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Gegenbauer Matrix Polynomials and Second Order Matrix Differential Equations

Polinomios Matriciales de Gegenbauer y Ecuaciones Diferenciales Matriciales de Segundo Orden

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Abstract

In this paper, we study the Gegenbauer matrix polynomials. An explicit representation, a three-term matrix recurrence relation and orthogonality property for the Gegenbauer matrix polynomials are given. These polynomials appear as finite series solutions of second-order matrix differential equations.
Key words and phrases: hypergeometric matrix function, orthogonal matrix polynomials, Gegenbauer matrix polynomials, three-terms matrix recurrence.

Resumen

En este artículo se estudian los polinomios matriciales de Gegenbauer. Se da una representación explícita, una relación de recurrencia matricial de tres términos y una propiedad de ortogonalidad para los polinomios matriciales de Gegenbauer. Estos polinomios aparecen como soluciones en series finitas de ecuaciones diferenciales matriciales de segundo orden.
Palabras y frases clave: función hipergeométrica matricial, polinomios matriciales ortogonales, polinomios matriciales de Gegenbauer, recurrencia matricial de tres términos.

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1 Introduction

It is well known that special matrix functions appear in the study in statistics, theoretical physics, groups representation theory and number theory [1, 2, 8, 15, 21]. Orthogonal matrix polynomials have been considered in the book on matrix polynomials by Gohberg, Lancaster and Rodman [7] and in the survey on orthogonal matrix polynomials by Rodman [17] and see more papers [4, 5, 6, 19, 20] and the references therein. During the last two decades the classical orthogonal polynomials have been extended to the orthogonal matrix polynomials see for instance [12, 13] . Hermite and Laguerre matrix polynomials was introduced and studied in [9, 10] and an accurate approximation of certain differential systems in terms of Hermite matrix polynomials was computed in [3]. Furthermore, a connection between Laguerre and Hermite matrix polynomials was established in [12]. Recently, the generalized Hermite matrix polynomials have been introduced and studied in [18]. Jódar and Cortés introduced and studied the hypergeometric matrix function F(*A,B;C;z*) and the hypergeometric matrix differential equation in [11] and the the explicit closed form general solution of it has been given in [14].

The primary goal of this paper is to consider a new system of matrix polynomials, namely the Gegenbauer matrix polynomials. The structure of this paper is the following. In section 2 a definition of Gegenbauer matrix polynomials is given. Some differential recurrence relations, in particular Gegenbauer's matrix differential equation are established in section 3. Moreover, hypergeometric matrix representations of these polynomials will be given in section 4. Finally in section 5 we obtain the orthogonality property of Gegenbauer matrix polynomials.

Throughout this study, consider the complex space $\mathbb{C}^{N\times N}$ of all square complex matrices of common order N. We say that a matrix A in $\mathbb{C}^{N\times N}$ is a positive stable if Re$(\lambda)>0$ for all $\lambda \in \sigma\ (A)$ where $\sigma\ (A)$ is the set of all eigenvalues of A. If $A_0, A_1, \cdots, A_n$ are elements of $\mathbb{C}^{N\times N}$ and $A_n \neq 0$, then we call

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0,$$

a matrix polynomial of degree n in x.

If $P+nI$ is invertible for every integer $n \geq 0$, then from [11] it follows that

$$(P)_n = P(P+I)(P+2I)...(P+(n-1)I);\ n \geq 1;\ (P)_0 = I. \tag{1}$$

From (1), it is easy to find that

$$(P)_{n-k} = (-1)^k (P)_n [(I-P-nI)_k]^{-1}; \quad 0 \leq k \leq n. \tag{2}$$

From the relation (3) of [16, pp. 58], one obtains

$$\frac{(-1)^k}{(n-k)!}I = \frac{(-n)_k}{n!}I = \frac{(-nI)_k}{n!}; \quad 0 \leq k \leq n. \tag{3}$$

The hypergeometric matrix function $F(A,B;C;z)$ has been given in the form [11, p. 210]

$$F(A,B;C;z) = \sum_{n\geq 0} \frac{1}{n!}(A)_n(B)_n[(C)_n]^{-1}z^n, \tag{4}$$

for matrices A, B and C in $\mathbb{C}^{N\times N}$ such that $C+nI$ is invertible for all integer $n \geq 0$ and for $|z| < 1$.

For any matrix P in $\mathbb{C}^{N\times N}$ we will exploit the following relation due to [11, p. 213]

$$(1-x)^{-P} = \sum_{n\geq 0} \frac{1}{n!}(P)_n x^n, \quad |x| < 1. \tag{5}$$

It has been seen by Defez and Jódar [3] that, for matrices $A(k,n)$ and $B(k,n)$ in $\mathbb{C}^{N\times N}$ where $n \geq 0$, $k \geq 0$, the following relations are satisfied:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k,n-2k), \tag{6}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n-k). \tag{7}$$

Similarly, we can write

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+2k), \tag{8}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} B(k,n-k), \tag{9}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n+k), \tag{10}$$

If A is a positive stable matrix in $\mathbb{C}^{N\times N}$, then the n^{th} Hermite matrix polynomials was defined by [10, pp.14]

$$H_n(x,A) = n!\sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k}; \quad n \geq 0. \tag{11}$$

The expansion of $x^n I$ in a series of Hermite matrix polynomials has been given in [3, pp. 14] in the form

$$\frac{x^n}{n!}I = (\sqrt{2A})^{-n}\sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)!}H_{n-2k}(x,A); \quad -\infty < x < \infty. \tag{12}$$

2 Gegenbauer matrix polynomials

Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$. We define the Gegenbauer matrix polynomials by means of the relation:

$$F = (1-2xt-t^2)^{-A} = \sum_{n=0}^{\infty} C_n^A(x)t^n. \tag{13}$$

By using (5) and (9), we have

$$(1-2xt-t^2)^{-A} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]} \frac{(-1)^k(A)_{n-k}}{k!(n-2k)!}(2x)^{n-2k}t^n. \tag{14}$$

By equating the coefficients of t^n in (13) and (14), we obtain an explicit representation of the Gegenbauer matrix polynomials in the form:

$$C_n^A(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(A)_{n-k}}{k!(n-2k)!}(2x)^{n-2k}. \tag{15}$$

Clearly, $C_n^A(x)$ is a matrix polynomial of degree n in x. Replacing x by -x and t by -t in (13), the left side does not exchange. Therefore

$$C_n^A(-x) = (-1)^n C_n^A(x). \tag{16}$$

For $x = 1$ we have

$$(1-t)^{-2A} = \sum_{n=0}^{\infty} C_n^A(1)t^n.$$

So that by (5) it follows

$$C_n^A(1) = \frac{(2A)_n}{n!}. \tag{17}$$

For $x = 0$ it follows

$$(1+t^2)^{-A} = \sum_{n=0}^{\infty} C_n^A(0)t^n.$$

Also, by (5) one gets

$$(1+t^2)^{-A} = \sum_{n=0}^{\infty} \frac{(-1)^n (A)_n}{n!} t^{2n}.$$

Therefore, we have

$$C_{2n}^A(0) = \frac{(-1)^n (A)_n}{n!}, \quad C_{2n+1}^A(0) = 0. \tag{18}$$

The explicit representation (15) gives

$$C_n^A(x) = \tfrac{2^n (A)_n}{n!} x^n + \prod_{n-2},$$

where $\prod_{n-2}$ is a matrix polynomial of degree (n - 2) in x. Consequently, if $D = \frac{d}{dx}$, then it follows that

$$D^n C_n^A(x) = 2^n (A)_n.$$

3 Differential recurrence relations

By differentiating (13) with respect to x and t yields respectively

$$\frac{\partial F}{\partial x} = \frac{t}{1-2xt-t^2} 2AF. \tag{19}$$

and

$$\frac{\partial F}{\partial t} = \frac{(x-t)}{1-2xt-t^2} 2AF. \tag{20}$$

So that the matrix function F satisfies the partial matrix differential equation:

$$(x-t)\frac{\partial F}{\partial x} - t\frac{\partial F}{\partial t} = 0.$$

Therefore by (13) we get

$$\sum_{n=0}^{\infty} xDC_n^A(x)t^n - \sum_{n=0}^{\infty} nC_n^A(x)t^n = \sum_{n=1}^{\infty} DC_{n-1}^A(x)t^n,$$

where $D = \frac{d}{dx}$.

Since $DC_0^A(x) = 0$ and for $n \geq 1$, then we obtain the differential recurrence relation:

$$xDC_n^A(x) - nC_n^A(x) = DC_{n-1}^A(x). \tag{21}$$

From (19) and (20) with the aid of (13) we get respectively the following

$$\frac{1}{1-2xt-t^2} 2A(1-2xt-t^2)^{-A} = \sum_{n=1}^{\infty} DC_n^A(x)t^{n-1}, \tag{22}$$

and

$$\frac{x-t}{1-2xt-t^2} 2A(1-2xt-t^2)^{-A} = \sum_{n=1}^{\infty} nC_n^A(x)t^{n-1}. \tag{23}$$

Note that $1-t^2-2t(x-t) = 1-2xt-t^2$. Thus by multiplying (22) by $(1-t^2)$ and (23) by $2t$ and subtracting (23) from (22) we obtain

$$2(A+nI)C_n^A(x) = DC_{n+1}^A(x) - DC_{n-1}^A(x). \tag{24}$$

From (21) and (24), one gets

$$xDC_n^A(x) = DC_{n+1}^A(x) - (2A+nI)C_n^A(x). \tag{25}$$

Substituting $(n-1)$ for n in (25) and putting the resulting expression for $DC_{n-1}^A(x)$ into (21), gives

$$(x^2-1)DC_n^A(x) = nxC_n^A(x) - (2A+(n-1)I)C_{n-1}^A(x). \tag{26}$$

Now, by multiplying (21) by (x^2-1) and substituting for $(x^2-1)DC_n^A(x)$ and $(x^2-1)DC_{n-1}^A(x)$ from (26) to obtain the three terms recurrence relation in the form

$$nC_n^A(x) = (2A+2(n-1)I)xC_{n-1}^A(x) - (2A+(n-2)I)C_{n-2}^A(x). \tag{27}$$

Write (22) in the form:

$$2A(1-2xt-t^2)^{-(A+I)} = \sum_{n=0}^{\infty} DC_{n+1}^A(x)t^n. \tag{28}$$

By applying (13) it follows

$$2A(1-2xt-t^2)^{-(A+I)} = \sum_{n=0}^{\infty} 2AC_n^{A+I}(x)t^n. \tag{29}$$

Identification of the coefficients of t^n in (28) and (29) yields

$$DC_{n+1}^{A}(x) = 2AC_n^{A+I}(x),$$

which gives

$$DC_n^{A}(x) = 2AC_{n-1}^{A+I}(x). \tag{30}$$

Iteration (30) yields, for $0 \le r \le n$,

$$D^r C_n^{A}(x) = 2^r (A)_r C_{n-r}^{A+rI}(x). \tag{31}$$

The first few Gegenbauer matrix polynomials are listed here,

$$\begin{aligned}
C_0^A(x) &= I,\\
C_1^A(x) &= 2Ax,\\
C_2^A(x) &= 2(A)_2x^2 - A,\\
C_3^A(x) &= \frac{4}{3}(A)_3x^3 - 2(A)_2x,
\end{aligned}$$

and

$$C_4^A(x) = \frac{2}{3}(A)_4x^4 - 2(A)_3x^2 + \frac{1}{2}(A)_2.$$

We conclude this section introducing the Gegenbauer's matrix differential equation as follows:

In (25), replace n by $(n-1)$ and differentiate with respect to x to find

$$xD^2C_{n-1}^A(x) = D^2C_n^A(x) - (2A+nI)DC_{n-1}^A(x). \tag{32}$$

Also, by differentiating (21) with respect to x we have

$$xD^2C_{n-1}^A(x) - (n-1)DC_n^A(x) = D^2C_{n-1}^A(x). \tag{33}$$

From (21) and (33) by putting $DC_{n-1}^A(x)$ and $D^2C_{n-1}^A(x)$ into (32) and rearrangement terms we obtain the Gegenbauer's matrix differential equation in the form:

$$(1-x^2)D^2C_n^A(x) - (2A+I)xDC_n^A(x) + n(2A+nI)C_n^A(x) = 0. \tag{34}$$

4 Hypergeometric matrix representations of $C_n^A(x)$

From (1) it is easy to find that

$$[(A)_{2n}]^{-1} = 2^{-2n}[(\frac{1}{2}(A+I))_n]^{-1}[(\frac{1}{2}A)_n]^{-1}, \tag{35}$$

and

$$(A)_{n+k} = (A)_n(A+nI)_k \tag{36}$$

Note that

$$(1-2xt-t^2)^{-A} = [1-\frac{2t(x-1)}{(1-t)^2}]^{-A}(1-t)^{-2A}.$$

Therefore, by using and (5) and (36) we have

$$\begin{aligned}\sum_{n=0}^{\infty} C_n^A(x)t^n &= \sum_{k=0}^{\infty}\frac{(A)_k 2^k t^k (x-1)^k}{k!(1-t)^{2k}}(1-t)^{-2A}\\ &= \sum_{k=0}^{\infty}(A)_k(1-t)^{-(2A+2kI)}\frac{2^k t^k (x-1)^k}{k!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(A)_k(2A+2kI)_n\frac{2^k(x-1)^k}{k!n!}t^{n+k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(A)_k[(A)_{2k}]^{-1}(A)_{n+2k}\frac{2^k(x-1)^k}{k!n!}t^{n+k},\end{aligned}$$

which by inserting (35) and applying (7) with the help of (3) yields

$$\sum_{n=0}^{\infty} C_n^A(x)t^n = \sum_{n=0}^{\infty}\frac{(2A)_n}{n!}\sum_{k=0}^{n}\frac{(-nI)_k(2A+nI)_k}{k!}[(A+\frac{1}{2}I)_{2k}]^{-1}(\frac{x-1}{2})^k t^n.$$

Thus, the hypergeometric matrix representation follows by equating the coefficients of t^n in the form

$$C_n^A(x) = \frac{(2A)_n}{n!}F(-nI, 2A+nI; A+\frac{1}{2}I; \frac{1-x}{2}). \tag{37}$$

On applying (16), one gets

$$C_n^A(x) = (-1)^n\frac{(2A)_n}{n!}F(-nI, 2A+nI; A+\frac{1}{2}I; \frac{1+x}{2}). \tag{38}$$

It is evident that

$$\frac{1}{(n-2k)!}I = \frac{(-nI)_{2k}}{n!}; \quad 0 \le 2k \le n.$$

By using (2) and taking into account that

$$(-nI)_{2k} = 2^{-2k}(\frac{-n}{2}I)_k(\frac{-(n-1)}{2}I)_k,$$

the explicit representation (15) becomes

$$C_n^A(x) = \frac{(2x)^n}{n!}(A)_n \sum_{k=0}^{[n/2]} \frac{(-\frac{n}{2}I)_k(\frac{-(n-1)}{2}I)_k}{k!}[(I-A-nI)_k]^{-1}(\frac{1}{x})^{2k},$$

which gives another hypergeometric matrix representation in the form:

$$C_n^A(x) = \frac{(2x)^n}{n!}(A)_n F(\frac{-n}{2}I, \frac{1-n}{2}I; I-A-nI; \frac{1}{x^2}). \tag{39}$$

Now, we can write

$$(1-2xt-t^2)^{-A} = [1 - \frac{t^2(x^2-1)}{(1-xt)^2}]^{-A}(1-xt)^{-2A}.$$

Therefore, by using (5) and (6) we find that

$$\begin{aligned}
\sum_{k=0}^{\infty} C_n^A(x)t^n &= \sum_{k=0}^{\infty} \frac{(A)_k(x^2-1)^k t^{2k}}{k!(1-xt)^{2k}}(1-xt)^{-2A} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(A)_k(2A+2kI)_n \frac{(x^2-1)^k x^{n-2k}}{k!n!} t^{n+2k} \\
&= \sum_{n=0}^{\infty}(A)_n \sum_{k=0}^{[n/2]}[(A+\frac{1}{2}I)_k]^{-1} \frac{(x^2-1)^k x^{n-2k}}{2^{2k}k!(n-2k)!} t^n.
\end{aligned}$$

By identification of the coefficients of t^n, another form for the Gegenbauer matrix polynomials follows

$$C_n^A(x) = (A)_n \sum_{k=0}^{[n/2]}[(A+\frac{1}{2}I)_k]^{-1} \frac{(x^2-1)^k x^{n-2k}}{2^{2k}k!(n-2k)!}. \tag{40}$$

Moreover, by exploiting (40) and using (8) in the sum

$$\sum_{n=0}^{\infty}[(A)_n]^{-1}C_n^A(x)t^n = \sum_{n=0}^{\infty}\frac{x^nt^n}{n!}\sum_{k=0}^{\infty}\frac{1}{k!}[(A+\frac{1}{2}I)_k]^{-1}(\frac{1}{4}t^2(x^2-1))^k.$$

By identification of the coefficients of t^n, we obtain a generating relation for the Gegenbauer matrix polynomials in the form:

$$exp(xt)_0F_1(-;A+\frac{1}{2}I;\frac{1}{4}t^2(x^2-1)) = \sum_{n=0}^{\infty}[(A)_n]^{-1}C_n^A(x)t^n. \tag{41}$$

5 Orthogonality of Gegenbauer matrix polynomials

Here, we will obtain the most interesting property of the Gegenbauer matrix polynomials, namely the orthogonality of this system of polynomials. Let A be a positive stable matrix in $\mathbb{C}^{N\times N}$ such that

$$A+kI \text{ is invertible for every integer } k \geq 0. \tag{42}$$

By multiplying (34) by $(1-x^2)^{A-\frac{1}{2}I}$ we get

$$D[(1-x^2)^{A+\frac{1}{2}I}DC_n^A(x)] + n(1-x^2)^{A-\frac{1}{2}I}(2A+nI)C_n^A(x) = 0. \tag{43}$$

Similarly, when $C_m^A(x)$ satisfies (34) it follows

$$D[(1-x^2)^{A+\frac{1}{2}I}DC_m^A(x)] + m(1-x^2)^{A-\frac{1}{2}I}(2A+mI)C_m^A(x) = 0. \tag{44}$$

By multiplying the equation (43) by $C_m^A(x)$ and the equation (44) by $C_n^A(x)$ and subtracting gives

$$\begin{aligned} D[(1-x^2)^{A+\frac{1}{2}I}DC_n^A(x)]C_m^A(x) - D[(1-x^2)^{A+\frac{1}{2}I}DC_m^A(x)]C_n^A(x) + \\ n(1-x^2)^{A-\frac{1}{2}I}(2A+nI)C_n^A(x)C_m^A(x) - \\ m(1-x^2)^{A-\frac{1}{2}I}(2A+mI)C_m^A(x)C_n^A(x) = 0. \end{aligned} \tag{45}$$

Since the multiplication of the matrix in $(A)_n$ is commutative for every integer $n \geq 0$, then $C_n^A(x)C_m^A(x) = C_m^A(x)C_n^A(x)$. We can write

$$D[(1-x^2)^{A+\frac{1}{2}I}\{C_m^A(x)DC_n^A(x) - DC_m^A(x)C_n^A(x)\}]$$

$$= (1-x^2)^{A+\frac{1}{2}I}DC_m^A(x)DC_n^A(x) + D[(1-x^2)^{A+\frac{1}{2}I}DC_n^A(x)]C_m^A(x) -$$

$$(1-x^2)^{A+\frac{1}{2}I} DC_m^A(x) DC_n^A(x) + D[(1-x^2)^{A+\frac{1}{2}I} DC_m^A(x)] C_n^A(x).$$

Thus, the equation (45) becomes

$$(n-m)\{A+(n+m)I\}(1-x^2)^{A-\frac{1}{2}I} C_n^A(x) C_m^A(x) =$$

$$D[(1-x^2)^{A+\frac{1}{2}I}\{C_m^A(x) DC_n^A(x) - DC_m^A(x) C_n^A(x)\}].$$

Actually, m and n are non-negative integer and A is a positive stable matrix, hence $A+(m+n)I \neq 0$. Therefore, it follows

$$\int_{-1}^{1} (1-x^2)^{A-\frac{1}{2}I} C_n^A(x) C_m^A(x) dx = 0; \quad m \neq n. \tag{46}$$

That is, for A is a positive stable matrix in $\mathbb{C}^{N\times N}$, the Gegenbauer matrix polynomials form an orthogonal set over the interval (-1,1) with respect to the weight function $(1-x^2)^{A-\frac{1}{2}I}$.

One immediate consequence of (46) is

$$\int_{-1}^{1} (1-x^2)^{A-\frac{1}{2}I} C_m^A(x) dx = 0; \quad m \neq 0.$$

Now, by multiplying (27) by $(1-x^2)^{A-\frac{1}{2}I} C_n^A(x)dx$ and integrating between -1 and 1 and taking into account (46) we get

$$\int_{-1}^{1} (1-x^2)^{A-\frac{1}{2}I} [C_n^A(x)]^2 dx = \tag{47}$$

$$\frac{2}{n}(A+(n-1)I) \int_{-1}^{1} x(1-x^2)^{A-\frac{1}{2}I} C_n^A(x) C_{n-1}^A(x) dx.$$

Again, by replacing n by $n-1$ in (27) and multiply by $(1-x^2)^{A-\frac{1}{2}I} C_{n-1}^A(x)dx$ and integrating between -1 and 1 and taking into account (46) to obtain

$$2(A+nI) \int_{-1}^{1} x(1-x^2)^{A-\frac{1}{2}I} C_{n-1}^A(x) C_n^A(x) dx = \tag{48}$$

$$(2A+(n-1)I) \int_{-1}^{1} (1-x^2)^{A-\frac{1}{2}I} [C_{n-1}^A(x)]^2 dx.$$

Thus, from (47) and (48) we get

$$\int_{-1}^{1} (1-x^2)^{A-\frac{1}{2}I} [C_n^A(x)]^2 dx = \frac{1}{n}(A+nI)^{-1}(A+(n-1)I) \tag{49}$$

$$(2A+(n-1)I)\int_{-1}^{1}(1-x^2)^{A-\frac{1}{2}I}[C_{n-1}^{A}(x)]^2dx.$$

By substituting for n the values $n-1, n-2, \cdots, 1$ in (49) it follows

$$\int_{-1}^{1}(1-x^2)^{A-\frac{1}{2}I}[C_n^A(x)]^2dx =$$

$$\frac{1}{n!}(A+nI)^{-1}A(2A)_n\int_{-1}^{1}(1-x^2)^{A-\frac{1}{2}I}[C_0^A(x)]^2dx.$$

Note that, for A is a positive stable matrix in $\mathbb{C}^{N\times N}$ satisfies (42) it follows

$$\int_{-1}^{1}(1-x^2)^{A-\frac{1}{2}I}dx = \sqrt{\pi}\,\Gamma(A+\frac{1}{2}I)\Gamma^{-1}(A+I).$$

Therefore, we obtain

$$\int_{-1}^{1}(1-x^2)^{A-\frac{1}{2}I}[C_n^A(x)]^2dx = \frac{1}{n!}(A+nI)^{-1}A(2A)_n\sqrt{\pi}$$

$$\Gamma(A+\frac{1}{2}I)\Gamma^{-1}(A+I),$$

which can be be written with 46 in the form

$$\int_{-1}^{1}(1-x^2)^{A-\frac{1}{2}I}C_n^A(x)C_m^A(x)dx = \frac{1}{n!}(A+nI)^{-1}A(2A)_n\sqrt{\pi} \tag{50}$$

$$\Gamma(A+\frac{1}{2}I)\Gamma^{-1}(A+I)\delta_{mn},$$

where δ_{mn} is Kronecker's delta symbol.

Finally, we will expand the Gegenbauer matrix polynomials in series of Hermite matrix polynomials. By employing (15), (8) and (12) and taking into account that each matrix commutes with itself, one gets

$$\sum_{n=0}^{\infty}2^{-n}(\sqrt{2A})^nC_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{[n/2]}\frac{(-1)^k(A)_{n+k}}{s!k!(n-2s)!}H_{n-2s}(x,A)t^{n+2k},$$

which on applying (8) becomes

$$\sum_{n=0}^{\infty}2^{-n}(\sqrt{2A})^nC_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k(A)_{n+k+2s}}{s!k!n!}H_n(x,A)t^{n+2k+2s}.$$

By using (7) one gets

$$\sum_{n=0}^{\infty} 2^{-n}(\sqrt{2A})^n C_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k} \frac{(-1)^{k-s}(A)_{n+k+s}}{s!(k-s)!n!} H_n(x,A)t^{n+2k}.$$

Since $(A)_{n+k+s} = (A+(n+k)I)_s(A)_{n+k}$, then by using (3) it follows

$$\sum_{n=0}^{\infty} 2^{-n}(\sqrt{2A})^n C_n^A(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!n!} \\ {}_2F_0(-kI, A+(n+k)I; -; 1)(A)_{n+k}H_n(x,A)t^{n+2k}.$$

By (6) and then equating the coefficients of t^n we obtain an expansion of the Gegenbauer matrix polynomials as series of Hermite matrix polynomials in the form:

$$C_n^A(x) = 2^n(\sqrt{2A})^{-n}\sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!(n-2k)!}\, {}_2F_0(-kI, A+(n-k)I; -; 1) \\ (A)_{n-k}H_{n-2k}(x,A). \qquad (51)$$

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