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ON RICCI H-PSEUDOSYMMETRIC H-HYPERSURFACES OF SOME ANTI-KÄHLER MANIFOLDS

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A b s t r a c t. We adopt the notion of the pseudosymmetry and Ricci pseudosymmetry to the case of the anti-Kähler manifolds and then we extend the results of the paper [1] to the h-hypersurfaces of the anti-Kähler manifolds of the constant totally real sectional curvatures.

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1. *The object of the paper*

Let (M, g) be a semi-Riemannian manifold of dimension ≥ 3. The manifold (M, g) is locally symmetric if $\nabla R = 0$, on M, where ∇ is its Levi-Civita connection and R the curvature tensor. The proper generalization of locally symmetric manifolds form semi-symmetric manifolds. They are characterized by the condition

$$R \cdot R = 0,$$

which holds on M, where R acts as a derivation. Some of the investigations of such manifolds gave rise to the next generalization, namely to the pseudosymmetric manifolds, i.e., manifolds satisfying on some set $\mathcal{U} \subset M$ the condition

$$R \cdot R = \mathcal{L}\, Q(g, R), \tag{1.1}$$

where $\mathcal{L}$ is a function on U and Q is a special operator (see section 2).

A manifold (M, g), dim $M \geq 3$, is said to be Ricci pseudosymmetric, resp. Ricci semi-symmetric, if

$$R \cdot \rho = \mathcal{L}\, Q(g, \rho), \quad \text{resp. } R \cdot \rho = 0, \tag{1.2}$$

holds on the appropiate set $\mathcal{U} \subset M$, and ρ is the Ricci tensor.

For a survey of results on different aspects of pseudosymmetric manifolds, we refer to [3]; see also [2], [10], [11], [14]. Among other problems there were studied the extrinsic characterizations of Ricci pseudosymmetric hypersurfaces of semi-Riemannian spaces of constant curvature in terms of the shape operator. Namely, in [1] (see Theorems 3.1 and 3.2) the following result is proved

Let M be a hypersurface of a semi-Reimannian space of constant curvature and dimension $n \geq 3$. Then M is Ricci pseudosymmetric if and only if at every point $p \in M$, the second fundamental form h satisfies one of the following conditions

$$h^2 = \alpha h + \beta q, \qquad \alpha, \beta \in R, \tag{1.3}$$

or

$$h^3 = \text{tr}\, h\, h^2 + \lambda h, \qquad \lambda \in R.$$

In practicular, for semi-Euclidean space, the previous result imply

A hypersurface M of semi-Euclidean space of dimension $n \geq 3$ is Ricci pseudosymmetric if and only if for every point $p \in M$ the tensor $R \cdot \rho$ vanishes at p, or (1.3) holds.

In section 4 of the present paper, we adopt the notion of pseudosymmetry and Ricci pseudosymmetry to the complex structure of the anti-Kähler manifolds and then we extend the above theorems for the h-hypersurface of anti-Kähler manifold of constant totally real sectional curvature. To do this, we use two formulas proved in section 3, valid for h-hypersurface of the anti-Kähler manifold of constant totally real sectional curvature. In section 2, we explain notations used in the paper.

2. *Preliminaries*

Let $\widetilde{M}$ be a connected differentiable manifold endoved with pseudo-Riemannian metric G and a $(1,1)$ tensor field F such that, with respect to the local coordinates, holds

$$F_B^A F_C^B = -\delta_C^A, \quad F_A^E F_B^D G_{ED} = -G_{AB}, \quad \widetilde{\nabla}_D F_B^A = 0. \tag{2.1}$$

Here $\widetilde{\nabla}$ is the Levi-Civita connection of $(\widetilde{M}, G)$ and $A, B, C, D \in \{1, 2, \ldots, 2m\}$, $2m = \dim \widetilde{M}$. The manifold $(\widetilde{M}, G, F)$ is said to be anti-Kähler manifold [12]. In some papers $(\widetilde{M}, G, F)$ is named B-manifold ([6],[7],[13]) and in some others - the Kähler manifold with the Norden metric ([8],[9]).

The manifold $(\widetilde{M}, G, F)$ is orientable and evendimensional. The metric G of such a manifold is indefinite and the signature is (m, m). Also, tr $F = 0$.

We denote by

$\widetilde{R}_{ABCD}$ - the Riemannian curvature tensor,

$\widetilde{\rho}_{AB} = \widetilde{R}^C{}_{ABC}$ - the Ricci tensor,

$\widetilde{\overset{*}{\rho}}_{AB} = F_A^D \widetilde{\rho}_{DB}$ - the second Ricci tensor,

$\widetilde{\kappa} = G^{AB} \widetilde{\rho}_{AB}$ - the scalar curvature,

$\widetilde{\overset{*}{\kappa}} = G^{AB} \widetilde{\overset{*}{\rho}}_{AB}$ - the second scalar curvature.

Since $\widetilde{\nabla} F = 0$, the curvature tensor and the Ricci tensors satisfy

$$\left.\begin{aligned} F_A^L F_B^M \widetilde{R}_{LMCD} &= -\widetilde{R}_{ABCD}\,, \\ F_A^L F_B^M \widetilde{\rho}_{LM} &= -\widetilde{\rho}_{AB}\,, \\ F_A^L F_B^M \widetilde{\overset{*}{\rho}}_{LM} &= -\widetilde{\overset{*}{\rho}}_{AB}\,. \end{aligned}\right\} \tag{2.2}$$

The manifold $(\widetilde{M}, G, F)$ is of pointwise constant totally real sectional curvature if at $p \in M$, ([6], [7]):

$$\begin{aligned} &\widetilde{R}_{ABCD} = \\ &\frac{\widetilde{\kappa}(p)}{4m(m-1)}(G_{AD}G_{BC} - G_{AC}G_{BD} - F_A^L G_{LD} F_B^M G_{MC} + F_A^L G_{LC} F_B^M G_{MD}) \\ &-\frac{\widetilde{\overset{*}{\kappa}}(p)}{4m(m-1)}(G_{AD} F_B^L G_{LC} + G_{BC} F_A^L G_{LD} - G_{AC} F_B^L G_{LD} - G_{BD} F_A^L G_{LC}). \end{aligned} \tag{2.3}$$

If $m \geq 3$, both functions $\widetilde{\kappa}$ and $\widetilde{\overset{*}{\kappa}}$ are constants.

Now, we consider a differentiable submanifold M of $\widetilde{M}$, $\dim M = 2n$, $n = m - 1$. Suppose that M is expressed in each neighbourhood $\widetilde{U}$ of $\widetilde{M}$ by the equations

$$x^A = x^A(u^a) \ ,$$

where x^A are the local coordinates of $\widetilde{M}$ in $\widetilde{U}$ and u^a are the local coordinates in $U = \widetilde{U} \bigcap M$. Lowercase Latin indices $a, b, c, \ldots, i, j, k, \ldots$ run over the range $\{1, 2, \ldots, 2n\}$. M is said to be a h-hypersurface (holomorphic hypersurface) of $\widetilde{M}$ if the restriction g of G on M has the maximal rank and the complex structure F leaves invariant the tangent space of M at each point $p \in M$. F induces on M the complex structure f such that (M, g, f) itself is an anti-Kähler manifold [4]. Similarly to (2.1) and (2.2), we have

$$f_i^a f_a^j = -\delta_i^j, \quad f_i^a f_j^b g_{ab} = -g_{ij}, \quad \nabla_i f_j^k = 0 \ ,$$

$$\left.\begin{aligned} f_i^a f_j^b R_{ablm} &= -R_{ijlm}, & \overset{*}{\rho}_{ij} &= f_i^a \rho_{aj} \ , \\ f_i^a f_j^b \rho_{ab} &= -\rho_{ij} \ , & f_i^a f_j^b \overset{*}{\rho}_{ab} &= -\overset{*}{\rho}_{ij}, \end{aligned}\right\} \tag{2.4}$$

where ∇ is the Levi-Civita connection with respect to the metric g, and R_{ijlm}, ρ_{ij} and $\overset{*}{\rho}_{ij}$ denote the local components of the Riemannian curvature tensor, Ricci tensor and the second Ricci tensor, respectively. We denote by κ and $\overset{*}{\kappa}$ the scalar curvature and the second scalar curvature of (M, g, f).

Because F leaves invariant the tangent space of M, it leaves invariant the normal space, too. There exist locally vector fields $N_{1|}$ and $N_{2|}$ normal to M, such that ([4]):

$$G_{AB} {N_{1|}}^A {N_{1|}}^B = -G_{AB} {N_{2|}}^A {N_{2|}}^B = 1, \quad G_{AB} {N_{1|}}^A {N_{2|}}^B = 0,$$

$$F_B^A {N_{1|}}^B = -{N_{2|}}^A, \qquad F_B^A {N_{2|}}^B = {N_{1|}}^A.$$

Denoting by h and k the second fundamental forms corresponding to $N_{1|}$ and $N_{2|}$ respectively, we have

$$h_{ij} = f_i^a k_{aj}, \qquad k_{ij} = -f_i^a h_{aj} \ . \tag{2.5}$$

Also, we shall use

$$h^2{}_{ij} = h_i{}^a h_{aj}, \qquad h^3{}_{ij} = h_i{}^a h^2{}_{aj}.$$

It is easy to see that the following conditions are satisfied

$$\left.\begin{aligned} & f_i^a f_j^b h_{ab} = -h_{ij}, \qquad f_i^a f_j^b k_{ab} = -k_{ij}\ , \\ & f_i^a h_{aj} = f_j^a h_{ai}, \qquad f_i^a k_{aj} = f_j^a k_{ia}, \\ & h_{ij}^2 = h_{ji}^2, \quad f_i^a f_j^b h_{ab}^2 = -h_{ij}^2, \quad f_i^a h_{aj}^2 = f_j^a h_{ia}^2, \\ & h_{ij}^3 = h_{ji}^3, \quad f_i^a f_j^b h_{ab}^3 = -h_{ij}^3, \quad f_i^a h_{aj}^3 = f_j^a h_{ia}^3. \end{aligned}\right\} \tag{2.6}$$

Let at $p \in M$, A and D be two symmetric $(0,2)$ tensors and B the curvature like tensor, satisfying

$$f_i^a f_j^b A_{ab} = -A_{ij}, \qquad f_i^a f_j^b D_{ab} = -D_{ij}\ , \tag{2.7}$$

$$f_i^a f_j^b B_{ablm} = -B_{ijlm} \tag{2.8}$$

Let T be a $(0,4)$ tensor. We define the tensors $B \cdot A$, $B \cdot T$, $Q(A,D)$, $Q(A,B)$ by the formulas

$$(B \cdot A)_{rsij} = A_{aj} B^a{}_{irs} + A_{ia} B^a{}_{jrs}, \tag{2.9}$$

$$(B \cdot T)_{rsijlm} = T_{ajlm} B^a{}_{irs} + T_{ialm} B^a{}_{jrs} + T_{ijam} B^a{}_{lrs} + T_{ijla} B^a{}_{mrs}, \tag{2.10}$$

$$\begin{aligned} Q(A,D)_{rsij} = \ & A_{ri} D_{sj} + A_{rj} D_{si} - A_{si} D_{rj} - A_{sj} D_{ri} \\ & - f_r^a f_s^b (A_{ai} D_{bj} + A_{aj} D_{bi} - A_{bi} D_{aj} - A_{bj} D_{ai}), \end{aligned} \tag{2.11}$$

$$\begin{aligned} Q(A,B)_{rsijlm} = \ & A_{ri} B_{sjlm} + A_{rj} B_{islm} + A_{rl} B_{ijsm} + A_{rm} B_{ijls} \\ & - A_{si} B_{rjlm} - A_{sj} B_{irlm} - A_{sl} B_{ijrm} - A_{sm} B_{ijlr} \\ & - f_r^a f_s^b \left(A_{ai} B_{bjlm} + A_{aj} B_{iblm} + A_{al} B_{ijbm} + A_{am} B_{ijlb} \right. \\ & \left. - A_{bi} B_{ajlm} - A_{bj} B_{ialm} - A_{bl} B_{ijam} - A_{bm} B_{ijla} \right). \end{aligned} \tag{2.12}$$

Remark. The operator Q of a semi-Riemannian manifold (M,g) is defined in the following way (e.g. see [1],[2],[3]):

$$Q(A,D)_{rsij} = A_{ri} D_{sj} + A_{rj} D_{si} - A_{si} D_{rj} - A_{sj} D_{ri},$$

$$\begin{aligned} Q(A,B)_{rsijlm} = \ & A_{ri} B_{sjlm} + A_{rj} B_{islm} + A_{rl} B_{ijsm} + A_{rm} B_{ijls} \\ & - A_{si} B_{rjlm} - A_{sj} B_{irlm} - A_{sl} B_{ijrm} - A_{sm} B_{ijlr}. \end{aligned}$$

Thus, (2.11) and (2.12) are the same operators, but adopted to the complex structure of the manifold.

We note that

$$\left.\begin{array}{l} Q(A,D) = -Q(D,A) \quad \text{and therefore} \quad Q(A,A) = 0, \\ Q(fA,fD) = -Q(A,D) \quad \text{and therefore} \quad Q(fA,D) = Q(A,fD), \\ Q(A,fA) = 0, \quad Q(fD,B) = Q(D,fB). \end{array}\right\} \tag{2.13}$$

For the latter use, we present

Lemma 2.1 *([4]) Let as a point $p \in M$, A and D be two symmetric* $(0,2)$ *tensors satisfying (2.7). If*

$$Q(A,D) = 0\ , \tag{2.14}$$

then

$$D = \delta A + \bar{\delta} f A, \qquad \delta, \bar{\delta} \in R. \tag{2.15}$$

P r o o f. Let X be a vector such that

$$X^a X^b A_{ab} = \omega \neq 0, \qquad X^a \bar{X}^b A_{ab} = \bar{\omega} \neq 0,$$

where $\bar{X}^i = f^i_a X^a$. We put

$$\eta = X^a X^b D_{ab}, \qquad \bar{\eta} = X^a \bar{X}^b D_{ab}.$$

Transvecting (2.14) with $X^i X^r$, and symmetrizing the resulting equality, we get

$$\omega D_{sj} - \eta A_{sj} - \bar{\omega} f^a_s D_{aj} + \bar{\eta} f^a_s A_{aj} = 0\ ,$$

from which it follows that

$$\omega f^a_i D_{aj} - \eta f^a_i A_{aj} + \bar{\omega} D_{ij} - \bar{\eta} A_{ij} = 0\ .$$

These two relations imply

$$D_{ij} = \frac{\omega\eta + \bar{\omega}\bar{\eta}}{\omega^2 + \bar{\omega}^2} A_{ij} - \frac{\omega\bar{\eta} - \bar{\omega}\eta}{\omega^2 + \bar{\omega}^2} f^a_i A_{aj}\ .$$

But this is just the relation (2.15). □

3. *H-hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures*

The Gauss equation for an h-hypersurface (M, g, f) reads

$$\widetilde{R}_{ABCD}\frac{\partial x^A}{\partial u^i}\frac{\partial x^B}{\partial u^j}\frac{\partial x^C}{\partial u^l}\frac{\partial x^D}{\partial u^m} = R_{ijlm} - (h_{im}h_{jl} - h_{il}h_{jm}) + (k_{im}k_{jl} - k_{il}k_{jm}).$$

Now, we suppose that the ambient manifold $(\widetilde{M}, G, F)$ is a manifold of constant totally real sectional curvatures. Then, substituting (2.3) into above Gauss equation, and taking into account that $m = n + 1$, we get

$$R_{ijlm} = K\, G_{ijlm} + \overset{*}{K} f_i^a G_{ajlm} + E_{ijlm} \tag{3.1}$$

where

$$G_{ijlm} = g_{im}g_{jl} - g_{il}g_{jm} - f_{im}f_{jl} + f_{il}f_{jm}\ , \tag{3.2}$$

$$E_{ijlm} = h_{im}h_{jl} - h_{il}h_{jm} - k_{im}k_{jl} + k_{il}k_{jm} \tag{3.3}$$

$$K = \frac{\widetilde{\kappa}}{4n(n+1)}\ , \qquad \overset{*}{K} = -\frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)}\ , \tag{3.4}$$

and $f_{ij} = f_i^a g_{aj}$.

The relation (3.1) yields

$$\left.\begin{aligned} \rho_{im} &= 2(n-1)(Kg_{im} + \overset{*}{K} f_{im}) + \text{tr}\ h\ h_{im} + \text{tr}\ k\ f_i^a h_{am} - 2h^2_{im}\ , \\ \overset{*}{\rho}_{im} &= 2(n-1)(Kf_{im} - \overset{*}{K} g_{im}) + \text{tr}\ h\ f_i^a h_{am} - \text{tr}\ k\ h_{im} - 2f_i^a h^2_{am}\ , \end{aligned}\right\} \tag{3.5}$$

and therefore

$$\left.\begin{aligned} \kappa &= 4n(n-1)K + (\text{tr}\ h)^2 - (\text{tr}\ k)^2 - 2\text{tr}\ (h^2), \\ \overset{*}{\kappa} &= -4n(n-1)\ \overset{*}{K} - 2\text{tr}\ h\ \text{tr}\ k - 2\text{tr}\ (fh^2). \end{aligned}\right\} \tag{3.6}$$

We note that, because of $k_{ij} = -f_i^a h_{aj}$, we have tr $k = -$tr (fh). In view of (3.1), we have

$$R \cdot R = K\ G \cdot R + \overset{*}{K}(fG) \cdot R + E \cdot R\ .$$

Using (2.12), we can easy to see that

$$G \cdot R = Q(g, R), \qquad (fG) \cdot R = Q(fg, R)\ .$$

Therefore

$$R \cdot R = K\ Q(g,R) + \overset{*}{K} Q(fg,R) + E \cdot R\ . \tag{3.7}$$

On the other hand

$$E \cdot R = K\ (E \cdot G) + \overset{*}{K}(E \cdot fG) + E \cdot E\ .$$

But

$$\begin{aligned}(E \cdot G)_{rsijlm} &= \\ &= G_{ajlm}E^a_{\ irs} + G_{ialm}E^a_{\ jrs} + G_{ijam}E^a_{\ lrs} + G_{ijla}E^a_{\ mrs} \\ &= g_{jl}\,(E_{mirs} + E_{imrs}) - g_{jm}\,(E_{lirs} + E_{ilrs}) \\ &\quad + g_{im}\,(E_{ljrs} + E_{jlrs}) - g_{il}\,(E_{mjrs} + E_{jmrs}) \\ &\quad - f_{im}\left(f^a_l E_{ajrs} + f^a_j E_{alrs}\right) + f_{il}\left(f^a_m E_{ajrs} + f^a_j E_{amrs}\right) \\ &\quad - f_{jl}\,(f^a_m E_{airs} + f^a_i E_{amrs}) + f_{jm}\,(f^a_l E_{airs} + f^a_i E_{alrs}) = 0\ ,\end{aligned}$$

because of

$$E_{mirs} = -E_{mirs} \qquad \text{and} \qquad f^a_m E_{airs} = -f^a_i E_{amrs}\ .$$

Similary we have $E \cdot fG = 0$, and therefore (3.7) reduces to

$$R \cdot R = K\ Q(g,R) + \overset{*}{K}\ Q(fg,R) + E \cdot E\ .$$

Finally

$$\begin{aligned}(E \cdot E)_{rsijlm} &= \\ &= -\Big[h^2_{ri}E_{sjlm} + h^2_{rj}E_{islm} + h^2_{rl}E_{ijsm} + h^2_{rm}E_{ijls} \\ &\quad - h^2_{si}E_{rjlm} - h^2_{sj}E_{irlm} - h^2_{sl}E_{ijrm} - h^2_{sm}E_{ijlr} \\ &\quad - f^a_r f^b_s \Big(h^2_{ai}E_{bjlm} + h^2_{aj}E_{iblm} + h^2_{al}E_{ijbm} + h^2_{am}E_{ijlb} \\ &\quad - h^2_{bi}E_{ajlm} - h^2_{bj}E_{ialm} - h^2_{bl}E_{ijam} - h^2_{bm}E_{ijla}\Big)\Big] \\ &= -Q(h^2, E)_{rsijlm}.\end{aligned}$$

Thus, we can state

Proposition 3.1. *The relation*

$$(R \cdot R)_{rsijlm} = KQ(g,R)_{rsijlm} + \overset{*}{K}Q(fg,R)_{rsijlm} - Q(h^2,E)_{rsijlm} \tag{3.8}$$

holds good for any h-hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures.

Transvecting (3.8) with g^{jl} we get

$$R \cdot \rho = KQ(g, \rho) + \overset{*}{K}(fg, \rho) + Q(h, \mathrm{tr}\, h\, h^2 + \mathrm{tr}\, k\, fh^2 - 2h^3). \tag{3.9}$$

Thus, we have

Proposition 3.2. *The relation (3.9) holds good for any h-hypersurface of an anti-Kähler manifold of constant totally real sectional curvatures.*

4. *H-pseudosymmetry*

In the case of anti-Kähler manifolds, we adopt the conditions (1.1) and (1.2) to the complex structure of the manifold introducing the following

Definition. *The anti-Kähler manifold (M, g, f) is said to be h-pseudosymmetric if the condition*

$$R \cdot R = \mathcal{L}_1 Q(g, R) + \mathcal{L}_2 Q(fg, R) \tag{4.1}$$

is satisfied on some set $U \subset M$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are some scalar function on U.

The manifold (M, g, f) is said to be Ricci h-pseudosymmetric if the condition

$$R \cdot \rho = \mathcal{L}_1 Q(g, \rho) + \mathcal{L}_2 Q(fg, \rho) \tag{4.2}$$

is satisfied on U.

Now, we consider h-hypersurface (M, g, f) of the anti-Kähler manifold of constant totally real sectional curvatures. Then, according to the Proposition 3.2, the relation (3.9) holds good. Thus, if (M, g, f) is also Ricci h-pseudosymmetric, then we have

$$(\mathcal{L}_1 - K)Q(g, \rho) + (\mathcal{L}_2 - \overset{*}{K})Q(fg, \rho) = Q(h, \mathrm{tr}\, h\, h^2 + \mathrm{tr}\, k\, fh^2 - 2h^3). \tag{4.3}$$

We shall examine two cases.

Case (1). If

$$\mathcal{L}_1 = K \qquad \text{and} \qquad \mathcal{L}_2 = \overset{*}{K}, \tag{4.4}$$

then (4.3) reduces to

$$Q(h, \mathrm{tr}\ h\ h^2 + \mathrm{tr}\ k\ fh^2 - 2h^3) = 0$$

and, in view of Lemma 2.1, we have

$$h^3 = \frac{1}{2}\mathrm{tr}\ h\ h^2 + \frac{1}{2}\mathrm{tr}\ k\ fh^2 + \delta h + \bar{\delta} fh\ . \tag{4.5}$$

Conversely, if (4.5) holds, then

$$Q(h, \mathrm{tr}\ h\ h^2 + \mathrm{tr}\ k\ fh^2 - 2h^3) =$$
$$= -2\delta Q(h,h) - 2\bar{\delta} Q(h, fh) = 0\ ,$$

and (3.9) reduces to

$$R \cdot \rho = KQ(g,\rho) + \overset{*}{K}Q(fg,\rho)\ ,$$

i.e., (4.4) holds.

Thus, we can state

Theorem 4.1. *Let (M,g,f) be h-hypersurface of the anti-Kähler manifold (M,G,F) of constant totally real sectional curvatures. Then (4.5) is the necessary and the sufficient condition for (M,g,f) to be Ricci h-pseudosymmetric on the appropriate set $U \subset M$ such that (4.4) holds.*

Remark. According to (3.4), (4.4) turns into

$$\mathcal{L}_1 = \frac{\widetilde{\kappa}}{4n(n+1)}\ , \qquad \mathcal{L}_2 = -\frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)}\ ,$$

where $\widetilde{\kappa}$ and $\widetilde{\overset{*}{\kappa}}$ are the first and the second scalar curvatures of $\widetilde{M}$ and $\dim M = 2n$.

Corollary. *Let (M,g,f) be h-hypersurface of the flat anti-Kähler manifold. Then (4.5) is the necessary and the sufficient condition for (M,g,f) to be Ricci semisymmetric.*

Case (2) If

$$\lambda_1 = \mathcal{L}_1 - \frac{\widetilde{\kappa}}{4n(n+1)} \neq 0 \quad \text{and} \quad \lambda_2 = \mathcal{L}_2 + \frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)} \neq 0\ ,$$

then (4.3) gives

$$\lambda_1 Q(g,\rho) + \lambda_2 Q(fg,\rho) = Q(h, \mathrm{tr}\ h\ h^2 + \mathrm{tr}\ k\ fh^2 - 2h^3)\ , \tag{4.6}$$

from which it follows that

$$\lambda_1 Q(fg,\rho) - \lambda_2 Q(g,\rho) = -Q(h, \mathrm{tr}\ k\ h^2 - \mathrm{tr}\ h\ fh^2 + 2fh^3)\ . \tag{4.7}$$

In the local coordinates, the left hand side of (4.6) is the following

$$\lambda_1(g_{ri}\rho_{sj} + g_{rj}\rho_{si} - g_{si}\rho_{rj} - g_{sj}\rho_{ri} - f_{ri}\ \overset{*}{\rho}_{sj} - f_{rj}\ \overset{*}{\rho}_{si} + f_{si}\ \overset{*}{\rho}_{rj} + f_{sj}\ \overset{*}{\rho}_{ri})$$
$$+\lambda_2(f_{ri}\rho_{sj} + f_{rj}\rho_{si} - f_{si}\rho_{rj} - f_{sj}\rho_{ri} + g_{ri}\ \overset{*}{\rho}_{sj} + g_{rj}\ \overset{*}{\rho}_{si} - g_{si}\ \overset{*}{\rho}_{rj} - g_{sj}\ \overset{*}{\rho}_{ri}),$$

from which, by transvection with g^{ri} we get

$$\lambda_1(2n\rho - \kappa g + \overset{*}{\kappa} fg) + \lambda_2(2n\ \overset{*}{\rho} - \kappa fg - \overset{*}{\kappa} g).$$

In the similar way, we obtain from

$$Q(h, \mathrm{tr}\ h\ h^2 + \mathrm{tr}\ k\ fh^2 - 2h^3)$$

the following expression

$$-\left[\mathrm{tr}\ h\ \mathrm{tr}\ h^2 + \mathrm{tr}\ k\ \mathrm{tr}(fh^2) - 2\mathrm{tr}\ h^3\right] h + \left[\mathrm{tr}\ h\ \mathrm{tr}(fh^2) - \mathrm{tr}\ k\ \mathrm{tr}\ h^2 - 2\mathrm{tr}(fh^3)\right] fh$$
$$+\left[(\mathrm{tr}\ h)^2 - (\mathrm{tr}\ k)^2\right] h^2 + 2\mathrm{tr}\ h\ \mathrm{tr}\ k\ fh^2 - 2\mathrm{tr}\ h\ h^3 - 2\mathrm{tr}\ k\ fh^3\ .$$

Thus, as a consequence of (4.6), we have

$$\begin{aligned}
&\lambda_1 Q(2n\rho - \kappa g + \overset{*}{\kappa} fg, h) + \lambda_2 Q(2n\ \overset{*}{\rho} - \kappa fg - \overset{*}{\kappa} g, h) \\
&= -\left[(\mathrm{tr}h)^2 - (\mathrm{tr}k)^2\right] Q(h,h^2) - 2\mathrm{tr}h\ \mathrm{tr}k\ Q(h,fh^2) \\
&\qquad +2\mathrm{tr}h\ Q(h,h^3) + 2\mathrm{tr}k\ Q(h,fh^3)\ .
\end{aligned} \tag{4.8}$$

But, the right hand side of (4.8) can be written in the form

$$\begin{aligned}
&\left[-(\mathrm{tr}h)^2 + (\mathrm{tr}k)^2\right] Q(h,h^2) - 2\mathrm{tr}h\ \mathrm{tr}k\ Q(h,fh^2) \\
&\qquad +2\mathrm{tr}h\ Q(h,h^3) + 2\mathrm{tr}k\ Q(h,fh^3) \\
&= -\mathrm{tr}h\ \left[\mathrm{tr}h\ Q(h,h^2) + \mathrm{tr}k\ Q(h,fh^2) - 2Q(h,h^3)\right] \\
&\quad +\mathrm{tr}k\ \left[\mathrm{tr}k\ Q(h,h^2) - \mathrm{tr}h\ Q(h,fh^2) + 2Q(h,fh^3)\right] \\
&= -\mathrm{tr}h\ Q(h, \mathrm{tr}h\ h^2 + \mathrm{tr}k\ fh^2 - 2h^3) + \mathrm{tr}k\ Q(h, \mathrm{tr}k\ h^2 - \mathrm{tr}h\ fh^2 + 2fh^3) \\
&= -\mathrm{tr}h\ \left[\lambda_1 Q(g,\rho) + \lambda_2 Q(fg,\rho)\right] + \mathrm{tr}k\ \left[-\lambda_1 Q(fg,\rho) + \lambda_2 Q(g,\rho)\right]\ ,
\end{aligned}$$

because of (4.6) and (4.7). Thus, (4.8) is of the form

$$\begin{aligned}&\lambda_1 Q(2n\rho-\kappa g+\overset{*}{\kappa}fg,h)+\lambda_2 Q(2n\overset{*}{\rho}-\kappa fg-\overset{*}{\kappa}g,h)\\&=-\lambda_1\left[\mathrm{tr}h\, Q(g,\rho)+\mathrm{tr}k\, Q(fg,\rho)\right]+\lambda_2\left[\mathrm{tr}k\, Q(g,\rho)-\mathrm{tr}h\, Q(fg,\rho)\right]\ .\end{aligned}\tag{4.9}$$

On the other hand, using (3.5), we have

$$\begin{aligned}&Q(2n\rho-\kappa g+\overset{*}{\kappa}fg,h)\\&=\left[4n(n-1)K-\kappa\right]Q(g,h)+\left[4n(n-1)\overset{*}{K}+\overset{*}{\kappa}\right]Q(fg,h)+4nQ(h,h^2)\ .\end{aligned}$$

Similarly

$$\begin{aligned}&Q(2n\overset{*}{\rho}-\kappa fg-\overset{*}{\kappa}g,h)\\&=\left[4n(n-1)K-\kappa\right]Q(fg,h)-\left[4n(n-1)\overset{*}{K}+\overset{*}{\kappa}\right]Q(g,h)+4nQ(h,fh^2)\ ,\end{aligned}$$

such that (4.9) becomes

$$\begin{aligned}&\lambda_1\{[4n(n-1)K-\kappa]Q(g,h)+[4n(n-1)\overset{*}{K}+\overset{*}{\kappa}]Q(fg,h)\\&\qquad+4nQ(h,h^2)+\mathrm{tr}h\, Q(g,\rho)+\mathrm{tr}k\, Q(fg,\rho)\}\\&+\lambda_2\{[4n(n-1)K-\kappa]Q(fg,h)-[4n(n-1)\overset{*}{K}+\overset{*}{\kappa}]Q(g,h)\\&\qquad+4nQ(h,fh^2)-\mathrm{tr}k\, Q(g,\rho)+\mathrm{tr}h\, Q(fg,\rho)\}=0\ .\end{aligned}\tag{4.10}$$

If we set

$$\begin{aligned}P=\ &[4n(n-1)K-\kappa]Q(g,h)+[4n(n-1)\overset{*}{K}+\overset{*}{\kappa}]Q(fg,h)\\&+4nQ(h,h^2)+\mathrm{tr}h\, Q(g,\rho)+\mathrm{tr}k\, Q(fg,\rho)\ ,\end{aligned}$$

then

$$\begin{aligned}fP=\ &[4n(n-1)K-\kappa]Q(fg,h)-[4n(n-1)\overset{*}{K}-\overset{*}{\kappa}]Q(g,h)\\&+4nQ(fh,h^2)+\mathrm{tr}h\, Q(fg,\rho)-\mathrm{tr}k\, Q(g,\rho)\ .\end{aligned}$$

But

$$Q(h,fh^2)=Q(fh,h^2)\ .$$

This means that (4.10) can be expressed in the form

$$\lambda_1 P+\lambda_2 fP=0\ .$$

This relation, together with

$$-\lambda_2 P + \lambda_1 f P = 0 \ ,$$

yields

$$(\lambda_1^2 + \lambda_2^2) P = 0 \ ,$$

and $P = 0$ if at last one of the conditions $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ is satisfied. Thus we have

$$\begin{aligned} &[4n(n-1)K - \kappa]Q(g,h) + [4n(n-1) \overset{*}{K} + \overset{*}{\kappa}]Q(fg,h) \\ &+4nQ(h,h^2) + \mathrm{tr}h\ Q(g,\rho) + \mathrm{tr}k\ Q(fg,\rho) = 0 \ , \end{aligned}$$

or

$$\begin{aligned} &[4n(n-1)K - \kappa + (\mathrm{tr}h)^2 - (\mathrm{tr}k)^2]Q(g,h) \\ &+[4n(n-1) \overset{*}{K} + \overset{*}{\kappa} + 2\mathrm{tr}h\ \mathrm{tr}k]Q(fg,h) \\ &-2\mathrm{tr}h\ Q(g,h^2) - 2\mathrm{tr}k\ Q(g,fh^2) + 4nQ(h,h^2) = 0 \ , \end{aligned} \tag{4.11}$$

because of

$$\begin{aligned} Q(g,\rho) &= \mathrm{tr}h\ Q(g,h) + \mathrm{tr}k\ Q(g,fh) - 2Q(g,h^2) \ , \\ Q(fg,\rho) &= \mathrm{tr}h\ Q(fg,h) + \mathrm{tr}k\ Q(g,h) - 2Q(g,fh^2) \ . \end{aligned}$$

Finally, according to (3.6),

$$\begin{aligned} &4n(n-1)K - \kappa + (\mathrm{tr}h)^2 - (\mathrm{tr}k)^2 = 2\mathrm{tr}(h^2), \\ &4n(n-1) \overset{*}{K} + \overset{*}{\kappa} + 2\mathrm{tr}h\ \mathrm{tr}k = -2\mathrm{tr}(fh^2) \ , \end{aligned}$$

and (4.11) becomes

$$\begin{aligned} &\mathrm{tr}h^2 Q(g,h) - \mathrm{tr}(fh^2)Q(fg,h) \\ &-\mathrm{tr}h\ Q(g,h^2) - \mathrm{tr}k\ Q(g,fh^2) + 2nQ(h,h^2) = 0 \ . \end{aligned} \tag{4.12}$$

On the other hand,

$$\begin{aligned} &Q(h - \frac{\mathrm{tr}h}{2n}g + \frac{\mathrm{tr}(fh)}{2n}fg,\ h^2 - \frac{\mathrm{tr}h^2}{2n}g + \frac{\mathrm{tr}(fh^2)}{2n}fg) \\ &= Q(h,h^2) + Q(g, \frac{\mathrm{tr}h^2}{2n}h) - Q(g,\ \frac{\mathrm{tr}(fh^2)}{2n}fh) \\ &\quad -Q(g,\ \frac{\mathrm{tr}h}{2n}h^2) + Q(g,\ \frac{\mathrm{tr}fh}{2n}fh^2) \ . \end{aligned}$$

This, in view of (4.12), means that

$$Q(h - \frac{\mathrm{tr}\, h}{2n} g + \frac{\mathrm{tr}\,(fh)}{2n} fg,\ h^2 - \frac{\mathrm{tr}\, h^2}{2n} g + \frac{\mathrm{tr}\,(fh^2)}{2n} fg) = 0$$

from which, applying Lemma 2.1, we get

$$\begin{aligned} h^2 - \frac{\mathrm{tr}\, h^2}{2n} g + \frac{\mathrm{tr}\,(fh^2)}{2n} fg = \delta(h - \frac{\mathrm{tr}\, h}{2n} g + \frac{\mathrm{tr}\,(fh)}{2n} fg) \\ + \bar{\delta}(fh - \frac{\mathrm{tr}\, h}{2n} fg - \frac{\mathrm{tr}\,(fh)}{2n} g)\ , \end{aligned}$$

or

$$h^2 = \delta h + \bar{\delta} fh + \mu g + \bar{\mu} fg\ , \tag{4.13}$$

where

$$\begin{aligned} \mu &= \frac{\mathrm{tr}\, h^2}{2n} - \delta \frac{\mathrm{tr}\, h}{2n} - \bar{\delta} \frac{\mathrm{tr}\,(fh)}{2n}\ , \\ \bar{\mu} &= -\frac{\mathrm{tr}\,(fh^2)}{2n} + \delta \frac{\mathrm{tr}\,(fh)}{2n} - \bar{\delta} \frac{\mathrm{tr}\, h}{2n}\ . \end{aligned}$$

Conversely, if (4.13) holds, then

$$\begin{aligned} Q(h^2, E) &= Q(\delta h + \bar{\delta} fh + \mu g + \bar{\mu} fg, E) \\ &= \delta Q(h, E) + \bar{\delta} Q(fh, E) + \mu Q(g, E) + \bar{\mu} Q(fg, E). \end{aligned}$$

But

$$Q(h, E) = Q(fh, E) = 0\ ,$$

and therefore

$$Q(h^2, E) = \mu Q(g, E) + \bar{\mu} Q(fg, E)\ . \tag{4.14}$$

On the other hand, in view of $Q(g, G) = Q(fg, G) = 0$, we have

$$\mu Q(g,\ KG + \overset{*}{K} fG) = 0\ , \qquad \bar{\mu} Q(fg,\ KG + \overset{*}{K} fG) = 0\ ,$$

that is, the relation (4.14) is equivalent to

$$Q(h^2, E) = \mu Q(g, KG + \overset{*}{K} fG + E) + \bar{\mu} Q(fg, KG + \overset{*}{K} fG + E)\ .$$

In the other words

$$Q(h^2, E) = \mu Q(g, R) + \bar{\mu} Q(fg, R) . \tag{4.15}$$

According to the Proposition 3.1, for any h-hypersurface of the anti-Kähler manifold of constant totally real sectional curvatures, the relation (3.8) holds, which, in view of (4.15) becomes

$$R \cdot R = (K - \mu) Q(G, R) + (\overset{*}{K} - \bar{\mu}) Q(fg, R) .$$

This means that if (4.15) holds, then (M, g, f) is h-pseudosymmetric. But h-pseudosymmetric manifold is Ricci h-pseudosymmetric, too. Thus, we can state

Theorem 4.2. *Let (M, g, f), $\dim M = 2n$, be a h-hypersurface of the anti-Kähler manifold $(\widetilde{M}, G, F)$ of constant totally real sectional curvatures. Let $\widetilde{\kappa}$ and $\widetilde{\overset{*}{\kappa}}$ be the first and the second scalar curvatures of $(\widetilde{M}, G, F)$. Then (4.13) is the necessary and the sufficient condition for (M, g, f) to be, on the appropriate set $U \subset M$, Ricci h-pseudosymmetric such that at least one of the relations*

$$\mathcal{L}_1 \neq \frac{\widetilde{\kappa}}{4n(n+1)} , \qquad \mathcal{L}_2 \neq \frac{\widetilde{\overset{*}{\kappa}}}{4n(n+1)}$$

is satisfied.

5. Remark. H-pseudosymmetry is also considered in [5]. In that paper it is proved that every anti-Kähler manifold satisfying the Roter type equation

$$\begin{aligned} R(X,Y,Z,W) = \; & N_1 \Gamma(X,Y,Z,W) + N_2 \Gamma(fX,Y,Z,W) \\ & + N_3 G(X,Y,Z,W) + N_4 G(fX,Y,Z,W) , \end{aligned}$$

on some set $U \subset M$, is h-pseudosymmetric, where

$$\begin{aligned} \Gamma(X,Y,Z,W) = \; & \rho(X,W)\rho(Y,Z) - \rho(X,Z)\rho(Y,W) \\ & - \overset{*}{\rho}(X,W) \overset{*}{\rho}(Y,Z) + \overset{*}{\rho}(X,Z) \overset{*}{\rho}(Y,W) , \end{aligned}$$

and $N_1, \ldots, N_4$ are some scalar functions on U.

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