\Bigr|\,\mathrm{d}t \\ +GT^\varepsilon, \tag{3.2}$$

where $M < M' \le 2M$, $N < N' \le 2N$. The bound in (3.2) is the analogue of (2.3), if we notice that in (2.3) we have

$$\left(1+\frac{r}{n}\right)^{it} = \exp\left\{it\log\left(1+\frac{r}{n}\right)\right\} = \exp\left\{it\left(\frac{r}{n}-\frac{1}{2}\left(\frac{r}{n}\right)^2+\dots\right)\right\}, \tag{3.3}$$

so that the term itr/n dominates in the exponential in (3.3). Therefore we easily obtain from (3.2) the bound $\Delta(x) \ll_\varepsilon x^{1/3+\varepsilon}$. A short proof of the slightly sharper classical bound $\Delta(x) \ll x^{1/3}\log x$, by the use of the Voronoï formula (see (5.6)), is given by the author in [24], but the Voronoï formula may be considered as a non-elementary tool, while the preceding discussion was completely elementary.

3. *Sharper bounds*

It is worth remarking that both in the case of $\Delta(x)$ and in the case of the mean square of $|\zeta(\frac{1}{2}+it)|$ one can obtain notably better bounds. For the latter, as usual, we define

$$E(T) \;=\; \int_0^T |\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t - T\left(\log(\frac{T}{2\pi}) + 2\gamma - 1\right)$$

and note that we have the elementary inequalities (see the author's monograph [22, Lemma 4.1])

$$E(T) \le E(T+x) + Cx\log T, \quad E(T) \ge E(T-x) - Cx\log T \quad (0 \le x \le T). \tag{4.1}$$

We replace $T+x$ by t, multiply the first inequality in (4.1) by $f_0(t)$ and integrate, where $f_0(t)$ (≥ 0) is a smooth function supported in $[T,\, T+3G]$ such that $f_0(t) = 1$ when $T+G \le t \le T+2G$. It follows that, with suitable C_1, C_2,

$$E(T) \le \frac{C_1}{G}\int_T^{T+3G} f_0(t)E(t)\,\mathrm{d}t + C_2 G\log T \quad (C_1 > 0, C_2 > 0,\, T^\varepsilon \le G \le \sqrt{T}\,), \tag{4.2}$$

and also an analogous lower bound inequality holds. To deal with the integral in (4.2) we invoke the explicit formula of R. Balasubramanian [2], whose work is based on the use of the classical Riemann–Siegel formula for

$\zeta(\frac{1}{2}+it)$ (see [23, eq. (4.5)] or [41, Theorem 4.16]). This is

$$\begin{aligned} E(T) &= 2\sum_{n\le K}\sum_{m\le K, m\ne n}\frac{\sin(T\log n/m)}{\sqrt{mn}\log n/m} \\ &+ 2\sum_{n\le K}\sum_{m\le K, m\ne n}\frac{\sin(2\theta_1 - T\log mn)}{\sqrt{mn}\,(2\theta_1' - \log mn)} + O(\log^2 T), \end{aligned} \tag{4.3}$$

where $\theta_1 = \theta_1(T) = \frac{1}{2}T\log(T/(2\pi)) - \frac{1}{2}T - \frac{1}{8}\pi, K = \sqrt{T/(2\pi)}$. We insert (4.3) (with T replaced by t) in (4.2), and integrate by parts as before, using $f_0^{(j)}(t) \ll_j G^{-j}$. The double sums in (4.3) will yield exponential factors of the form

$$\exp\left(it\log\frac{m}{n}\right), \quad \exp\left(it\log\frac{tmn}{2\pi} - it\right) \quad (m \ne n), \tag{4.4}$$

and the sums over m and n are split in $\ll \log^2 T$ subsums with

$$M < m \le M' \le 2M, \quad N < n \le N' \le 2N, \quad M \ll \sqrt{T}, N \ll \sqrt{T}.$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\log\frac{m}{n}\right) = \log\frac{m}{n}, \ \frac{\mathrm{d}}{\mathrm{d}t}\left(t\log\frac{tmn}{2\pi} - t\right) = \log\frac{tmn}{2\pi}.$$

Thus, by the analysis that led to (2.3), the contribution of the second sum in (4.3) is clearly negligible. For the first exponential factor in (4.4) the non-negligible contribution will come from the values of M, N for which $M \ll N \ll M$. Further we set $n = m + r$ where $r \ne 0$ is an integer, and it follows that $M \le N^{1+\varepsilon}G^{-1}$ is the range for which the contribution is non-negligible. In view of (3.3) it transpires that an expression analogous to the right-hand side of (3.2) is obtained.

Both exponential sums in (2.3) and (3.2) are double (two-dimensional) exponential sums, and as such they may be treated by a variety of techniques and transformations developed for these types of sums (see e.g., the book of S.W. Graham and G. Kolesnik [5]). However, non-trivial results may be obtained if already the sum over n, say in (3.2), is estimated as $\ll (xM/N^2)^\kappa N^\lambda$ by the theory of one-dimensional exponent pairs (see e.g., [5, Chapter 3] or [23, Chapter 2]), where (κ, λ) is an exponent pair. Then the right-hand side of (3.2) is, by trivial estimation,

$$\ll_\varepsilon T^\varepsilon(T^{(\kappa+\lambda)/2}G^{-\kappa} + G) \ll_\varepsilon T^{\frac{\kappa+\lambda}{2+2\kappa}+\varepsilon}$$

with $G = T^{(\kappa+\lambda)/(2+2\kappa)}\ (\le \sqrt{T}\,)$, since $0 \le \kappa \le \frac{1}{2} \le \lambda \le 1$ has to hold for any exponent pair (κ, λ). Therefore from the preceding discussion we obtain a proof of the following

Theorem. *If* (κ, λ) *is an exponent pair, then*

$$E(T) \ll_\varepsilon T^{\frac{\kappa+\lambda}{2+2\kappa}+\varepsilon}, \qquad \Delta(T) \ll_\varepsilon T^{\frac{\kappa+\lambda}{2+2\kappa}+\varepsilon}. \tag{4.5}$$

From (2.1) and (4.5) we readily obtain then the following

Corollary.

$$\zeta(\tfrac{1}{2} + it) \ \ll_\varepsilon \ |t|^{\frac{\kappa+\lambda}{4+4\kappa}+\varepsilon}. \tag{4.6}$$

With the standard (elementary) exponent pair $(\frac{1}{2}, \frac{1}{2})$ we obtain again the exponent $1/3+\varepsilon$ in (4.5). The exponent is less than 1/3 if

$$3\lambda + \kappa < 2. \tag{4.7}$$

If we take, e.g., the exponent pair $(\frac{11}{30}, \frac{16}{30})$, then we obtain the exponent $27/82 + \varepsilon$ ($< 1/3$) in (4.5). Note that in (4.6) any non-trivial exponent pair (i.e., any pair (κ, λ) with $\lambda < 1$) beats convexity, namely produces the exponent in (4.6) that is strictly less than 1/4, while any exponent pair satisfying (4.7) improves (2.5). For some exponent pairs, obtained by the Bombieri–Iwaniec method, the reader is referred to M.N. Huxley [12], whilst the method itself will be analysed in Section 6. The results in the book [12] give the exponent $45/137+\varepsilon$ in (4.5), and there will be a small improvement from the latest work [14], and a better improvement from Sargos's programme, but the sum in (4.5) is precisely the one that the two-variable form of the Bombieri–Iwaniec method is designed to treat.

5. *Connections between* $E(T)$ *and* $\Delta(x)$

We conclude our discussion by analyzing more closely the connection between $E(T)$ and $\Delta(x)$. This has become well-known after the pioneering work of F.V. Atkinson [1], who established an explicit formula for $E(T)$ (different from (4.3)). Atkinson's result is the following: let $0 < A < A'$ be any two fixed constants such that $AT < N < A'T$, and let $N' = N'(T) = T/(2\pi) + N/2 - (N^2/4 + NT/(2\pi))^{1/2}$. Then

$$E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T), \tag{5.1}$$

where

$$\Sigma_1(T) = 2^{1/2}(T/(2\pi))^{1/4}\sum_{n\le N}(-1)^n d(n)n^{-3/4}e(T,n)\cos(f(T,n)), \tag{5.2}$$

$$\Sigma_2(T) = -2\sum_{n\le N'} d(n)n^{-1/2}(\log T/(2\pi n))^{-1}\cos\Bigl(T\log\Bigl(\frac{T}{2\pi n}\Bigr) - T + \tfrac14\pi\Bigr), \tag{5.3}$$

with

$$\begin{aligned} f(T,n) &= 2T\operatorname{ar\,sinh}\bigl(\sqrt{\pi n/(2T)}\,\bigr) + \sqrt{2\pi nT + \pi^2n^2} - \tfrac14\pi \\ &-\tfrac14\pi + 2\sqrt{2\pi nT} + \tfrac16\sqrt{2\pi^3}n^{3/2}T^{-1/2} + a_5n^{5/2}T^{-3/2} + a_7n^{7/2}T^{-5/2} + \dots, \end{aligned} \tag{5.4}$$

$$\begin{aligned} e(T,n) &= (1+\pi n/(2T))^{-1/4}\Bigl\{(2T/\pi n)^{1/2}\ \operatorname{ar\,sinh}\Bigl(\sqrt{\pi n/(2T)}\,\Bigr)\Bigr\}^{-1} \\ &= 1 + O(n/T) \qquad (1\le n<T), \end{aligned} \tag{5.5}$$

and $\operatorname{ar\,sinh} x = \log(x+\sqrt{1+x^2}\,)$. For $\Delta(x)$ we have the explicit, truncated Voronoï formula (see e.g., [23] or [42])

$$\Delta(x) = \frac{1}{\pi\sqrt2}x^{\frac14}\sum_{n\le N} d(n)n^{-\frac34}\cos(4\pi\sqrt{nx}-\tfrac14\pi) + O_\varepsilon(x^{\frac12+\varepsilon}N^{-\frac12}) \quad (2\le N\ll x). \tag{5.6}$$

A comparison between (5.2) and (5.6) reveals at once the similarities between $E(T)$ and $\Delta(x)$. This becomes even more pronounced if one considers

$$\Delta^*(x) \ := \ -\Delta(x) + 2\Delta(2x) - \tfrac12\Delta(4x) \tag{5.7}$$

instead of $\Delta(x)$. Then the arithmetic interpretation of $\Delta^*(x)$ (see T. Meurman [37]) is

$$\tfrac12\sum_{n\le 4x}(-1)^n d(n) \ = \ x(\log x + 2\gamma - 1) + \Delta^*(x). \tag{5.8}$$

One also has a Voronoï-type formula (see e.g., [23, eq. (15.68)]), for $2 \le N \ll x$,

$$\Delta^*(x) = \frac{1}{\pi\sqrt2}x^{\frac14}\sum_{n\le N}(-1)^n d(n)n^{-\frac34}\cos(4\pi\sqrt{nx} - \tfrac14\pi) + O_\varepsilon(x^{\frac12+\varepsilon}N^{-\frac12}), \tag{5.9}$$

which is completely analogous to (5.6). M. Jutila, in his works [28] and [29], investigated both the local and global behaviour of the difference

$$E^*(t) \;:=\; E(t) - 2\pi\Delta^*(\frac{t}{2\pi}).$$

He proved the mean square bound

$$\int_{T-H}^{T+H} (E^*(t))^2 \,\mathrm{d}t \ll_\varepsilon HT^{1/3}\log^3 T + T^{1+\varepsilon} \quad (1 \ll H \le T), \tag{5.10}$$

which in particular yields

$$\int_0^T (E^*(t))^2 \,\mathrm{d}t \ll T^{4/3}\log^3 T. \tag{5.11}$$

One conjectures that $\Delta(x) \ll_\varepsilon x^{1/4+\varepsilon}$ and $E(T) \ll_\varepsilon T^{1/4+\varepsilon}$ both hold (the exponent 1/4 is in both cases best possible). A weaker conjecture, still unproved but supported by the bounds in (4.5), is that $\alpha = \gamma$, where

$$\alpha = \inf\{\, a > 0 \;:\; \Delta(x) \ll x^a \,\}, \quad \gamma = \inf\{\, g > 0 \;:\; E(T) \ll T^g \,\}. \tag{5.12}$$

For sharp upper bounds on α and γ, obtained by the so-called Bombieri–Iwaniec method, which are better than those in (4.5), the reader is referred to the works of M.N. Huxley [8], [14]. This method will be briefly described in Section 6. It may be remarked that the conjectural values $\alpha = 1/4$ or $\gamma = 1/4$ cannot be attained even if one assumes the Riemann hypothesis. This clearly shows the difficulty of this subject. On the other hand, the strong conjecture that $(\varepsilon\,, \frac{1}{2}+\varepsilon)$ is an exponent pair (this implies the hitherto unproved Lindelöf hypothesis) does yield the conjectural values $\alpha = 1/4$ and $\gamma = 1/4$.

For an extensive discussion of $E^*(T)$ see the author's monographs [22], [23]. The significance of (5.11) is that it shows that $E^*(t)$ is in the mean square sense much smaller than either $E(t)$ or $\Delta^*(t)$. It is expected that $E(t)$ and $\Delta^*(t)$ are 'close' to one another in order, but this has never been satisfactorily established, although M. Jutila [28] obtained significant results in this direction. It is also conjectured that $E^*(T) \ll_\varepsilon T^{1/4+\varepsilon}, \Delta^*(x) \ll_\varepsilon x^{1/4+\varepsilon}$, which is the analogue of the classical conjecture $\Delta(x) \ll_\varepsilon x^{1/4+\varepsilon}$ in the Dirichlet divisor problem. Recently Y.-K. Lau and K.-M. Tsang [35] proved that $\alpha = \alpha^*$, where α is as in (5.12) and

$$\alpha^* \;=\; \inf\Big\{a^* > 0 \;:\; \Delta^*(x) \ll x^{a*}\Big\}.$$

In the first part of the second author's work [25] the bound in (5.11) was complemented with the new bound

$$\int_0^T (E^*(t))^4 \,\mathrm{d}t \ \ll_\varepsilon \ T^{16/9+\varepsilon}; \tag{5.13}$$

neither (5.11) or (5.13) seem to imply each other. In the second part of the same work [25], it was proved that

$$\int_0^T |E^*(t)|^5 \,\mathrm{d}t \ \ll_\varepsilon \ T^{2+\varepsilon},$$

and some further results on higher moments of $|E^*(t)|$ were obtained as well. In [26] it was shown that the bound in (5.11) is of the correct order of magnitude. The result is the asymptotic formula

$$\int_0^T (E^*(t))^2 \,\mathrm{d}t \ = \ T^{4/3}P_3(\log T) + O_\varepsilon(T^{7/6+\varepsilon}), \tag{5.14}$$

where $P_3(y)$ is a polynomial of degree three in y with positive leading coefficient, and all the coefficients may be evaluated explicitly. The formula (5.14) is the limit of the method.

6. *The Bombieri-Iwaniec method in a nutshell*

The sharpest subconvexity bounds for the Riemann zeta-function are obtained by a method which has still not found a descriptive name. The originators prefer 'Bombieri-Iwaniec method'; Huxley suggested 'the discrete Hardy-Littlewood method'; and Sargos suggested 'Poisson summation modulo one'. It has been successfully applied to four types of sum:

$$S_1 \ = \ \sum_{n\asymp N} b(n)\mathrm{e}(f(n)) \qquad \Big(\ \mathrm{e}(z) \ := \ \exp(2\pi i z)\Big),$$

where $b(n)$ are the coefficients of a modular form, $n \asymp N$ means that $N < n \le N'$ with $N < N' \le 2N$, and $f(x)$ is a real function growing faster than $O(N)$,

$$\begin{aligned} S_2 \ &= \sum_{n\asymp N} \ \mathrm{e}(f(n)), \\ S_3 \ &= \sum_{h\asymp H}\sum_{n\asymp N} \ \mathrm{e}(hf(n)), \end{aligned}$$

and

$$S_4 = \sum_{h \asymp H} \sum_{n \asymp N} \mathrm{e}\Big(f(n+h) - f(n-h)\Big).$$

Putting

$$f(x) = -\frac{t}{2\pi} \log \frac{x}{N} \tag{6.1}$$

makes S_1 corresponds to partial sum of the Hecke L-series $L(s,f)$, S_2 to those of the Riemann zeta-function, and S_4 to the short interval mean square of the Riemann zeta-function according to (2.4). Putting

$$f(x) = \sqrt{R^2 - x^2} \tag{6.2}$$

makes S_3 correspond to the Gauss circle problem of the lattice points in the circle $m^2 + n^2 \le R^2$, and putting

$$f(x) = T/x \tag{6.3}$$

makes S_3 correspond to the Dirichlet divisor problem according to (3.2). The sum

$$S_5 = \sum_{h \asymp H} \sum_{k \asymp K} \sum_{n \asymp N} \mathrm{e}\Big(kf(n+h) - kf(n-h)\Big)$$

could be used to investigate the short interval mean square in the circle problem, but the traditional Fourier series methods already give results which are in some sense best possible [11,39].

The method is bedevilled by technicalities and delicate estimates. There are two great ideas: the local lattice basis, and the large sieve. First we divide the sum into short intervals according to Diophantine approximation to $f'(x)$ in S_1 and S_3, or $f''(x)/2$ in S_2 and S_4. On a subinterval of S_1 where $f'(x)$ is close to an integer, the sequence $f(n)$ is slowly varying modulo one, and we use the Voronoï-Wilton summation formula [47]. If $f'(x)$ is close to a rational number a/q, then the sequence $f(n) - an/q$ is slowly varying modulo one, and we can use the Voronoï-Wilton formula for the coefficients $b(n)\,\mathrm{e}(an/q)$ of the modular form twisted by an additive character, or by the action of the matrix

$$\begin{pmatrix} a & -\bar{q} \\ q & \bar{a} \end{pmatrix}, \tag{6.4}$$

where $\bar{a}$ and $\bar{q}$ are integers with $a\bar{a} + q\bar{q} = 1$, which correspond to a change of basis in the underlying lattice. This idea was used by Jutila [30, Theorem 4.7] to obtain the result

$$\int_0^T \left| L\left(\tfrac{1}{2}\kappa + it, f\right)\right|^6 \,\mathrm{d}t \ \ll_\varepsilon\ T^{2+\varepsilon},$$

where $L(s, f)$ is the zeta-function of a holomorphic cusp form of weight κ for the full modular group. The method was developed in [31] and many subsequent papers.

Soon afterwards Bombieri and Iwaniec wrote their great paper [3] on the size of the Riemann zeta-function on the critical line. The first idea in [3] can be interpreted as the local lattice basis step for the sum

$$S_2' = \sum_{m \asymp N^2} q(m) \, \mathrm{e}(f(\sqrt{m})),$$

where $q(m) = 2$ if m is a perfect square, 0 otherwise, which is a lacunary sum of the form S_1, with the coefficients $b(m)$ those of a Jacobi theta function.

Iwaniec and C.J. Mozzochi [27] considered the more complicated sum S_3, which corresponds to a two-dimensional lattice point problem, with a/q a rational approximation to the gradient of the boundary curve, in the case (6.3). The sum S_4 was treated by Heath-Brown and Huxley [7]. The rational number a/q in (6.4) is an approximation to $2f'(x)$, and the local lattice basis step follows that for S_3.

The local lattice basis step on its own leads to the estimates

$$\begin{aligned} L(\tfrac{1}{2}\kappa + it, f) &\ll_\varepsilon t^{1/3+\varepsilon}, \\ \zeta\left(\tfrac{1}{2} + it\right) &\ll_\varepsilon t^{1/6+\varepsilon}, \\ \Delta(x) &\ll_\varepsilon x^{1/3+\varepsilon}, \\ E(T) &\ll_\varepsilon T^{1/3+\varepsilon}. \end{aligned}$$

Further savings come from estimating the transformed sums after Voronoï or Poisson summation. The transformed sum can be estimated directly when the denominator q is small (the ‘major arc’ case).

Bombieri and Iwaniec’s second innovation was to use the large sieve inequality in a general form of their devising. The first three or four terms of the power series for the function in the exponent are regarded as a vector inner product between a coefficient vector which depends only on the short interval, and a variable vector such as $(h^2, h^{3/2}, h, \sqrt{h})$, whose entries are monomials in the new variables introduced by Voronoï or Poisson summation.

The variable vectors form a thin set, and Hölder’s inequality is used to pass to powers of the transformed sums, so that four or five or six variable vectors with different integer parameters are added. Large sieves usually require the vectors to be distinct, but the Bombieri-Iwaniec form has a sum over a 'neighbourhood of the diagonal', that is, over pairs of vectors which are close enough together for their inner products with coordinate vectors to be approximately equal. The First Spacing Problem is to estimate this sum. Usually the parameters are chosen so that the order of magnitude of the sum is essentially that of the diagonal contribution. The First Spacing Problem for the sum S_3 (a sum of four vectors) was settled by Iwaniec and Mozzochi. For the sum S_2, Bombieri and Iwaniec [3, 4] took a sum of four vectors in the First Spacing Problem. N. Watt [43, 44] simplified the treatment, and Watt [45] and Huxley and Kolesnik [15] found partial results for a sum of five vectors; the optimal use of Hölder's inequality would give a sum of six vectors. The First Spacing Problem for S_4 is like that for S_2, but with extra low order terms. Heath-Brown and Huxley [7] had a partial result, and Watt [46] has recently obtained the expected bound.

The Second Spacing Problem is to estimate a sum over pairs of coefficient vectors which form a neighbourhood of the diagonal. The coefficients depend on the function $f(x)$. The problem is essentially the same for the sums S_2, S_3, and S_4. Huxley and Watt [18] and independently Kolesnik (unpublished) generalised Bombieri and Iwaniec's bound from the special case (6.1) in S_2 to a general function $f(x)$. Huxley [8], and independently Li Hongquan (unpublished) generalised the treatment of S_3 from Iwaniec and Mozzochi's special case (6.3) to a general function $f(x)$. The large sieve is applied to the transformed sum on the minor arcs, and the conditions for a pair of minor arc coefficient vectors labelled a/q and a'/q' to lie in a neighbourhood of the diagonal are expressed in terms of the lattice base change matrix

$$M = \begin{pmatrix} a & -\bar{q} \\ q & \bar{a} \end{pmatrix} \begin{pmatrix} a' & -\bar{q}' \\ q' & \bar{a}' \end{pmatrix}^{-1}. \tag{6.5}$$

The basic counting idea is that M acts only for short ranges of the rationals a/q and a'/q'; this idea uses only two of the four entries of the coefficient vector. Huxley [10, 14] has small improvements in which some use is made of the other two entries of the coefficient vector. For a fixed base change matrix M, the other two conditions can be interpreted as saying that an integer point in some dual plane lies close to a curve determined by the matrix M. The parameters are chosen so that the order of magnitude of the sum is essentially that of the diagonal contribution.

Jutila [32, 33], treating the sum S_1, introduced a variant where the minor arcs overlap, and so there are more denominators q in the Second Spacing Problem.

Huxley [9] considered the sum S_2 over a short interval. The matrices M are conjugates by a fixed matrix of matrices with small entries. Huxley and Watt [19, 20, 21] considered the sums S_2 and S_3 with congruence conditions; the matrices M lie in a congruence subgroup of the modular group.

These improvements relate to the largest range of sums in the approximate functional equation for $\zeta(s)$, or for the Fourier series in the lattice point problems. The methods become weaker for smaller ranges, when the parameters have different sizes. P. Sargos [40] gave a variation of the method which works well at $n = t^\alpha$ in the approximate functional equation for $\zeta(s)$, with $\alpha = 0.4$, and Huxley and Kolesnik gave two versions [16, 17] of an iterative method which works well near $\alpha = 0.42$.

The current best results by this method are

$$\begin{aligned} \zeta(\tfrac{1}{2} + it) &\ll_\varepsilon t^{32/205+\varepsilon} \qquad (32/205 = 0.15609\ldots), \\ \Delta(x) &\ll_\varepsilon x^{131/416+\varepsilon}, \qquad (131/416 = 0.314903\ldots), \\ E(T) &\ll_\varepsilon T^{131/416+\varepsilon}, \end{aligned}$$

due to Huxley [14, 13] and Watt (unpublished), respectively.

The Jutila uniform bound
Is something not easily found.
  With different cases
  In different places,
When transforms go round and around.

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