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TIME-DEPENDENT PERTURBATIONS OF ABSTRACT VOLTERRA EQUATIONS

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A b s t r a c t. The main purpose of this review is to provide a detailed analysis of results on time-dependent perturbations of abstract Volterra equations and abstract time-fractional equations with Caputo fractional derivatives. The results obtained are illustrated with some examples.

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1. *Introduction and Preliminaries*

In recent years, considerable research efforts have been directed towards the development of the theory of ill-posed abstract Volterra equations ([2], [6]-[7], [10]-[15], [19]-[20], [23]-[24], [31], [35]). For general information about

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the theory of abstract Volterra integrodifferential equations, the reader may consult the monograph [27] of J. Prűss. The purpose of this paper is to give an exposition of results on time-dependent perturbations of abstract Volterra equations and abstract time-fractional equations with Caputo fractional derivatives ([4]-[5], [15]-[18], [21], [26] and [32]-[34]). For further information concerning time-dependent perturbations of abstract time-fractional equations with Riemann-Liouville fractional derivatives, we refer the reader to the recent paper [3] by Kh. K. Avad and A. V. Glushak.

Now we will briefly describe the notion and terminology used throughout the paper. Henceforth E denotes a complex Banach space and $||x||$ denotes the norm of an element $x \in E$. If X, Y are Banach spaces, then $L(X, Y)$ denotes the space of all continuous linear mappings from X into Y; $L(E) := L(E, E)$. We assume that A is a closed linear operator acting on E and that $L(E) \ni C$ is an injective operator with $CA \subseteq AC$. The domain and range of A are denoted by $D(A)$ and $R(A)$, respectively. Recall that the *C-resolvent set of A*, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\}$. By $[R(C)]$ we denote the Banach space $R(C)$ equipped with the norm $||x||_{[R(C)]} := ||C^{-1}x||$, $x \in R(C)$. The convolution like mapping $*$ is given by $f * g(t) := \int_0^t f(t-s)g(s)ds$ and the principal branch is always used to take the powers. If $\beta \in (0, \pi]$ and $s \in \mathbb{R}$, put $\Sigma_\beta := \{z \in \mathbb{C} : z \neq 0, \ |\arg(z)| < \beta\}$ and $\lceil s \rceil := \inf\{k \in \mathbb{Z} : k \geq s\}$. Given $T > 0$ in advance, $BV[0, T]$ ($AC[0, T]$) denotes the space of all scalar-valued functions that are of bounded variation on $[0, T]$ (the space of all scalar-valued absolutely continuous functions on $[0, T]$). The Sobolev space $W^{1,1}([0, T] : E)$ is defined by $W^{1,1}([0, T] : E) = \{f \in L^1([0, T] : E) : f(s) = f(s_0) + \int_{s_0}^s g(\sigma)d\sigma$ for some $s_0 \in [0, T]$ and $g \in L^1([0, T] : E)\}$. Let $\alpha > 0$, let $\beta > 0$ and let the *Mittag-Leffler function* $E_{\alpha,\beta}(z)$ be defined by $E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$, $z \in \mathbb{C}$. Set, for short, $E_\alpha(z) := E_{\alpha,1}(z)$, $z \in \mathbb{C}$.

Definition 1.1. Let $0 < \tau \leq \infty$ and let $a \in L^1_{loc}([0, \tau))$, $a \neq 0$. A strongly continuous operator family $(R(t))_{t \in [0,\tau)}$ is called a (*local*, if $\tau < \infty$) *(a, C)-regularized resolvent family* having A as a *subgenerator* iff the following holds:

(a) $R(t)A \subseteq AR(t), \ t \in [0, \tau)$, $R(0) = C$ and $CA \subseteq AC$,

(b) $R(t)C = CR(t), \ t \in [0, \tau)$ and

(c) $R(t)x = Cx + \int_0^t a(t-s)AR(s)xds, \ t \in [0, \tau), \ x \in D(A)$;

if $\tau = \infty$, then $(R(t))_{t\geq 0}$ is said to be *exponentially bounded* if there exist $M \geq 1$ and $\omega \geq 0$ such that $||R(t)|| \leq Me^{\omega t}$, $t \geq 0$.

An (a, C)-regularized resolvent family is a special case of the notion of an (a, k)*-regularized* C*-resolvent family* (cf. [11]-[12] and [19]-[20] for the notion and explicit examples). We would like to note, here, that the obtained results cannot be so easily interpreted in this general framework. Concerning the C-wellposedness of the following abstract time-fractional equation with $\alpha > 0$:

$$\mathbf{D}_t^\alpha u(t) = Au(t),\ t > 0;\ u^{(k)}(0) = x_k,\ k = 0, 1, \cdots, \lceil\alpha\rceil - 1, \qquad (1)$$

where $x_k \in D(A)$, $k = 0, 1, \cdots, \lceil\alpha\rceil - 1$ and $\mathbf{D}_t^\alpha$ denotes the *Caputo fractional derivative of order* α ([4]), the reader is referred to [4], [11]-[13] and [19].

2. *Time-Dependent Perturbations of Abstract Volterra Equations*

Multiplicative (time-dependent) perturbations of abstract Volterra equations have been considered in [5], [11], [21], [24] and [32]-[34]. We start this section by strengthening results on multiplicative time-dependent Desch-Schappacher type perturbations of abstract Volterra equations established by T.-J. Xiao, J. Liang and J. van Casteren in [32], the paper of fundamental importance in our work. Keeping in mind the argumentation given in this paper, the proofs of subsequent assertions become straightforward and therefore omitted. Our standing hypothesis will be:

(H): A is a subgenerator of an (a, C)-regularized resolvent family $(V(t))_{t\in[0,\tau)}$ such that:

$$V(t)x = Cx + A\int_0^t a(t-s)V(s)x\,ds,\ t \in [0, \tau),\ x \in E, \qquad (2)$$

where $0 < \tau \leq \infty$. Unless stated otherwise, we assume that $T \in (0, \tau)$.

If (H) holds with $a(t) = 1$ ($a(t) = t$), then it is also said that A is a subgenerator of a (local) C*-regularized semigroup* $(V(t))_{t\in[0,\tau)}$ (C*-regularized cosine function* $(V(t))_{t\in[0,\tau)}$).

Theorem 2.1. *Assume* (H) *holds,* $a \in BV[0, T]$, $G_0 \in L(C([0, T] : E))$, $G = CG_0 + I$ *and the following conditions:*

(a) $G_0(\psi) \in W^{1,1}([0, T] : E)$ *for all* $\psi \in C^1([0, T] : E)$.

(b) $||G_0(\psi)(t)|| \leq M_0 \sup_{0\leq s\leq t} ||\psi(s)||$, $\psi \in C([0,T]:E)$, $t \in [0,T]$, *for an appropriate constant* $M_0 > 0$.

(c) *For every* $\psi \in C([0,T]:E)$, $\int_0^t \tilde{V}(t-\sigma)G_0(\psi)(\sigma)d\sigma \in D(A)$ *and there exists* $M > 0$ *such that, for every* $t \in [0,T]$ *and* $\psi \in C([0,T]:E)$,

$$\Big\| A\int_0^t \tilde{V}(t-\sigma)G_0(\psi)(\sigma)d\sigma \Big\| \leq M\int_0^t \sup_{0\leq s\leq\sigma} \|\psi(s)\|d\sigma, \tag{3}$$

where $\tilde{V}(\sigma)x := a(0)V(\sigma)x + \int_0^\sigma V(\sigma-\tau)x da(\tau)$, $\sigma \in [0,T]$, $x \in E$.

Then the following holds:

(i) *If* $C^{-1}f \in W^{1,1}([0,T]:[D(A)])$, *then there exists a unique solution* $\mathcal{V}_f \in C([0,T]:[D(A)])$ *of the integral equation*

$$\mathcal{V}(t) = f(t) + \int_0^t a(t-s)G(A\mathcal{V})(s)ds,\ t \in [0,T], \tag{4}$$

which is given by $\mathcal{V}_f(t) := \sum_{m=0}^\infty v_m(t)$, $t \in [0,T]$, *where*

$$v_0(t) := V(t)C^{-1}f(0) + \int_0^t V(t-s)(C^{-1}f)'(s)ds,\ t \in [0,T]$$

and

$$v_m(t) := \int_0^t \tilde{V}(t-s)G_0(Av_{m-1})(s)ds,\ m \in \mathbb{N},\ t \in [0,T]. \tag{5}$$

(ii) *If* $C^{-1}f \in W^{1,1}([0,T]:E)$, *then there exists a unique solution* $\mathcal{W}_f \in C([0,T]:E)$ *of the integral equation*

$$\mathcal{W}(t) = f(t) + A\int_0^t a(t-s)G(\mathcal{W})(s)ds,\ t \in [0,T], \tag{6}$$

which is given by $\mathcal{W}_f(t) := \sum_{m=0}^\infty w_m(t)$, $t \in [0,T]$, *where* $w_0(t) := v_0(t)$, $t \in [0,T]$ *and*

$$w_m(t) := A\int_0^t \tilde{V}(t-s)G_0(w_{m-1})(s)ds,\ m \in \mathbb{N},\ t \in [0,T].$$

Corollary 2.1. *Assume* (H) *holds,* $a \in BV[0,T]$, *the function* $B_0 : [0,T] \to L(E)$ *is strongly continuously differentiable and* $B(\sigma) := CB_0(\sigma) + I$, $\sigma \in [0,T]$. *Suppose* $M > 0$, $\int_0^t \tilde{V}(t-\sigma)B_0(\sigma)\psi(\sigma)d\sigma \in D(A)$, $\psi \in C([0,T] : E)$ *and* (3) *holds with* $G_0(\psi)(\cdot)$ *replaced by* $B_0(\cdot)\psi(\cdot)$. *Then* (4) *has a unique solution* $\mathcal{V}_f$ *provided* $C^{-1}f \in W^{1,1}([0,T] : [D(A)])$ *and* (6) *has a unique solution* $\mathcal{W}_f$ *provided* $C^{-1}f \in W^{1,1}([0,T] : E)$.

Suppose $0 < \epsilon < T < \tau$ and $a(t) > 0$, $t \in (0,\epsilon)$. Then the *Favard class of* $(V(t))_{t\in[0,\tau)}$ is defined by

$$F_V := \Big\{x \in E : \overline{\lim}_{t\to 0+}\Big\|\Big(\int_0^t a(s)ds\Big)^{-1}(V(t)x - Cx)\Big\| < \infty\Big\}.$$

Equipped with the norm

$$\|x\|_{F_V} := ||x|| + \overline{\lim}_{t\to 0+}\Big\|\Big(\int_0^t a(s)ds\Big)^{-1}(V(t)x - Cx)\Big\|,$$

the Favard class F_V becomes a Banach space (cf. also [11, (2.51), Theorem 2.26] and [25, Section 3]).

Corollary 2.2. *Assume* (H) *holds,* $\epsilon \in (0,T)$ *and* $a \in BV[0,T]$.

(i) *Let* $B_0 : [0,T] \to L(E, [D(A)])$ *be strongly continuous, or*

(ii) *Let* $a(t) > 0$, $t \in (0,\epsilon)$, *let* $B_0 : [0,T] \to L(E, F_V)$ *be strongly continuous and let* $a(t) - \alpha t^k = o(t^k)$ $(t \to 0+)$ *for certain* $k \in \mathbb{N}_0$ *and* $\alpha \neq 0$.

Then the conclusions of Corollary 2.1 hold.

Corollary 2.3. *Assume* (H) *holds and* $\epsilon \in (0,T)$.

(i) *Let* $B_0 : [0,T] \to L(E, [D(A)])$ *be strongly measurable and* $||B_0||_{E\to[D(A)]} \in L^\infty[0,T]$, *or*

(ii) *Let* $a(t) > 0$, $t \in (0,\epsilon)$, *let* $B_0 : [0,T] \to L(E, F_V)$ *be strongly measurable,* $||B_0||_{E\to F_V} \in L^\infty[0,T]$, *and let* $a(t) - \alpha t^k = o(t^k)$ $(t \to 0+)$ *for certain* $k \in \mathbb{N}_0$ *and* $\alpha \neq 0$.

If $a \in AC[0,T]$, *then the conclusions of Corollary 2.1 hold.*

The following corollary is an extension of [5, Theorem 2.2], [17, Theorem 2.1], [21, Theorem 2.1], [32, Corollary 2.6] and [34, Theorem 3] (cf. also [18, Theorem 2.3]). The existence of a unique strongly continuous operator family $(V_{B,C}(t))_{t\in[0,\tau)}$ satisfying $A(I+B)\int_0^t a(t-s)V_{B,C}(s)xds = V_{B,C}(t)x - Cx$, $t \in [0,\tau)$, $x \in E$ can be proved even if the condition $C_1A(I+B) \subseteq A(I+B)C_1$ is not included in the analysis; in such a way, we obtain an extension of [21, Theorem 2.3]. Notice also that the condition (7) holds provided $R(C^{-1}B) \subseteq D(A)$.

Corollary 2.4. *Let* (H) *hold and let* $a \in BV_{loc}([0,\tau))$. *Suppose* $B \in L(E)$, $R(B) \subseteq R(C)$, *there exists an injective operator* $C_1 \in L(E)$ *satisfying* $R(C_1) \subseteq R(C)$, $C_1A(I+B) \subseteq A(I+B)C_1$ *and, for every* $T \in (0,\tau)$, *there exists* $M_T > 0$ *such that, for every* $\psi \in C([0,T]:E)$,

$$\Big\|A\int_0^t \tilde{V}(t-\sigma)C^{-1}B\psi(\sigma)d\sigma\Big\| \leq M_T\int_0^t \sup_{0\leq s\leq\sigma}\|\psi(s)\|d\sigma. \tag{7}$$

Then $A(I+B)$ *is a subgenerator of an* (a,C_1)*-regularized resolvent family* $(V_B(t))_{t\in[0,\tau)}$ *satisfying*

$$V_B(t)x = V(t)C^{-1}C_1x + A\int_0^t \tilde{V}(t-s)C^{-1}BV_B(s)xds,\ x \in E,\ t \in [0,\tau),$$

and (2) *with* A, $(V(t))_{t\in[0,\tau)}$ *and* C *replaced by* $A(I+B)$, $(V_B(t))_{t\in[0,\tau)}$ *and* C_1, *respectively. Furthermore, if* $\rho((I+B)A) \neq \emptyset$ *and* $BC_1 = C_1B$, *then* $(I+B)A$ *is a subgenerator of an* (a,C_1)*-regularized resolvent family* $(V_B^1(t))_{t\in[0,\tau)}$ *satisfying* (2) *with* A, $(V(t))_{t\in[0,\tau)}$ *and* C *replaced by* $(I+B)A$, $(V_B^1(t))_{t\in[0,\tau)}$ *and* C_1, *respectively.*

The following is an insignificant modification of [21, Example 2.10].

Example 2.1. Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary, let $\alpha \geq 1$ and let the Dirichlet Laplacian $A := \Delta$ on $E := L^2(\Omega)$ be defined by $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$ and $Af := \Delta f$, $f \in D(A)$. Assume $\gamma \in (0, \frac{\pi}{2})$, $d \in (0,1]$ and $\frac{1}{\alpha} < \beta < \frac{\pi}{2\gamma}$. Denote by Γ_γ the boundary of $\Sigma_\gamma \cup \{z \in \mathbb{C} : |z| \leq d\}$ and assume that Γ_γ is oriented in such a way that $\Im\lambda$ decreases along Γ_γ. Define, for every $\varepsilon > 0$,

$$S_\varepsilon(t)f := \frac{1}{2\pi i}\int_{\Gamma_\gamma} E_\alpha(t^\alpha\lambda)e^{-\varepsilon\lambda^\beta}(\lambda + A)^{-1}fd\lambda,\ f \in E,\ t \geq 0.$$

Then one can simply prove that, for every $\varepsilon > 0$, $(S_\varepsilon(t))_{t\geq 0}$ is a global (not exponentially bounded) $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, S_\varepsilon(0))$-regularized resolvent family with a subgenerator $-A$ and that $R(S_\varepsilon(0))$ is dense in E. Let $n \in \mathbb{N}$, $\lambda_i \in (-\infty, 0)$, $g_i,\ \omega_i \in E$, $Ag_i = \lambda_i g_i$ $(1 \leq i \leq n)$,

$$B_0 u := \sum_{i=1}^{n} (u, \omega_i)_{L^2(\Omega)} g_i \text{ and } B := A^{-1} B_0.$$

Then $R(S_\varepsilon(0)^{-1}B_0) \subseteq D(A)$, $A(I + B) = A + B_0$ and $R(B_0) \subseteq R(S_\varepsilon(0)^{-1})$ for all $\varepsilon > 0$. Applying Corollary 2.4 we get that, for every $\alpha \geq 1$ and $\varepsilon > 0$, there exists a unique strongly continuous operator family $(V_{B,\varepsilon}(t))_{t\geq 0}$ satisfying $(A + B_0)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} V_{B,\varepsilon}(s) f ds = V_{B,\varepsilon}(t) f - S_\varepsilon(0) f$, $t \geq 0$, $f \in E$. Assume now $\varepsilon > 0$, $x_i \in D(A)$ and $Ax_i \in R(S_\varepsilon(0))$ $(0 \leq i \leq \lceil\alpha\rceil)$. Define

$$u_\varepsilon(t) := \sum_{i=0}^{\lceil\alpha\rceil-1} \Big[\frac{t^i}{i!} x_i + \int_0^t \frac{(t-s)^{\alpha+i-1}}{\Gamma(\alpha+i)} S_\varepsilon(s) S_\varepsilon(0)^{-1} A x_i\Big] ds,\ t \geq 0.$$

Keeping in mind the representation $S_\varepsilon(t) f = \sum_{n=0}^{\infty} \frac{t^{\alpha n}}{\Gamma(\alpha n+1)} A^n S_\varepsilon(0) f$, $t \geq 0$, $f \in E$, it readily follows that $u_\varepsilon(t)$ is a unique solution of (1).

In a similar manner, one can prove the following results on time-dependent additive perturbations of integral Volterra equations (cf. [32, Section 3]).

Theorem 2.2. *Assume* (H) *holds,* $a \in BV[0,T]$, $G_0 \in L(C([0,T] : [D(A)]), C([0,T] : E))$ *and the following conditions hold:*

(a) $G_0(\psi) \in W^{1,1}([0,T] : E)$ *for all* $\psi \in C^1([0,T] : E)$.

(b) $||G_0(\psi)(t)|| \leq M_0 \sup_{0\leq s\leq t} ||\psi(s)||_{[D(A)]}$, $\psi \in C([0,T] : [D(A)])$, $t \in [0,T]$, *for an appropriate constant* $M_0 > 0$.

(c) *For every* $\psi \in C([0,T] : [D(A)])$, $\int_0^t \tilde{V}(t-\sigma) G_0(\psi)(\sigma) d\sigma \in D(A)$ *and there exists* $M > 0$ *such that, for every* $t \in [0,T]$ *and* $\psi \in C([0,T] : E)$,

$$\Big\|A \int_0^t \tilde{V}(t-\sigma) G_0(\psi)(\sigma) d\sigma\Big\| \leq M \int_0^t \sup_{0\leq s\leq\sigma} \|\psi(s)\|_{[D(A)]} d\sigma. \tag{8}$$

If $C^{-1} f \in W^{1,1}([0,T] : E)$, *then the integral equation*

$$u(t) = f(t) + \int_0^t a(t-s)(Au(s) + CG_0(u)(s)) ds,\ t \in [0,T], \tag{9}$$

has a unique solution in $C([0,T]:[D(A)])$, *which is given by* $u(t)=\sum_{m=0}^{\infty}v_m(t)$, $t\in[0,T]$, *where* $v_m(t)$ $(m\in\mathbb{N},\ t\in[0,T])$ *is given by replacing* $G_0(Av_{m-1})$ *in* (5) *by* $G_0(v_{m-1})$.

Example 2.2. Assume (H) holds, $a,\ b\in BV[0,T]$, $C^{-1}B_1:[0,T]\to L([D(A)]:E)$ is strongly continuous and $G_0(\psi)(t)=(b*C^{-1}B_1\psi)(t)$, $t\in[0,T]$, $\psi\in C([0,T]:[D(A)])$. If $C^{-1}f\in W^{1,1}([0,T]:[D(A)])$, then the integral equation

$$u(t)=f(t)+(a*(Au+b*B_1u))(t),\ t\in[0,T],$$

has a unique solution in $C([0,T]:[D(A)])$.

Corollary 2.5. *Assume* (H) *holds,* $a\in BV[0,T]$, $M>0$ *and* $B_0:[0,T]\to L([D(A)]:E)$ *is strongly continuously differentiable. If* $C^{-1}f\in W^{1,1}([0,T]:[D(A)])$, $\int_0^t\tilde{V}(t-s)B_0(s)\psi(s)ds\in D(A)$, $t\in[0,T]$, $\psi\in C([0,T]:[D(A)])$ *and* (8) *holds with* $G_0(\cdot)$ *replaced by* $B_0(\cdot)$, *then the integral equation* (9), *with* $CG_0(\cdot)$ *replaced by* $CB_0(\cdot)$ *therein, has a unique solution in* $C([0,T]:[D(A)])$, *which is given by* $u(t)=\sum_{m=0}^{\infty}v_m(t)$, $t\in[0,T]$, *where* $v_m(t)$ $(m\in\mathbb{N},\ t\in[0,T])$ *is given by replacing* $G_0(Av_{m-1})(s)$ *in* (5) *by* $B_0(s)v_{m-1}(s)$.

Corollary 2.6. *Assume* (H) *holds,* $a\in BV[0,T]$ *and:*

(i) $B_0:[0,T]\to L([D(A)])$ *is strongly continuous, or*

(ii) *There exists* $\epsilon\in(0,T)$ *such that* $a(t)>0$, $t\in(0,\epsilon)$, $B_0:[0,T]\to L([D(A)],F_V)$ *is strongly continuous, and* $a(t)-\alpha t^k=o(t^k)$ $(t\to0+)$ *for certain* $k\in\mathbb{N}_0$ *and* $\alpha\neq0$.

Then the conclusions of Corollary 2.5 hold.

Corollary 2.7. *Assume* (H) *holds and:*

(i) $B_0:[0,T]\to L([D(A)])$ *is strongly measurable and* $||B(\cdot)||_{L([D(A)])}\in L^{\infty}[0,T]$, *or*

(ii) *There exists* $\epsilon\in(0,T)$ *such that* $a(t)>0$, $t\in(0,\epsilon)$, $B_0:[0,T]\to L([D(A)],F_V)$ *is strongly measurable,* $||B(\cdot)||_{L([D(A)],F_V)}\in L^{\infty}[0,T]$, *and* $a(t)-\alpha t^k=o(t^k)$ $(t\to0+)$ *for certain* $k\in\mathbb{N}_0$ *and* $\alpha\neq0$.

Then the conclusions of Corollary 2.5 hold provided $a\in AC[0,T]$.

Assuming $s=0$, the following corollary can be simply reformulated for fractional resolvent families.

Corollary 2.8.

(i) *Assume A is a subgenerator of a C-regularized semigroup $(V(t))_{t\in[0,\tau)}$, $C^{-1}B : [0,T] \to L([D(A)], F_V)$ is strongly measurable, $||C^{-1}B(\cdot)||_{L([D(A)],F_V)} \in L^\infty[0,T]$ and $B(\cdot)x \in C([0,T] : E)$, $x \in D(A)$. Then, for every $s \in [0,T]$ and $x \in D(A)$, the following initial value problem*

$$\begin{cases} u'(t) = (A + B(t))u(t), \ t \in [s,T], \\ u(s) = Cx, \end{cases}$$

has a unique solution $\mathcal{U}(\cdot, s) \in C^1([s,t] : E) \cap C([s,t] : [D(A)])$, which is given by $\mathcal{U}(t,s) := \sum_{m=0}^{\infty} u_m(t,s)x$, $s \le t \le T$, where $u_0(t,s)x := V(t-s)x$, $s \le t \le T$ and $u_m(t,s) := \int_0^t V(t-\sigma)C^{-1}B(\sigma)u_{m-1}(\sigma,s)d\sigma$, $m \in \mathbb{N}$, $s \le t \le T$.

(ii) *Assume A is a subgenerator of a C-regularized cosine function $(V(t))_{t\in[0,\tau)}$, $C^{-1}B : [0,T] \to L([D(A)], F_V)$ is strongly measurable, $||C^{-1}B(\cdot)||_{L([D(A)],F_V)} \in L^\infty[0,T]$ and $B(\cdot)x \in C([0,T] : E)$, $x \in D(A)$. Then, for every $s \in [0,T]$ and x, $y \in D(A)$, the following initial value problem*

$$\begin{cases} u''(t) = (A + B(t))u(t), \ t \in [s,T], \\ u(s) = Cx, \ u'(s) = Cy, \end{cases}$$

has a unique solution $\mathcal{C}(\cdot, s) \in C^2([s,t] : E) \cap C([s,t] : [D(A)])$, which is given by $\mathcal{C}(t,s) := \sum_{m=0}^{\infty}(c_m(t,s)x + s_m(t,s)y)$, $s \le t \le T$, where

$$\begin{cases} s_0(t,s)x := \int\limits_0^{t-s} V(\sigma)xd\sigma, \ 0 \le s \le t \le T, \\ s_m(t,s)x := \int\limits_s^t s_0(t,\sigma)C^{-1}B(\sigma)s_{m-1}(\sigma,s)xd\sigma, \ m \in \mathbb{N}, \ 0 \le s \le t \le T \end{cases}$$

and

$$\begin{cases} c_0(t,s)x := V(t-s)x, \ s \le t \le T, \\ c_m(t,s)x := \int\limits_s^t s_0(t,\sigma)C^{-1}B(\sigma)c_{m-1}(\sigma,s)xd\sigma, \ m \in \mathbb{N}, \ 0 \le s \le t \le T. \end{cases}$$

The subsequent theorem is closely related to [4, Theorem 2.26] and can be applied to coercive differential operators considered by F.-B. Li, M. Li and Q. Zheng in [19, Section 4].

Theorem 2.3. *Suppose $\alpha > 1$, $M \geq 1$, $\omega \geq 0$ and A is a subgenerator of a (local) $(\frac{t^{\alpha-1}}{\Gamma(\alpha)}, C)$-regularized resolvent family $(S_\alpha(t))_{t\in[0,\tau)}$ satisfying $||S_\alpha(t)|| \leq Me^{\omega t}$, $t \in [0,\tau)$, and (2) with $V(\cdot)$ and $a(t)$ replaced by $S_\alpha(\cdot)$ and $\frac{t^{\alpha-1}}{\Gamma(\alpha)}$, respectively.*

(i) *Let $(B(t))_{t\in[0,\tau)} \subseteq L(E)$, $R(B(t)) \subseteq R(C)$, $t \in [0,\tau)$ and $C^{-1}B(\cdot) \in C([0,\tau) : L(E))$. If $C^{-1}f \in W^{1,1}_{loc}([0,\tau) : E)$, then there exists a unique solution of the integral equation*

$$u(t,f) = f(t) + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s,f)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}B(s)u(s,f)ds \tag{10}$$

in $C([0,\tau) : E)$. The solution $u(t,f)$ is given by $u(t,f) := \sum_{n=0}^{\infty} S_{\alpha,n}(t)$, $t \in [0,\tau)$, where we define $S_{\alpha,n}(t)$ $(t \in [0,\tau))$ recursively by $S_{\alpha,0}(t) := v_0(t)$ (cf. the formulation of Theorem 2.1) and

$$S_{\alpha,n}(t) := \int_0^t \int_0^{t-\sigma} \frac{(t-\sigma-s)^{\alpha-2}}{\Gamma(\alpha-1)}S_\alpha(s)C^{-1}B(\sigma)S_{\alpha,n-1}(\sigma)dsd\sigma.$$

Denote, for every $T \in (0,\tau)$, $K_T := \max_{t\in[0,T]} ||C^{-1}B(t)||$ and $F_T := ||C^{-1}f(0)|| + \int_0^T e^{-\omega s}||(C^{-1}f)'(s)||ds$. Then

$$||u(t,f)|| \leq Me^{\omega t}E_\alpha(MK_T t^\alpha)F_T,\ t \in [0,T] \tag{11}$$

and

$$\|u(t,f) - v_0(t)\| \leq Me^{\omega t}(E_\alpha(MK_T t^\alpha) - 1)F_T,\ t \in [0,T]. \tag{12}$$

(ii) *Let $(B(t))_{t\in[0,\tau)} \subseteq L([D(A)])$ be strongly continuous and let $C^{-1}B(\cdot) \in C([0,\tau) : L([D(A)]))$. If $C^{-1}f \in W^{1,1}_{loc}([0,\tau) : [D(A)])$, then there exists a unique solution of the integral equation (10) in $C([0,\tau) : [D(A)])$. Denote, for every $T \in (0,\tau)$, $K_{T,A} := \max_{t\in[0,T]} ||C^{-1}B(t)||_{L([D(A)])}$ and $F_{T,A} := ||C^{-1}f(0)||_{[D(A)]} + \int_0^T e^{-\omega s}||(C^{-1}f)'(s)||_{[D(A)]}ds$. Then*

$$||u(t,f)||_{[D(A)]} \leq Me^{\omega t}E_\alpha(MK_{T,A}t^\alpha)F_{T,A},\ t \in [0,T]$$

and

$$\|u(t,f) - v_0(t)\|_{[D(A)]} \leq Me^{\omega t}(E_\alpha(MK_{T,A}t^\alpha) - 1)F_{T,A},\ t \in [0,T].$$

P r o o f. We will only prove the first part of theorem. Inductively, we obtain that $||S_{\alpha,n}(t)|| \leq M^{n+1}K_T^nF_Te^{\omega t}\frac{t^{\alpha n}}{\Gamma(\alpha n+1)}$, $t \in [0,T]$, $n \in \mathbb{N}_0$, which implies that the series $\sum_{n=0}^{\infty} S_{\alpha,n}(t)$ converges uniformly on compact subsets of $[0,\tau)$ and that (11)-(12) hold. Clearly, $u(t,f) = v_0(t) + \int_0^t (\frac{\cdot^{\alpha-1}}{\Gamma(\alpha-2)} * S_\alpha)(t-s)C^{-1}B(s)u(s,f)ds$, $t \in [0,T]$. This implies

$$u(t,f) = f(t) + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}v_0(s)ds + [\frac{\cdot^{\alpha-1}}{\Gamma(\alpha-2)} * S_\alpha * C^{-1}B(\cdot)u(\cdot,f)](t)$$

$$= f(t) + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[u(s,f) - (\frac{\cdot^{\alpha-1}}{\Gamma(\alpha-2)} * S_\alpha * C^{-1}B(\cdot)u(\cdot,f))(s)]ds + [\frac{\cdot^{\alpha-1}}{\Gamma(\alpha-2)} * S_\alpha * C^{-1}B(\cdot)u(\cdot,f)](t)$$

$$= f(t) + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s,f)ds - A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(\frac{\cdot^{\alpha-1}}{\Gamma(\alpha-2)} * S_\alpha * C^{-1}B(\cdot)u(\cdot,f))(s)ds + [\frac{\cdot^{\alpha-1}}{\Gamma(\alpha-2)} * S_\alpha * C^{-1}B(\cdot)u(\cdot,f)](t)$$

$$= f(t) + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s,f)ds - [\frac{\cdot^{\alpha-2}}{\Gamma(\alpha-1)} * (S_\alpha(\cdot) - C) * C^{-1}Bu(\cdot)u(\cdot,f)](t) + [\frac{\cdot^{\alpha-1}}{\Gamma(\alpha-2)} * S_\alpha * C^{-1}B(\cdot)u(\cdot,f)](t)$$

$$= f(t) + A\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s,f)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}B(s)u(s,f)ds,\ t \in [0,\tau).$$

Therefore, $u(t,x)$ is a solution of (10). The uniqueness of solutions is left to the reader as an easy exercise.

The basic properties of hyperbolic Volterra equations of non-scalar type have been recently considered by the author in [14] (cf. also [27]). With the notion explained in [14], we have the following theorem.

Theorem 2.4.

(i) *Assume* $L^1_{loc}([0,\tau)) \ni a$ *is a kernel,* (H) *holds,*

$$A(t) = a(t)A + (a * B_1)(t) + B_0(t),\ t \in [0,\tau),$$

where $B_0(\cdot)$ *and* $B_1(\cdot)$ *satisfy the following conditions:* $(B_0(t))_{t\in[0,\tau)} \subseteq L([D(A)]) \cap L(E,[R(C)])$, $(B_1(t))_{t\in[0,\tau)} \subseteq L([D(A)],[R(C)])$,

(i) $C^{-1}B_0(\cdot)y \in BV_{loc}([0,\tau) : [D(A)])$ *for all* $y \in D(A)$, $C^{-1}B_0(\cdot)x \in BV_{loc}([0,\tau) : E)$ *for all* $x \in E$,

(ii) $C^{-1}B_1(\cdot)y \in BV_{loc}([0,\tau) : E)$ *for all* $y \in D(A)$, *and*

(iii) $CB(t)y = B(t)Cy$, $y \in D(A)$, $t \in [0,\tau)$.

Then there exists an a-regular A-regularized C-resolvent family $(R(t))_{t\in[0,\tau)}$.

(ii) *Assume A is a subgenerator of a C-regularized semigroup $(S(t))_{t\in[0,\tau)}$. If $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions stated in* (i), *then for every $x \in D(A)$ there exists a unique solution of the problem*

$$\begin{cases} u \in C^1([0,\tau) : E) \ \cap \ C([0,\tau) : [D(A)]), \\ u'(t) = Au(t) + (dB_0 * u)(t)x + (B_1 * u)(t) + Cx, \ t \in [0,\tau), \\ u(0) = 0. \end{cases}$$

Furthermore, the mapping $t \mapsto u(t)$, $t \in [0,\tau)$ is locally Lipschitz continuous in $[D(A)]$.

(iii) *Assume A is a subgenerator of a C-regularized cosine function $(C(t))_{t\in[0,\tau)}$. If $B_0(\cdot)$ and $B_1(\cdot)$ satisfy the assumptions stated in* (i), *then for every $x \in D(A)$ there exists a unique solution of the problem*

$$\begin{cases} u \in C^2([0,\tau) : E) \ \cap \ C([0,\tau) : [D(A)]), \\ u''(t) = Au(t) + (dB_0 * u')(t)x + (B_1 * u)(t) + Cx, \ t \in [0,\tau), \\ u(0) = u'(0) = 0. \end{cases}$$

Furthermore, the mapping $t \mapsto u(t)$, $t \in [0,\tau)$ is continuously differentiable in $[D(A)]$ and the mapping $t \mapsto u'(t)$, $t \in [0,\tau)$ is locally Lipschitz continuous in $[D(A)]$. iz (i)

Before proceeding further, we would like to note that the existing theory of time-dependent perturbations for abstract evolution equations of second order ([22], [30]) leans heavily on the notion of Kisyński's space [9]. Contrary to Corollary 2.8, the results obtained in the aforementioned papers cannot be transferred to abstract time-fractional equations without further non-trivial analyses. Further on, it does not seem plausible that the subsequent assertions [11] can be formulated in the context of time-dependent perturbations.

Theorem 2.5.

(i) *Assume $C([0,\infty)) \ni a$ satisfies* (P1), *$B \in L(E)$, $R(B) \subseteq R(C)$ and A is a subgenerator of an exponentially bounded (a,a)-regularized C-resolvent family $(R(t))_{t\geq 0}$ satisfying*

$$V(t)x = a(t)Cx + A\int_0^t a(t-s)V(s)x\,ds, \ t \geq 0, \ x \in E.$$

Assume, further, that there exists $\omega \geq 0$ such that, for every $h \geq 0$ and for every function $f \in C([0,\infty) : E)$, (Ma) $\int_0^h R(h-s)C^{-1}Bf(s)ds \in D(A)$,

(Mb) $\|A\int_0^h R(h-s)C^{-1}Bf(s)ds\| \leq e^{\omega t}\mu_B(h)||f||_{[0,h]}$, $t \geq 0$, *where* $||f||_{[0,h]} := \sup_{t\in[0,h]}||f(t)||$, $\mu_B(t) : [0,\infty) \to [0,\infty)$ *is continuous, non-decreasing and satisfies* $\mu_B(0) = 0$, *and*

(Mc) *There exists an injective operator* $C_1 \in L(E)$ *such that* $R(C_1) \subseteq R(C)$ *and that* $C_1A(I+B) \subseteq A(I+B)C_1$.

Then $A(I+B)$ *is a subgenerator of an exponentially bounded* (a,a)*-regularized* C_1*-resolvent family* $(S(t))_{t\geq 0}$ *which satisfies the following integral equation:*

$$S(t)x = R(t)C^{-1}C_1x + A\int_0^t R(t-s)C^{-1}BS(s)xds,\ t \geq 0,\ x \in E.$$

(ii) *Let* A *be a subgenerator of an exponentially bounded, once integrated* C*-cosine function and let* ω, B *and* C_1 *have the same meaning as in* (i). *Then* $A(I+B)$ *is a subgenerator of an exponentially bounded, once integrated* C_1*-cosine function.*

Proposition 2.1. *Let* $B \in L(E)$ *and* $BC = CB$.

(i) *Assume* BA *is a subgenerator of an* (a,k)*-regularized* C*-resolvent family* $(V(t))_{t\in[0,\tau)}$ *satisfying*

$$V(t)x = k(t)Cx + A\int_0^t a(t-s)V(s)xds,\ t \in [0,\tau),\ x \in E. \tag{13}$$

Then AB *is a subgenerator of an* (a,k)*-regularized* C*-resolvent family* $(V(t))_{t\in[0,\tau)}$.

(ii) *Assume* AB *is a subgenerator of an* (a,k)*-regularized* C*-resolvent family* $(V(t))_{t\in[0,\tau)}$ *satisfying* (13). *Then* BA *is a subgenerator of an* (a,k)*-regularized* C*-resolvent family, provided* $\rho(BA) \neq \emptyset$.

Recall that V. Keyantuo and M. Warma analyzed in [8] the generation of fractionally integrated cosine functions in L^p-spaces by elliptic differential operators with variable coefficients. Notice finally that Proposition 2.1(i) can be applied to these operators (cf. [8, Theorem 2.2 and pp. 78-79] and [29, Example 3.1] for more details).

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