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Comments on "A New Method for the Explicit Integration of Lotka-Volterra Equations"

Comentarios sobre "Un Nuevo Método para la Integración Explícita de las Ecuaciones de Lotka-Volterra

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Abstract

The equation for the orbit of the classical Lotka-Volterra oscillator is solved for one of two variables in terms of the other by using two inverse functions of $x\exp(x)$. Moreover, the period of the orbit is expressed as an integral, which is approximated numerically by Gauss-Tschebyscheff integration rule of the first kind.
Key words and phrases: Lotka-Volterra oscillator, Lambert W-function, period, Gauss-Tschebyscheff integration rule of the first kind.

Resumen

La ecuación de la órbita del oscilador clásico de Lotka-Volterra se resuelve para una de las variables en términos de la otra usando dos funciones inversas de $x\exp(x)$. Más aun, el período de la órbita se expresa como una integral, que se aproxima numéricamente usando la fórmula de integración de Gauss-Tschebyscheff de primera clase.
Palabras y frases clave: oscilador de Lotka-Volterra, función W de Lambert, periodo, regla de integración de Gauss-Tschebyscheff de primera clase.

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1 Introduction

Scarpello and Ritelli [11] studied the classical Lotka-Volterra system

$$\begin{cases} v'(t) &= v\{a - bu\} \\ u'(t) &= u\{dv - c\} \end{cases} \tag{1.1}$$

where a, b, c, d are given positive numbers. The authors state in p. 6 that "In spite of its simple formulation, no closed form solution of (1.1) is known."

Since 1920, the system (1.1), which was studied independently by Lotka [6] and Volterra [14], has been known to have a functional relationship between two dependent variables, from which one has *graphically* a one-parameter family of periodic solutions with the critical point as the center in the first quadrant. Existence of periodic solutions for the system (1.1) may be justified *mathematically* by Morse lemma [7].

Lotka linearized this equation at the critical point of this system; while Volterra analyzed this equation through an auxiliary variable, see also Davis [1, pp. 102-109] for more information. The work of Scarpello and Ritelli may be considered as an extension of Lotka's procedure to higher degree polynomials.

The aim of this short note is to make a remark that the functional relationship between two dependent variables can be solved *directly* for one variable in terms of the other. Such procedure is based on two inverse functions of $x\exp(x)$. As a byproduct, the period of each orbit can be expressed as an integral. On the other hand, solving this equation explicitly gives rise to a nonlinear transformation between Lotka-Volterra oscillators and the harmonic oscillators (more precisely, circles). These results echo what Davis states in the Preface of his book [1]:

> .....
>
> Substantial progress could be made only when clever transformation had reduced the nonlinear problems to linear ones, or to problems asymptotic to some linear algorithm.
>
> .....
>
> The solutions of nonlinear equations still possess singularities, which only the analytical method can discover and describe.
>
> .....

In fact, each end of the integral for the period of each orbit is revealed to have a weak singularity of the square-root type through the asymptotics of two

inverse functions of $x\exp(x)$ at the branch point $-\exp(-1)$. Thus, numerical approximations of such integral may be obtained by Gauss-Tschebyscheff integration rule of the first kind.

In the computer algebra systems, two inverse functions of $x\exp(x)$ are denoted by LambertW$(-k,x)$ in Maple, and ProductLog$[-k,x]$ in Mathematica, respectively. A series expansion for one of these inverse functions was employed when Euler solved in 1779 the trinomial equation proposed in 1758 by Lambert $x^\alpha - x^\beta = (\alpha-\beta)vx^{\alpha+\beta}$ in the limiting case as $\alpha \to \beta$. These functions will appear [10] in the chapter on elementary functions, by R. Roy, F. Olver and S. Krantz, of the *Digital Library of Mathematical Functions*, an updated version of Abramowitz and Stegun's Handbook of Mathematical Functions (with Formulas, Graphs, and Mathematical Tables):

> The Lambert W-function $W(x)$ is the solution of the equation $We^W = x$. The solution for which $W(x) \geq W(-\exp(-1))$ is called the principal branch and denoted by $W_p(x)$. The other solution is denoted by $W_m(x)$.

See Weisstein [15] for more information about these functions. This work follows these notations.

For the sake of comparison, some related procedures in Scarpello and Ritelli [11] are reviewed briefly as follows. The system (1.1) is first written as

$$\begin{cases} \dfrac{d}{dt}\log(v(t)) &= a - bu \\ \dfrac{d}{dt}\log(u(t)) &= dv - c, \end{cases}$$

which is reduced via the change of variables $x = \log(\dfrac{d}{c}v), y = \log(\dfrac{b}{a}u)$ to

$$\begin{cases} x'(t) &= a\{1-\exp(y)\} \\ y'(t) &= c\{\exp(x)-1\}. \end{cases} \tag{1.2}$$

For the initial conditions $x(0) = x_0, y(0) = y_0$, one has the orbit differential equation in the xy-plane

$$\frac{dy}{dx} = k\frac{1-\exp(x)}{1-\exp(y)} \qquad y(x_0) = y_0 \qquad k = -\frac{c}{a}, \tag{1.3}$$

which is solved to give

$$y - \exp(y) = k\{x - \exp(x)\} + C \tag{1.4}$$

with $C = y_0 - \exp(y_0) - k\{x_0 - \exp(x_0)\}$. Then the authors say "If (1.4) could be solved explicitly for y, the orbit of the Lotka-Volterra system would be known: but unfortunately the transcendental nonlinearity of (1.4) prevents this." Scarpello and Ritelli [11] then solved (1.4) *approximately* for y in terms of x via Taylor series and computer algebra software Mathematica.

2 Main results

The importance of (1.1) can be found with its applications in population dynamics [8], chemical dynamics [3], laser dynamics [5], economics [4], and virus dynamics [9].

Equations in (1.1) gives the separable differential equation

$$\frac{du}{dv} = \frac{u\{dv - c\}}{v\{a - bu\}},$$

which is integrated to yield

$$dv - c\log(v) - a\log(u) + bu = H, \tag{2.1}$$

where H is the constant of integration defined by

$$H = dv_0 - c\log(v_0) - a\log(u_0) - bu_0$$

for the initial conditions $v(0) = v_0, u(0) = u_0$.

It is easy, by linearizing (2.1) at the critical point $v = c/d, u = a/b$, to see that (2.1) gives periodic solutions in a neighborhood of this point in the vu-plane. On the other hand, by using Morse lemma, (2.1) indeed gives periodic orbits in the vu-plane, see for example Verhulst [13, p. 21].

An elementary technique in calculus shows that

$$H \geq a + c - a\log(\frac{a}{b}) - c\log(\frac{c}{d}),$$

and the minimum value takes place at $v = c/d, u = a/b$. In the notation of Hamiltonian systems, we write $H = a + c - a\log(a/b) - c\log(c/d) + E$ in (2.1) to obtain

$$dv - c - c\log(\frac{d}{c}v) + bu - a - a\log(\frac{b}{a}u) = E \tag{2.2}$$

with the energy $E \geq 0$ and $E = 0$ at $v = c/d, u = a/b$.

Lokta [6] linearized (2.1) at the equilibrium point $v = c/d, u = a/b$ to obtain an equation of the ellipse in order to find an approximation of the period of the Lotka-Volterra oscillator. Volterra [14] introduced an auxiliary variable in treating this equation in order to construct an integral representation for the period of the Lotka-Volterra oscillator.

Shih [12] solved (2.2) for u in terms of v to obtain $u = g_k(v)$ in $v \in [v_0, v_1]$ for $k = 0, 1$, with $W_p = W_0, W_m = W_1$ for the clarity of the presentation

$$g_k(v) = -\frac{a}{b} W_k(-(\frac{d}{c}v)^{-c/a} \exp(\frac{d}{a}v - 1 - \frac{c}{a} - \frac{E}{a})), \tag{2.3}$$

$$v_k = -\frac{c}{d} W_k(-\exp(-1 - \frac{E}{c})). \tag{2.4}$$

Note that $0 < g_0(v) < a/b < g_1(v) < \infty$. In other words, the function $u = g_0(v) \in (0, a/b]$ gives the lower branch of the orbit (2.2) in the vu-plane by traveling from the point $(v_0, a/b)$ to the point $(v_1, a/b)$ in the counterclockwise direction; while the function $u = g_1(v) \in [a/b, \infty)$ describes the upper branch by traveling from the point $(v_1, a/b)$ to the point $(v_0, a/b)$ in the counterclockwise direction.

Next, traveling along the lower branch described by $u = g_0(v)$ from the point $(v_0, a/b)$, with $t = t|_{P_w}$, to the point $(v_1, a/b)$, with $t = t|_{P_e}$, in the counterclockwise direction yields

$$t|_{P_e} - t|_{P_w} = \int_{v_0}^{v_1} \frac{dv}{v\{a - b\, g_0(v)\}};$$

while traveling along the upper branch described by $v = g_1(u)$ from the point $(v_1, a/b)$, with $t = t|_{P_e}$, to the point $(v_0, a/b)$, with $t = t|_{P_w}$, in the counterclockwise direction yields

$$t|_{P_w} - t|_{P_e} = \int_{v_1}^{v_0} \frac{dv}{v\{a - b\, g_1(v)\}}.$$

Thus we obtain an integral representation of the period

$$T(E) = \int_{v_0}^{v_1} \left\{ \frac{1}{v\{a - b\, g_0(v)\}} + \frac{-1}{v\{a - b\, g_1(v)\}} \right\} du, \tag{2.5}$$

where the functions $g_0(v), g_1(v)$ are given by (2.3), and two endpoints v_0, v_1 of the integral are defined by (2.4).

Splitting the integration interval in (2.5) into two halves and performing substitutions give

$$T(E) = \frac{1}{ac}\int_0^E \Phi(\frac{s}{c})\Phi(\frac{E-s}{a})ds, \tag{2.6}$$

with

$$\Phi(s) = \frac{1}{1+W_p(-\exp(-1-s))} - \frac{1}{1+W_m(-\exp(-1-s))}.$$

Asymptotic behavior of $\Phi(s) \sim \sqrt{2/s}$, as $s \downarrow 0$, shows that the integral (2.6) possesses a weak singularity of the square root type at each endpoint of the integration, and is thus computed numerically by the Gauss-Tschebyscheff integration rule of the first kind. For more on this numerical quadrature, see for example Davis and Rabinowitz [2]. To proceed further, the integral (2.6) is converted to the form

$$T = \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx,$$

with

$$f(x) = \frac{E}{2ac}\Phi(\frac{E}{2c}(1+x))\Phi(\frac{E}{2a}(1-x))\sqrt{1-x^2}.$$

A numerical approximation to T by using the Gauss-Tschebyscheff integration rule of the first kind is

$$T_{\text{sgt}} = \frac{\pi}{n}\sum_{i=1}^{n} f(x_i) \qquad \text{with} \quad x_i = \cos(\frac{(2i-1)\pi}{2n}); \tag{2.7}$$

for $i = 1, \ldots, n$, as well as the error term

$$T - T_{\text{sgt}} = \frac{\pi}{2^{2n-1}(2n)!} f^{(2n)}(\xi) \qquad \text{for some} \quad \xi \in (-1, 1).$$

Scarpello and Ritelli [11, p. 8] has a numerical value 5.27 for the period of the system (1.2) subject to the initial conditions $x(0) = 0, y(0) = 1$. This value has an excellent good agreement with the results 5.270242963, 5.270242894, 5.270242858, obtained from (2.7) in Maple 9.5 with $n = 3, 5, 10$, respectively.

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