Divulgaciones Matemáticas Vol. 14 No. 1(2006), pp. 11–29

Hamiltonian Cycles and Hamiltonian-biconnectedness in Bipartite Digraphs

Ciclos Hamiltonianos y Biconectividad Hamiltoniana en Digrafos Bipartitos

Denise Amar (amar@labri.u-bordeaux.fr)
LaBRI, Université de Bordeaux I
351 Cours de la Libération 33405 Talence Cedex, France
Daniel Brito
Mathematics Department, School of Science
Universidad de Oriente, Cumaná, Venezuela
Oscar Ordaz
Mathematics Department, Faculty of Science
Universidad Central de Venezuela
AP. 47567, Caracas 1041-A, Venezuela

Abstract

Let D denote a balanced bipartite digraph with $2n$ vertices and for each vertex x, $d^+(x) \geq k$, $d^-(x) \geq k$, $k \geq 1$, such that the maximum cardinality of a balanced independent set is 2β and $n = 2\beta + k$. We give two functions $F(n, \beta)$ and $G(n, \beta)$ such that if D has at least $F(n, \beta)$ (resp. $G(n, \beta)$) arcs, then it is hamiltonian (resp. hamiltonian-biconnected).
Key words and phrases: hamiltonian cycles, bipartite digraphs, hamiltonian-biconnectedness.

Resumen

Sea D un digrafo bipartito balanceado de orden $2n$. Supongamos que para todo vértice x, $d^+(x) \geq k$, $d^-(x) \geq k$, $k \geq 1$. Sea 2β la máxima cardinalidad de los conjuntos independientes balanceados y sea $n = 2\beta + k$. Damos dos funciones $F(n, \beta)$ y $G(n, \beta)$ tal que si D tiene al

Received 2004/10/21. Revised 2006/06/15. Accepted 2006/06/17.
MSC (2000): Primary 05C45, 05C40.

menos $F(n,\beta)$ (resp. $G(n,\beta)$) arcos, entonces D es hamiltoniano (resp. hamiltoniano biconectado).
Palabras y frases clave: ciclos hamiltonianos, digrafos bipartitos, digrafos hamiltonianos biconectados.

1 Introduction

Many conditions involving the number of arcs, the minimum half-degree, and the independence number for a digraph to be hamiltonian or hamiltonian-connected are known (see [1], [3], [4], [6], [9], [10], [11] , [13], [15], [16], [17], [18], [19].

The parameter 2β, defined as the maximum cardinality of a balanced independent set, has been introduced by P. Ash [5] and B. Jackson and O. Ordaz [14] where a balanced independent set in D is an independent subset S such that $|S \cap X| = |S \cap Y|$.

In this paper we give conditions involving the number of arcs, the minimum half-degree, and the parameter 2β for a balanced bipartite digraph to be hamiltonian or hamiltonian-biconnected, i.e. such that for any two vertices x and y which are not in the same partite set, there is a hamiltonian path in D from x to y.

Let $D = (X, Y, E)$ denote a balanced bipartite digraph with vertex-set $X \cup Y$, X and Y being the two partite sets.

In a digraph D, for $x \in V(D)$, let $N_D^+(x)$ (resp. $N_D^-(x)$) denote the set of the vertices of D which are dominated by (resp. dominate) x; if no confusion is possible we denote them by $N^+(x)$ (resp. $N^-(x)$).

Let H be a subgraph of D, $E(H)$ denotes the set of the arcs of H, and $|E(H)|$ the cardinality of this set; if $x \in V(D)$, $d_H^+(x)$ (resp. $d_H^-(x)$) denotes the cardinality of the set of the vertices of H which are dominated by (resp. dominate) x; if $x \in V(D)$, $x \notin V(H)$, $E(x, H)$ denotes the set of the arcs between x and $V(H)$.

If C is a cycle (resp. if P is a path) in D, and $x \in V(C)$ (resp. $x \in V(P)$), x^+ denotes the successor of x on C (resp. on P) according to the orientation of the cycle (resp. of the path).

If $x,\ y \in V(C)$ (resp. $x,\ y \in V(P)$), x, C, y (resp. x, P, y) denotes the part of the cycle (resp. the path) starting at x and terminating at y.

The following results will be used :

Theorem 1.1. (N. Chakroun, M. Manoussakis, Y. Manoussakis [8])

Let $D = (X, Y, E)$ be a bipartite digraph with $|X| = a$, $|Y| = b$, $a \leq b$. If $|E| \geq 2ab - b + 1$, then D has a cycle of length $2a$.

Theorem 1.2. (N. Chakroun, M. Manoussakis, Y. Manoussakis [8])

Let $D = (X, Y, E)$ be a balanced bipartite digraph with $|X| = |Y| = n$. If $|E| \geq 2n^2 - n + 1$ then D is hamiltonian. If $|E| \geq 2n^2 - n + 2$, D is hamiltonian-biconnected.

Theorem 1.3. (N. Chakroun, M. Manoussakis, Y. Manoussakis [8])

Let $D = (X, Y, E)$ be a bipartite digraph with $|X| = a$, $|Y| = b$, $a \leq b$, such that for every vertex x, $d^+(x) \geq k$, $d^-(x) \geq k$. Then:
(i) If $|E| \geq 2ab - (k + 1)(a - k) + 1$, D has a cycle of length $2a$,
(ii) If $|E| \geq 2ab - k(a - k) + 1$, for any two vertices x and y which are not in the same partite set, there is a path from x to y of length $2a - 1$.

If $b \geq 2k$, for $k \leq p \leq b - k$, let $K^*_{k,p}$,, (resp. $K^*_{k-1,b-p}$) be a complete bipartite digraph with partite sets (X_1, Y_1) (resp. (X_2, Y_2)); for $a = 2k - 1$ and $b > a$, $\Gamma_1(a, b)$ consists of the disjoint union of $K^*_{k,p}$ and $K^*_{k-1,b-p}$ by adding all the arcs between exactly one vertex of X_1 and all the vertices of Y_2.

$\Gamma_2(3, b)$ is a bipartite digraph with vertex-set $X \cup Y$, where $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, ..., y_b\}$, and arc-set
$E(D) = \{(x_1y_1), (x_2y_2), (y_1x_2)(y_2x_1)\} \cup \{(x_3y_i), (y_ix_3), 1 \leq i \leq 2\} \cup \{(x_jy_i), (x_jy_i),\ 3 \leq i \leq b,\ 1 \leq j \leq 2\}$.

Theorem 1.4. (D. Amar, Y. Manoussakis [2])
Let $D = (X, Y, E)$ be a bipartite digraph with $|X| = a$, $|Y| = b$, $a \leq b$, such that for every vertex x, $d^+(x) \geq k$, $d^-(x) \geq k$. Then if $a \leq 2k-1$ D has a cycle of length $2a$, unless
(i) $b > a = 2k - 1$ and D is isomorphic to $\Gamma_1(a, b)$ or
(ii) $k = 2$ and D is isomorphic to $\Gamma_2(3, b)$.

Theorem 1.5. (N.Chakroun, M. Manoussakis, Y. Manoussakis [8])
Let $D = (X, Y, E)$ be a hamiltonian bipartite digraph of order $2n$ such that $|E| \geq n^2 + n - 2$; then D is bipancyclic.

2 Main Results

Let $f(n, \beta) = 2n^2 - 2\beta^2 - (n - \beta) + 1$, $F(n, \beta) = 2n^2 - 2\beta^2 - \beta(n - 2\beta + 1) + 1$, $G(n, \beta) = F(n, \beta) + \beta$.

We prove the following Theorems and their immediate Corollaries:

Theorem 2.1.
Let $D = (X, Y, E)$ be a balanced bipartite digraph with $|X| = |Y| = n$, and let 2β be the maximum cardinality of a balanced independent set in D. If $n \geq 2\beta + 1$ and
(i) If $|E| \geq f(n, \beta)$, D is hamiltonian.
(ii) If $|E| \geq f(n, \beta) + 1$, D is hamiltonian-biconnected.

Corollary 2.2. *Let $D = (X, Y, E)$ be a balanced bipartite digraph with $|X| = |Y| = n$, and let 2β be the maximum cardinality of a balanced independent set in D. If $n \geq 2\beta + 1$ and $|E| \geq f(n, \beta)$ then D is bipancyclic*

Theorem 2.3.
Let $D = (X, Y, E)$ be a balanced bipartite digraph with $|X| = |Y| = n$, such that for every vertex x, $d^+(x) \geq k$, $d^-(x) \geq k$, $k \geq 1$. Let 2β be the maximum cardinality of a balanced independent set in D. If $n = 2\beta + k$ and
(i) If $|E| \geq F(n, \beta)$, D is hamiltonian.
(ii) If $|E| \geq G(n, \beta)$, D is hamiltonian-biconnected.

Using Theorems 1.5, 2.1 and 2.3 we obtain the following:

Corollary 2.4. *Let $D = (X, Y, E)$ be a balanced bipartite digraph with $|X| = |Y| = n$, such that for every vertex x, $d^+(x) \geq k$, $d^-(x) \geq k$, $k \geq 1$. Let 2β be the maximum cardinality of a balanced independent set in D. If $n = 2\beta + k$ and $|E| \geq F(n, \beta)$ then D is bipancyclic.*

Proof of the corollaries:
Since $n \geq 2\beta + 1$, then $f(n, \beta) - (n^2 + n - 2) = n^2 - 2\beta^2 - 2n + \beta + 3$
$= (n-1)^2 - 2\beta^2 + \beta + 2 \geq 4\beta^2 - 2\beta^2 + \beta + 2 = 2\beta^2 + \beta + \beta + 2 > 0.$

resp.
$$\begin{aligned} F(n,\beta) - (n^2+n-2) &= 2n^2 - 2\beta^2 - \beta(n - 2\beta + 1) + 1 - (n^2 + n - 2) \\ &= n^2 - n(\beta+1) - \beta + 3 \\ &\geq n(2\beta + 1 - \beta - 1) - \beta + 3 = \beta(n-1) + 3 > 0. \end{aligned}$$

3 Definitions and a basic lemma

Before proving Theorem 2.1 and Theorem 2.3, we give some definitions and a basic lemma.

Definition 3.1. *$\mathcal{D}(n,\beta,k)$ denotes the set of balanced bipartite digraphs of order $2n$, with $k \geq 1$, $n = 2\beta + k$, such that $\forall x \in V(D)$, $d^+(x) \geq k$, $d^-(x) \geq k$, and for which the maximum cardinality of a balanced independent set is 2β.*

Definition 3.2. *In the following, if $D = (X, Y, E) \in \mathcal{D}(n,\beta,k)$, denote by S a balanced independent set of cardinality 2β.*

D_1 is the induced subgraph of D with partite sets (X_1, Y_1), $X_1 = X \cap S$, $Y_1 = Y \backslash S$,

D_2 is the induced subgraph of D with partite sets (X_2, Y_2), $X_2 = X \backslash S$, $Y_2 = Y \cap S$.

Lemma 3.3. *Let $D = (X, Y, E)$ be a balanced bipartite digraph with $|X| = |Y| = n$. Suppose that D contains a cycle C and a path P such that C and P are disjoint and $|V(C)| = 2p$,*

$|V(P)| = 2(n-p)$. If the beginning-vertex a and the end-vertex b of P satisfy the condition $d^-_C(a) + d^+_C(b) \geq p + 1$, then D has a hamiltonian cycle containing P.

Proof:

W.l.o.g. we may assume that $a \in X$ and $b \in Y$. Set $C = (y_1, x_1 ... y_p, x_p, y_1)$ with $x_i \in X$, $y_i \in Y$. The condition $d^-_C(a) + d^+_C(b) \geq p + 1$ implies that there exists i, $1 \leq i \leq p$, such that $y_i \in N^-(a)$, $x_i \in N^+(b)$; then the cycle $(a, P, b, x_i, C, y_i, a)$ is a hamiltonian cycle of D containing P.

4 Proof of Theorem 2.1

Let $D = (X, Y, E)$ be a bipartite digraph such that the maximum cardinality of a balanced independent set is 2β.

For $\beta = 0$, if $|E| \geq f(n, 0) = 2n^2 - n + 1$, (resp. $|E| \geq g(n, 0) = 2n^2 - n + 2$),

by Theorem 1.2, D is hamiltonian (resp. hamiltonian-biconnected).

Thus we assume $\beta \geq 1$.

4.1 Proof of (i)

As $|E| \geq f(n,\beta)$,

$|E(D_1)| + |E(D_2)| \geq f(n,\beta) - 2(n-\beta)^2 = -4\beta^2 + 4n\beta - (n-\beta) + 1$.

Therefore w.l.o.g.,

$|E(D_1)| \geq \frac{1}{2}\Big(|E(D_1)| + |E(D_2)|\Big) \geq 2\beta(n-\beta) - (n-\beta)/2 + 1/2 \geq (2\beta - 1)(n-\beta) + 1$.

Thus by Theorem 1.1, D_1 contains a cycle C of length 2β. Clearly C saturates $X \cap S$.

Let Γ be the subgraph induced by the vertex-set $V(D)\backslash V(C)$.

If $|E(\Gamma)| \geq 2(n-\beta)^2 - (n-\beta) + 2$, by Theorem 1.2, Γ is hamiltonian-biconnected. As D has at most $2\beta^2 + (n-\beta) - 1$ less arcs than the corresponding complete digraph, the number of arcs between C and Γ is

(1) $\displaystyle\sum_{x \in V(C)} d^+_\Gamma(x) + d^-_\Gamma(x^+) \geq 4\beta(n-\beta) - 2\beta^2 - (n-\beta) + 1.$

If for every $x \in C$ either $N^+_\Gamma(x) = \emptyset$ or $N^-_\Gamma(x^+) = \emptyset$ then

(2) $\displaystyle\sum_{x \in V(C)} d^+_\Gamma(x) + d^-_\Gamma(x^+) \leq 2\beta(n-\beta).$

As $4\beta(n-\beta) - 2\beta^2 - (n-\beta) + 1 > 2\beta(n-\beta)$ by (1) and (2), there exist $x \in V(C)$, $a \in V(\Gamma)$, $b \in V(\Gamma)$ such that x dominates a and x^+ is dominated by b.

Let P be a hamiltonian path in Γ from a to b. Then (x, a, P, b, x^+, C, x) is a hamiltonian cycle in D.

If $E(\Gamma) = 2(n-\beta)^2 - (n-\beta) + 1$, Γ is hamiltonian. Moreover, if $x \in V(C)$, $z \in V(\Gamma)$, then both (x, z) and (z, x) are in $E(D)$ unless $x \in X \cap S$, then $d^+_\Gamma(x) = n - 2\beta$, $d^-_\Gamma(x^+) = n - \beta$. Thus $d^+_\Gamma(x) + d^-_\Gamma(x^+ = n - 3\beta \geq (n-\beta) + 1$. Hence, by Lemma 3.3, D is hamiltonian. □

4.2 Proof of (ii)

We assume $n \geq 2\beta + 1$ and $|E| \geq f(n, \beta) + 1$.

Let $x \in V(D)$, $y \in V(D)$, x and y not in the same partite set. We want to prove that there exists a hamiltonian path from x to y. W.l.o.g. we can suppose $x \in X$ and $y \in Y$.

Case 1: $x \in X \cap S$, $y \in Y \cap S$.

By similar arguments as in part (i), we may assume that D_1 contains a cycle C of length 2β. As C saturates $X \cap S$, $x \in V(C)$.

If Γ denotes the subgraph of D induced by the vertex-set $V(D)\backslash V(C)$, $|E(\Gamma)| \geq 2(n-\beta)^2 - (n-\beta) + 2$, then by Theorem 5.4 it is hamiltonian-biconnected.

Let x^- be the predecessor of x on C; as in part (i) we can prove that x^- has at least one neighbor $a \in V(\Gamma)$.

Let P be a hamiltonian path of Γ from a to y. Then (x, C, x^-, a, P, y) is a hamiltonian path in D from x to y.

Thus there exists in D a hamiltonian path from x to y.

Case 2: $x \in X \cap S$, $y \in Y \cap (D \backslash S)$.

Let D_3 be the subgraph induced by the set of vertices $(X \cap S) \cup (Y \cap (D \backslash S) - \{y\})$. As $E(D) \geq f(n, \beta) + 1$, D_3 has at most $(n - \beta + 2)$ arcs less than the corresponding complete digraph, then $|E(D_3)| \geq 2\beta(n - \beta - 1) - (n - \beta - 1) + 1$; by Theorem 1.1, D_3 contains a cycle C of length 2β, with $x \in V(C)$, $y \notin V(C)$.

If, as in case 1, Γ denotes the subgraph of D induced by the vertex-set $V(D) \backslash V(C)$, Γ is hamiltonian-biconnected; similar arguments as in case 1 prove that there exists a hamiltonian path from x to y.

Case 3: $x \notin X \cap S$, $y \notin Y \cap S$.

As in case 2, the subdigraph D_3 induced by the set of vertices $(X \cap S) \cup (Y \cap (D \backslash S) - \{y\})$ contains a cycle C of length 2β.

The subgraph Γ of D induced by the vertex-set $V(D) \backslash V(C)$ is, as in case 1, hamiltonian-biconnected. The vertices x and y are in $V(\Gamma)$; let P be a hamiltonian path in Γ from x to y.

If we assume that for any $a \in V(P) \backslash \{y\}$, $d^+_C(a) + d^-_C(a^+) \leq \beta$, D has at least $\beta(n - \beta) + \beta(n - \beta - 1)$ arcs less than the corresponding complete digraph; the condition $|E| \geq f(n, \beta)$ implies :

$2\beta(n-\beta)-\beta \leq n-\beta-2+2\beta^2 \Leftrightarrow 2\beta n \leq 4\beta^2+n-2 \Leftrightarrow (2\beta-1)(n-2\beta) \leq 2\beta - 2$, a contradiction.

Hence there exists $a \in V(P)$, $a \neq y$, such that $d^+_C(a) + d^-_C(a^+) \geq \beta + 1$.

By Lemma 3.3, there exists in D a hamiltonian path from x to y.

Theorem 2.1 is proved. □

5 Proof of Theorem 2.3

5.1 Strategy of the proof

The proof of Theorem 2.3 is by induction on k.

In sub-section 5.2, we shall prove the Theorem for $k = 1$.

Then we shall do the following induction hypothesis:

Induction Hypothesis 5.1.

For $1 \leq p \leq k - 1$, let $D = (X, Y, E) \in \mathcal{D}(n, \beta, p)$.

(i) The condition $|E| \geq F(n, \beta)$, implies that D is hamiltonian.

(ii) The condition $|E| \geq G(n, \beta)$, implies that D is hamiltonian-biconnected.

In sub-section 5.3, we shall prove Proposition 5.2:

Proposition 5.2. *Under the induction hypothesis* 5.1*, if* $D \in \mathcal{D}(n, \beta, k)$ *satisfies* $|E| \geq G(n, \beta)$*, then* D *is hamiltonian-biconnected.*

In sub-section 5.4, we shall prove Proposition 5.3**:**

Proposition 5.3. *Under the induction hypothesis* 5.1*, if* $D \in \mathcal{D}(n, \beta, k)$ *satisfies* $|E| \geq F(n, \beta)$*, then* D *is hamiltonian.*

Proposition 5.2 and Proposition 5.3 will imply Theorem 2.3.

5.2 Proof of Theorem 2.3 when $k = 1$.

We need two general lemmas:

Lemma 5.4. *We suppose that for any digraph* $D' = (X', Y', E') \in \mathcal{D}(n, \beta, k)$*, the condition* $|E'| \geq G(n, \beta)$ *implies that* D' *is hamiltonian-biconnected, then*

If $D = (X, Y, E) \in \mathcal{D}(n, \beta, k)$ *satisfies the condition* $|E| \geq G(n, \beta) - p$*, and if there is no hamiltonian path from a vertex* y *to a vertex* x *not in the same partite set then:*

(i) If $x \in S$*,* $y \notin S$*, then* $d^+(x) + d^-(y) \geq 2n - \beta - p + 1$*,* $d^+(x) \geq n - \beta - p + 1$*,* $d^-(y) \geq n - p + 1$*.*

(ii) If $x \notin S$*,* $y \in S$*, then* $d^+(x) + d^-(y) \geq 2n - \beta - p + 1$*,* $d^+(x) \geq n - p + 1$*,* $d^-(y) \geq n - \beta - p + 1$*.*

(iii) If $x \notin S$*,* $y \notin S$*, then* $d^+(x) + d^-(y) \geq 2n - p + 1$*,* $d^+(x) \geq n - p + 1$*,* $d^-(y) \geq n - p + 1$*.*

(iv) If $x \in S$*,* $y \in S$*, then* $d^+(x) + d^-(y) \geq 2n - 2\beta - p + 1$*,* $d^+(x) \geq n - \beta - p + 1$*,* $d^-(y) \geq n - \beta - p + 1$*.*

Lemma 5.5. *Under the same hypothesis as in Lemma* 5.4*, if* D *is not hamiltonian then:*

(i) $\forall x \in S$*,* $d^+(x) \geq n - \beta - p + 1$*,* $d^-(x) \geq n - \beta - p + 1$*,*

(ii) $\forall x \notin S$*,* $d^+(x) \geq n - p + 1$*,* $d^-(x) \geq n - p + 1$

Proof of Lemma 5.4:

Let $D = (X, Y, E) \in \mathcal{D}(n, \beta, k)$. We assume $|E| \geq G(n, \beta) - p$.

If one of the following cases happen:

1) $x \in S$, $y \notin S$, $d^+(x) + d^-(y) \leq 2n - \beta - p$,

2) $x \notin S$, $y \in S$, $d^+(x) + d^-(y) \leq 2n - \beta - p$,

3) $x \notin S$, $y \notin S$, $d^+(x) + d^-(y) \leq 2n - p$,

4) $x \in S$, $y \in S$, $d^+(x) + d^-(y) \leq 2n - 2\beta - p$,

we can add p arcs to $N^+(x) \cup N^-(y)$ to obtain a digraph $D' = (X', Y', E') \in \mathcal{D}(n, \beta, k)$ such that $|E(D')| \geq G(n, \beta)$; then $D' \in \mathcal{D}(n, \beta, k)$ and satifies: $|E'| \geq G(n, \beta)$; then under the assumption of Lemma 5.4 D' is hamiltonian-biconnected, and a hamiltonian path from y to x in D' is a hamiltonian path from y to x in D. □

To prove Lemma 5.5, we apply Lemma 5.4 to any vertices x and y such that the arc $(xy) \in E(D)$. □

Lemma 5.6. *For $D \in \mathcal{D}(n, \beta, 1)$, (i) If $|E| \geq F(n, \beta)$, D is hamiltonian,*
(ii) If $|E| \geq G(n, \beta)$, D is hamiltonian-biconnected.

Proof:

(*ii*) For $k = 1$, $f(n, \beta) + 1 = G(n, \beta)$, then if $|E| \geq G(n, \beta)$, by Theorem 2.1, D is hamiltonian-biconnected. □

(*i*) If $|E| \geq F(n, \beta)$, as $F(n, \beta) = G(n, \beta) - \beta$, if we assume that D is not hamiltonian we can apply Lemma 5.5 with $p = \beta$ and, as $n = 2\beta + 1$, obtain:

(*) $\forall x \in S$, $d^+(x) \geq 2$, $d^-(x) \geq 2$, $\forall x \notin S$, $d^+(x) \geq \beta + 2$, $d^-(x) \geq \beta + 2$.

D has at most $2\beta^2 + 2\beta - 1$ arcs less than the corresponding complete digraph, then $D_1 \cup D_2$ have at most $2\beta - 1$ arcs less than the union of corresponding complete digraphs; w.l.o.g. we may assume $|E(D_1)| \geq 2\beta(\beta + 1) - \beta + 1$; then, by Theorem 1.1, D_1 contains a cycle C of length 2β; C saturates $X \cap S$. If Γ denotes the subgraph of D induced by the vertex-set $V(D) \backslash V(C)$, $|E(\Gamma)| \geq 2(\beta + 1)^2 - 2\beta + 1$.

If $x \in V(\Gamma) \cap S$ all the neighbors of x are in Γ; if $y \in V(\Gamma) \cap (D \backslash S)$, $d^+_\Gamma(y) \geq d^+(y) - \beta$, $d^-_\Gamma(y) \geq d^-(y) - \beta$; in every case:

The conditions (*) imply: $\forall x \in V(\Gamma)$, $d^+_\Gamma(x) \geq 2$, $d^-_\Gamma(x) \geq 2$.

Hence, by Theorem 1.3, Γ is hamiltonian. Moreover

$|E(H, \Gamma)| \geq F(n, \beta) - |E(H)| - |E(\Gamma)| \geq F(n, \beta) - 2\beta^2 - 2(\beta + 1)^2 \geq 2\beta(\beta + 1) + 1$.

The subdigraph Γ is hamiltonian-biconnected unless

$$|E(\Gamma)| \leq 2(\beta + 1)^2 - 2\beta + 2.$$

If Γ is hamiltonian-biconnected, as $|E(H, \Gamma)| \geq 2\beta(\beta + 1) + 1$, there exist $x \in V(C)$, $a \in V(\Gamma)$, $b \in V(\Gamma)$ such that x dominates a and x^+ is dominated by b; let P be a hamiltonian path in Γ from a to b. Then (x, a, P, b, x^+, C, x) is a hamiltonian cycle in D.

If Γ is not hamiltonian-biconnected, as $|E(\Gamma)| \leq 2(\beta + 1)^2 - 2\beta + 2$, the subgraph H induced by $V(C)$ satisfies $|E(H)| \geq 2\beta^2 - 1$; then H is hamiltonian-biconnected. Let C_Γ be a hamiltonian cycle of Γ; as $|E(H, \Gamma)| \geq 2\beta(\beta + 1) + 1$, there exist $a \in C_\Gamma$ and $a^+ \in C_\Gamma$, such that a dominates a vertex

$c \in V(H)$ and a^+ is dominated by a vertex $d \in V(H)$; let P be a hamiltonian path in H from c to d, then $(c, P, d, a^+, C_\Gamma, a, c)$ is a hamiltonian cycle of D.

In both cases, D is hamiltonian. □

5.3 Proof of Proposition 5.2

The induction hypothesis 5.1 is satisfied for $k = 2$.

Proposition 5.2 *Under the induction hypothesis* 5.1, *if* $D \in \mathcal{D}(n, \beta, k)$ *satisfies* $|E| \geq G(n, \beta)$, *D is hamiltonian-biconnected.*

Proof:

We assume $k \geq 2$.

Let $D = (X, Y, E) \in \mathcal{D}(n, \beta, k)$ and suppose $|E| \geq G(n, \beta)$.

For any $x \in V(D)$, $y \in V(D)$ not in the same partite set, we prove that there exists a hamiltonian path from x to y. W.l.o.g. we can suppose $x \in X$, $y \in Y$.

Claim 5.7. *There exist at least* $\beta + 1$ *vertices* $u \in X \cap (D \backslash S)$, *and* $\beta + 1$ *vertices* $v \in Y \cap (D \backslash S)$, *such that* $d^+(u) \geq \beta + k$, $d^-(u) \geq \beta + k$, $d^+(v) \geq \beta + k$, $d^-(v) \geq \beta + k$.

Proof:

If Claim 5.7 is not true, w.l.o.g. we may assume $d^+(u) \leq \beta + k - 1$ for k vertices $u \in X \cap (D \backslash S)$. As $n = 2\beta + k$, the subgraph of D induced by the vertex-set $X \cap (D \backslash S) \cup Y$ has at leat $(\beta + 1)k$ arcs less than the corresponding complete graph. Hence, S being an independent set, the inequality $|E(D)| \leq 2n^2 - 2\beta^2 - (\beta + 1)k$ would be satisfied.

As $(\beta + 1)k < \beta k$, $2n^2 - 2\beta^2 - (\beta + 1)k < G(n, \beta)$, a contradiction with the hypothesis

$|E(D)| \geq G(n, \beta)$. □

Then let $u_0 \in X \cap (D \backslash S)$, $u_0 \neq x$, and $v_0 \in Y \cap (D \backslash S)$, $v_0 \neq y$, be vertices satisfying $d^+(u_0) \geq \beta + k$, $d^-(u_0) \geq \beta + k$, $d^+(v_0) \geq \beta + k$, $d^-(v_0) \geq \beta + k$.

Let $\epsilon = 1$ if $(xy) \in E$, $\epsilon = 0$ if $(xy) \notin E$, and $\epsilon' = 1$ if $(yx) \in E$, $\epsilon' = 0$ if $(yx) \notin E$.

Let D'_i be a bipartite digraph of order $2(n-1)$ with vertex-set $V(D'_i) = V(D) \backslash \{x, y\}$ and edge-set $E(D'_i)$ defined as follows:

Case 1 If $x \notin S$, $y \notin S$, D'_1 is the subgraph of D induced by $V(D) \backslash \{x, y\}$; then

$|E(D'_1)| = |E(D)| - d(x) - d(y) + \epsilon + \epsilon' \geq G(n,\beta) - d(x) - d(y) + \epsilon + \epsilon'$, hence $|E(D'_1)| \geq G(n,\beta) - (4n-2) = F(n-1,\beta)$.

Case 2 If $x \in S$, $y \notin S$, $E(D'_2) = E(D'_1)\backslash\Big(E(u_0, Y \cap S)\Big)$; then
$|E(D'_2)| = |E(D)| - d(x) - d(y) + \epsilon + \epsilon' - |E(u_0, Y \cap S)| \geq$
$G(n,\beta) - d(x) - d(y) + \epsilon + \epsilon' - |E(u_0, Y \cap S)|$,
hence $|E(D'_2)| \geq G(n,\beta) - (4n-2) = F(n-1,\beta)$.

Case 3 If $x \in S$, $y \in S$, $E(D'_3) = E(D'_1)\backslash\Big(E(u_0, Y \cap S) \cup E(v_0, X \cap S) \cup E(u_0, v_0)\Big)$; then
$|E(D'_3)| = \quad |E(D)| - d(x) - d(y) - |E(u_0, (Y \cap S\backslash\{y\}))| - |E(v_0, (X \cap S\backslash\{x\}))| - |E(u_0, v_0)| \geq$
$G(n,\beta) - 4(n-\beta) - 4(\beta - 1) - 2$,
hence $|E(D'_3)| \geq G(n,\beta) - (4n-2) = F(n-1,\beta)$.

Moreover S (resp. $S\backslash\{x\} \cup \{u_0\}$, resp. $S\backslash\{x,y\} \cup \{u_0, v_0\}$) is a balanced independent set of D'_1 (resp. of D'_2, resp. of D'_3) of order 2β.

For every $z \in V(D'_1)$, for $z \neq u_0$ in D'_2 and for $z \neq u_0$ and $z \neq v_0$ in D'_3, the conditions $d^+(z) \geq k$, $d^-(z) \geq k$ imply $d^+_{D'_i}(z) \geq k-1$, $d^-_{D'_i}(z) \geq k-1$,

In **Case 2** the conditions $d^+(u_0) \geq \beta + k$, $d^-(u_0) \geq \beta + k$, imply $d^+_{D'_2}(u_0) \geq k-1$, $d^-_{D'_2}(u_0) \geq k-1$.

In **Case 3** the conditions $d^+(u_0) \geq \beta + k$, $d^-(u_0) \geq \beta + k$, $d^+(v_0) \geq \beta + k$, $d^-(v_0) \geq \beta + k$ imply $d^+_{D'_3}(u_0) \geq k-1$, $d^-_{D'_3}(u_0) \geq k-1$, $d^+_{D'_3}(v_0) \geq k-1$, $d^-_{D'_3}(v_0) \geq k-1$.

At least the equality $n - 1 = 2\beta + k - 1$ is satisfied.

We can conclude that in every case $D'_i \in \mathcal{D}(n-1, \beta, k-1)$, and satisfies $|E(D'_i)| \geq F(n-1,\beta)$.

By the induction hypothesis 5.1, D'_i is hamiltonian.

Let C be a hamiltonian cycle in D'_i.

If $d^+(x) + d^-(y) \geq n + 2\epsilon$, let $a \in V(C)$ such that $a \in N^-(y)$, $a^+ \in N^+(x)$, then the path (x, a^+, C, a, y) is a hamiltonian path in D from x to y.

If D'_i is hamiltonian-biconnected, let c and d be vertices in $V(D'_i)$ such that $d \in N^+(x)$, $c \in N^-(y)$, and let P be a hamiltonian path in D'_i from d to c; then (x, d, P, c, y) is a hamiltonian path in D from x to y.

Then we may assume that $d^+(x) + d^-(y) \leq n - 1 + 2\epsilon$ and that D'_i is hamiltonian but not hamiltonian-biconnected, and by the induction hypothesis 5.1 that $|E(D'_i)| < G(n-1,\beta)$.

Then $|E(D)| - |E(D'_i)| \geq G(n,\beta) - G(n-1,\beta) + 1 = 4n - 1 - \beta$.

This inequality implies:

Case 1: $|E(D)| - |E(D_1')| = d(x) + d(y) - \epsilon - \epsilon' \geq 4n - 1 - \beta$**.**

As $d^-(x) + d^+(y) \leq 2(n-1) + 2\epsilon'$, $d^+(x) + d^-(y) \geq 2n + 1 - 2\beta + \epsilon - \epsilon' = n + k + 1 + \epsilon - \epsilon' \geq n + 2\epsilon$, a contradiction with the assumption $d^+(x) + d^-(y) \leq n - 1 + 2\epsilon$.

Case 2: $|E(D)| - |E(D_2')| = d(x) + d(y) - \epsilon - \epsilon' + |E(u_0, Y \cap S)| \geq 4n - 1 - \beta$, then

$d(x) + d(y) \geq 4n - 1 - 3\beta + \epsilon + \epsilon'$.

As $d^-(x) + d^+(y) \leq 2(n-1) - \beta + 2\epsilon'$, $d^+(x) + d^-(y) \geq 2n + 1 - 2\beta + \epsilon - \epsilon' = n + k + 1 + \epsilon - \epsilon' \geq n + 2\epsilon$, a contradiction with the assumption $d^+(x) + d^-(y) \leq n - 1 + 2\epsilon$.

Case 3: $|E(D)| - |E(D_3')| =$

$d(x) + d(y) + |E(u_0, (Y \cap S \backslash \{y\}))| + |E(v_0, (X \cap S \backslash \{x\}))| + |E(u_0, v_0)| \geq 4n - 1 - \beta$.

As $d^-(x) + d^+(y) \leq 2(n - \beta)$, $d^+(x) + d^-(y) \geq 2n + 1 - 3\beta$.

If $x \in V(S)$, and $y \in V(S)$, $\epsilon = \epsilon' = 0$.

$d(x) + d(y) \geq 4n - 1 - \beta - 4(\beta - 1) - 2 = 4n - 5\beta + 1$.

The only remaining problem is **Case 3**, when $2n + 5 - 3\beta \leq d^+(x) + d^-(y) \leq n - 1$.

As $d^+(x) + d^-(y) \geq 2n + 1 - 3\beta = \beta + 2k + 1$, $d^+(x) \leq \beta + k \Rightarrow d^-(y) \geq k + 1$, and $d^-(y) \leq \beta + k \Rightarrow d^+(x) \geq k + 1$. Moreover the condition $d^+(x) + d^-(y) \leq n - 1$ implies

$d(x) + d(y) \leq 2(n - \beta) + n - 1 = 3n - 2\beta - 1$; then:

$|E(D_3')| \geq G(n, \beta) - (3n - 2\beta - 1) - 4\beta + 2 = G(n, \beta) - 4n + 2 + k + 1 = G(n-1, \beta) - (\beta - k - 1)$.

We obtain the following

Claim 5.8. *If there is no hamiltonian path in D from x to y, then $\forall a \in N^-(y)$, and $\forall b \in N^+(x)$, $d^+(a) + d^-(b) \geq 2n - \beta + k + 2$.*

Proof**:**

If $a \neq u_0$ and $b \neq v_0$, Claim 5.8 follows from Lemma 5.4 applied to D_3', the vertices $b \in N^+(x)$ and $a \in N^-(y)$ and $p = \beta - k - 1$.

If $u_0 \in N^-(y)$ or $v_0 \in N^+(x)$, the condition $\beta \geq k + 2$ implies that there exist u and v, $u \neq u_0$, or $v \neq v_0$, satisfying $d^+(u) \geq \beta + k$, $d^{+-}(u) \geq \beta + k$ or $d^+(v) \geq \beta + k$, $d^-(v) \geq \beta + k$.

We can consider for D_3' : $D_3' = D \backslash (\{x, y\} \cup E(u, Y \cap S) \cup E(v, X \cap S) \cup E(u, v))$ and Claim 5.8 follows in all cases. □

Conditions $d^+(x) \geq k+1$, $d^-(y) \geq k+1$ imply, by a counting argument and Claim 5.7, that there exists a vertex $a_1 \in N^-(y)$, $a_1 \neq u_0$ and a vertex $b_1 \in N^+(x)$, $b_1 \neq v_0$ which satisfy the conditions $d^+(b_1) \geq \beta + k$, $d^-(a_1) \geq \beta + k$ and by Claim 5.8, $d^+(a_1) + d^-(b_1) \geq 2n - \beta + k + 2$.

Let us consider the digraph Δ obtained from D by contracting the vertices x and a_1, and the vertices y and b_1, i.e.:
$V(\Delta) = V(D) \backslash \{x, y, a_1, b_1\} \cup \{A, B\}$ with :
$N^+_\Delta(A) = N^+(x) \backslash \{b_1\}$; $N^-_\Delta(A) = N^-(a_1) \backslash ((Y \cap S) \cup \{b_1\})$;
$N^+_\Delta(B) = N^+(b_1) \backslash ((X \cap S) \cup \{a_1\})$; $N^-_\Delta(B) = N^-(y) \backslash \{a_1\}$;
for $z \notin \{A, B\}$, $N^+_\Delta(z) = N^+(z) \backslash \{x, y, a_1, b_1\} \cup \{B\}$ if $(zy) \in E(D)$,
$N^+_\Delta(z) = N^+(z) \backslash \{x, y, a_1, b_1\} \cup \{A\}$ if $(za_1) \in E(D)$,
$N^-_\Delta(z) = N^-(z) \backslash \{x, y, a_1, b_1\} \cup \{A\}$ if $(xz) \in E(D)$,
$N^-_\Delta(z) = N^-(z) \backslash \{x, y, a_1, b_1\} \cup \{B\}$ if $(b_1 y) \in E(D)$.

Then $d^+_\Delta(A) = d^+(x) - 1$, $d^-_\Delta(A) \geq d^-(a_1) - (\beta + 1)$, that implies $d^+_\Delta(A) \geq k - 1$, $d^-_\Delta(A) \geq k - 1$,

$d^+_\Delta(B) \geq d^+(b_1) - (\beta+1)$, $d^-_\Delta(B) = d^-(y) - 1$, that implies $d^+_\Delta(B) \geq k-1$, $d^-_\Delta(B) \geq k - 1$,

$\forall z \in V(\Delta) \backslash \{A, B\}$, $d^+_\Delta(z) \geq d^+(z) - 1$, $d^-_\Delta(z) \geq d^-(z) - 1$, then
$\forall x \in V(\Delta)$, $d^+_\Delta(x) \geq k - 1$, $d^-_\Delta(x) \geq k - 1$.

The digraph Δ is a balanced bipartite digraph of order $2(n - 1)$.

The set $S \backslash \{x, y\} \cup \{A, B\}$ is a balanced independent set of cardinality 2β in Δ.

Hence $\Delta \in \mathcal{D}(n - 1, \beta, k - 1)$ and $|E(\Delta)| \geq G(n, \beta) - d^-(x) - d^+(y) - d^+(a_1) - d^-(b_1) + \eta - \eta' - 2\beta + 2$, with $\eta = 1$ if $(a_1 b_1) \in E$, $\eta = 0$ if $(a_1 b_1) \notin E$, and $\eta' = 1$ if $(b_1 a_1) \in E$, $\eta' = 0$ if $(b_1 a_1) \notin E$.

Then $|E(\Delta)| \geq G(n, \beta) - 4n + 2 = F(n - 1, \beta)$.

By the induction hypothesis 5.1, Δ is hamiltonian, and from a hamiltonian cycle in Δ, we can deduce two disjoint paths P_1 from x to y, and P_2 from b_1 to a_1 with $V(P_1) \cup V(P_2) = V(D)$.

Let $|V(P_1)| = 2n_1$ and $|V(P_2)| = 2n_2$.

As $d^+(a_1) + d^-(b_1) \geq 2n - \beta + k + 2$ the following inequality is satisfied: $d^+_{P_1}(a_1) + d^-_{P_1}(b_1) \geq 2n - \beta + k + 2 - 2n_2 = 2n_1 - \beta + k + 2$.

If $d^+_{P_1}(a_1) + d^-_{P_1}(b_1) \geq n_1 + 1$, let $v \in V(P_1) \cap N^-(b_1)$ such that $v^+ \in N^+(a_1)$;

$(x, P_1, v, b_1, P_2, a_1, v^+, P_1, y)$ is a hamiltonian path from x to y.

If $d^+_{P_1}(a_1) + d^-_{P_1}(b_1) \leq n_1$, then $n_1 \leq \beta - k - 2$, and $n_2 \geq \beta + 2k + 2$;

If y^- is the predecessor of y on P_1 and x^+ is the successor of x on P_1, by Claim 5.8:

$$d^+(y^-) + d^-(x^+) \geq 2n - \beta + k + 2 \ ;$$

$$d^+_{P_1}(y^-) + d^-_{P_1}(x^+) \leq 2n_1 \Rightarrow d^+_{P_2}(y^-) + d^-_{P_2}(x^+) \geq 2n_2 - \beta + k + 2 \geq n_2 + 1.$$

Let $\alpha \in N^-_{P_2}(x^+)$ such that $\alpha^+ \in N^+_{P_2}(y^-)$;

$(x, b_1, P_2, \alpha, x^+, P_1, y^-, \alpha^+, P_2, a_1, y)$ is a hamiltonian path from x to y.

Proposition 5.2 is proved. □

5.4 Proof of Proposition 5.3

Proposition 5.3 *Under the induction hypothesis* 5.1*, if* $D \in \mathcal{D}(n, \beta, k)$ *satisfies* $|E| \geq F(n, \beta)$*,* D *is hamiltonian.*

Let $D \in \mathcal{D}(n, \beta, k)$ satisfy $|E| \geq F(n, \beta)$. If we assume that D is not hamiltonian, for any arc $(x, y) \in E(D)$ there is no hamiltonian path in D from y to x; as $|E| \geq F(n, \beta) = G(n, \beta) - \beta$, we can apply Lemma 5.5 with $p = \beta$ and obtain the following Claim:

Claim 5.9. *If* $D \in \mathcal{D}(n, \beta, k)$ *satisfying* $|E| \geq F(n, \beta)$ *is not hamiltonian, then for any arc* $(xy) \in E$*:*

(i) If $x \in S$, $y \notin S$, *or* $x \notin S$, $y \in S$, $d^+(x) + d^-(y) \geq 2n - 2\beta + 1$,

(ii) If $x \notin S$, $y \notin S$, $d^+(x) + d^-(y) \geq 2n - \beta + 1$,

(iii) $\forall x \in S$, $d^+(x) \geq k + 1$, $d^-(x) \geq k + 1$,

(iv) $\forall x \notin S$, $d^+(x) \geq \beta + k + 1$, $d^-(x) \geq \beta + k + 1$.

5.4.1 Preliminary Lemma

Lemma 5.10. *If* $D \in \mathcal{D}(n, \beta, k)$ *satisfying* $|E| \geq F(n, \beta)$ *is not hamiltonian, there exists in* D *a cycle* C *of length* 2β *which saturates* $X \cap S$ *or* $Y \cap S$*.*

The proof is based on the following Claim:

Claim 5.11. *If* $D \in \mathcal{D}(n, \beta, k)$*, and if* $|E| \geq F(n, \beta)$*, there exists a perfect matching of* $X \cap S$ *into* $Y \cap (D \backslash S)$*, and a perfect matching of* $Y \cap S$ *into* $X \cap (D \backslash S)$

Proof:

We use the Hall-Konig Theorem (see [7] p 128) to prove Claim 5.11:

Theorem 5.12. (Hall-Konig) *Let $G = (U, V, E)$ be a bipartite digraph with partite sets U and V; if for any subset $A \subset U$, $|N^+(A)| \geq |A|$, then there exists a perfect matching of U into V.*

We assume there exists $A \subset X \cap S$, such that if $B = N^+(A)$, $|B| < |A|$; the condition $d^+(x) \geq k$ for any $x \in A$ implies the inequality:

$$k \leq |B| \leq |A| - 1 \leq \beta - 1$$

and at least $|A|(\beta+k-|B|)$ arcs are missing between $X \cap S$ and $Y \cap (D \backslash S)$; let $t = |B|$.

$|A|(\beta + k - |B|) \geq (t+1)(\beta + k - t)$, with $k \leq t \leq \beta - 1$.

$|A|(\beta + k - |B|) \geq \min_{k \leq t \leq \beta-1}((t+1)(\beta + k - t)) = \beta(k+1)$.

Then at least $\beta(k+1)$ arcs are missing between $X \cap S$ and $Y \cap (D \backslash S)$, then

$|E(D)| \leq 2n^2 - 2\beta^2 - \beta(k+1) < F(n, \beta)$, a contradiction with the condition $|E(D)| \geq F(n, \beta)$.

Claim 5.11 is proved. □

Proof of Lemma 5.10:

Set $l = \min(k, \lfloor \frac{\beta}{2} \rfloor)$; we consider the two following cases:

Case 1. There exists a vertex $x_0 \notin S$ with $|E(x_0, S)| \leq \beta + l$,

Case 2. For any vertex $x \notin S$, $|E(x, S)| > \beta + l$.

Case 1: W.l.o.g. we can assume $|E(x_0, S)| \leq \beta + l$ for a vertex $x_0 \in X \backslash S$.

Let $(x_i y_i)$, $1 \leq i \leq \beta$, be a matching from $X \cap S$ into $Y \cap (D \backslash S)$.

For $1 \leq i \leq \beta$ let $D'_i = D \backslash (\{x_i, y_i\} \cup E(x_0, S))$; $D'_i \in \mathcal{D}(n-1, \beta, k-1)$ and *:*

$|E(D'_i)| \geq F(n, \beta) - d(x_i) - d(y_i) + 1 + \epsilon_i - |E(x_0, S)|$, with $\epsilon_i = 1$ if $(y_i x_i) \in E$, $\epsilon_i = 0$ if $(y_i x_i) \notin E$.

Case 1-1: $\exists i, 1 \leq i \leq \beta$ such that:

$d(x_i) + d(y_i) - 1 - \epsilon_i + |E(x_0, S)| \leq F(n, \beta) - F(n-1, \beta) = 4n - 2 - \beta$.

Then $|E(D'_i)| \geq F(n-1, \beta)$ and by the induction hypothesis 5.1 D'_i is hamiltonian.

If $d^-(x_i) + d^+(y_i) \geq n + 2\epsilon_i$, by Lemma 3.3, D is hamiltonian.

If $d^-(x_i) + d^+(y_i) \leq n - 1 + 2\epsilon_i$, by Claim 5.9, the arc $(y_i x_i) \notin E(D)$, then $\epsilon_i = 0$.

As $d^+(x_i) + d^-(y_i) \leq 2n - \beta$, then $d(x_i) + dy_i) \leq 3n - 1 - \beta$.

$|E(D'_i)| \geq F(n, \beta) - (3n-1-\beta) - \beta - l \geq F(n, \beta) - (3n+k-2) = G(n-1, \beta)$; then D'_i is hamiltonian-biconnected.

Let $b \in N^-(x_i)$, $a \in N^+(y_i)$ and let P be a hamiltonian path in D_i from a to b; the cycle (a, P, b, x_i, y_i, a) is a hamiltonian cycle in D.

Case 1-2: $\forall i, 1 \le i \le \beta$:

$d(x_i) + d(y_i) - 1 - \epsilon_i + |E(x_0, S)| > F(n, \beta) - F(n-1, \beta) = 4n - 2 - \beta$.

Then $d(x_i)+d(y_i) > 4n-2\beta-l-1+\epsilon_i$; the conditions $d(y_i) \le 2n-1+\epsilon_i$ and $d(x_i) \le 2n - 2\beta - 1 + \epsilon_i$ imply $d(x_i) > 2n - 2\beta - l$ and $d(y_i) > 2n - l$.

As $d^+(x_i) \le n - \beta$, $d^-(x_i) \le n - \beta$, $d^+(y_i) \le n$, $d^-(y_i) \le n$, then we have :

$d^+(x_i) > n - \beta - l \ge \beta + k - \lfloor \frac{\beta}{2} \rfloor$; $d^-(x_i) > n - \beta - l \ge \beta + k - \lfloor \frac{\beta}{2} \rfloor$;

$d^+(y_i) > n - l \ge n - \lfloor \frac{\beta}{2} \rfloor$; $d^-(y_i) > n - l \ge n - \lfloor \frac{\beta}{2} \rfloor$.

Let H be the subgraph induced by $\{x_i, y_i, 1 \le i \le \beta\}$;

$\forall i, 1 \le i \le \beta$, the following inequalities are satisfied:

$d^+_H(x_i) > \beta + k - \lfloor \frac{\beta}{2} \rfloor - k = \lfloor \frac{\beta+1}{2} \rfloor$; $d^-_H(x_i) > \lfloor \frac{\beta+1}{2} \rfloor$;

$d^+_H(y_i) > n - \lfloor \frac{\beta}{2} \rfloor - \beta - k = \lfloor \frac{\beta+1}{2} \rfloor$; $d^-_H(y_i) > \lfloor \frac{\beta+1}{2} \rfloor$.

By Theorem 1.4, H is hamiltonian, and a hamiltonian cycle of H is a cycle of length 2β that saturates $X \cap S$.

Case 2 : $\forall x \in S$, $|E(x, S)| > \beta + l$ with $l = \min(k, \lfloor \frac{\beta}{2} \rfloor)$.

As in Definition 3.2, let D_1 (resp. D_2) denote the subgraph induced by the set of vertices $(X \cap S) \cup (Y \cap (D \backslash S))$, (resp. $(X \cap (D \backslash S) \cup (Y \cap S))$.

As $|E(D_1)|+|E(D_2)| \ge F(n, \beta) - 2(n-\beta)^2 = 2\beta(n-\beta)+\beta(n-2\beta+1)+1$, w.o.l.g. we may assume $|E(D_1)| \ge 2\beta(n-\beta) - \frac{1}{2}\beta(n-2\beta+1) + \frac{1}{2}$.

Case 2-1 : $\beta \ge 2k+1$, then $l = k$, and $\forall y \in V(D_1) \cap Y$, $d^+_{D_1}(y) \ge l+1 = k+1$, $d^-_{D_1}(y) \ge k+1$; by Claim 5.9, $\forall x \in V(D_1) \cap S$, $d^+_{D_1}(x) \ge k+1$, $d^-_{D_1}(x) \ge k+1$ and

$|E(D_1)| \ge 2\beta(n-\beta) - \frac{1}{2}\beta(n-2\beta+1) + \frac{1}{2} \ge 2\beta(n-\beta) - (k+1)(\beta-k) + 1$; by Theorem 1.3, D_1 has a cycle of length 2β, hence a cycle that saturates $X \cap S$.

Case 2-2 : $\beta \le 2k$, then $l = \lfloor \frac{\beta}{2} \rfloor$, and $\forall y \in V(D_1) \cap Y$, by the assumption of case 2,

$d^+_{D_1}(y) > \beta + l - \beta \ge \lfloor \frac{\beta}{2} \rfloor$, $d^-_{D_1}(y) > \lfloor \frac{\beta}{2} \rfloor$, and by Claim 5.9,

$\forall x \in V(D_1) \cap S$, $d^+_{D_1}(x) \ge k+1 \ge \lfloor \frac{\beta}{2} \rfloor + 1$, $d^-_{D_1}(x) \ge \lfloor \frac{\beta}{2} \rfloor + 1$.

By Theorem 1.4, D_1 has a cycle of length 2β that saturates $X \cap S$.

Lemma 5.10 is proved.□

5.4.2 Proof of Proposition 5.3

Claim 5.13. *Under the assumption of Lemma* 5.10*, let C be a cycle of length 2β in D that saturates $X \cap S$ or $Y \cap S$ and let Γ be the subgraph of D induced by $V(D)\backslash V(C)$, then Γ is hamiltonian.*

Proof:

The subgraph Γ satisfies: $|V(\Gamma)| = 2(n-\beta)$, $|E(\Gamma)| \geq |E(D)| - 2\beta n$.

By Claim 5.9, $\forall x \in S$, $d^+(x) \geq k+1$, $d^-(x) \geq k+1$ then

$\forall x \in V(\Gamma) \cap S$, $d^+_\Gamma(x) \geq k+1$, $d^-_\Gamma(x) \geq k+1$,

and $\forall x \notin S$, $d^+(x) \geq \beta + k + 1$, $d^-(x) \geq \beta + k + 1 \Rightarrow$

$\forall x \in V(\Gamma) \cap (D\backslash S)$, $d^+_\Gamma(x) \geq k+1$, $d^-_\Gamma(x) \geq k+1$.

Moreover $|E(\Gamma)| \geq 2(n-\beta)^2 - \beta(n - 2\beta + 1) + 1 = 2(n-\beta)^2 - (k+1)(n-\beta-k)+1$.

By Theorem 1.3 Γ is hamiltonian.□

Proof of Proposition 5.3 **:**

If $D \in \mathcal{D}(n,\beta,k)$ satisfying $|E| \geq F(n,\beta)$ is not hamiltonian, by Lemma 5.10 there exists in D a cycle C of length 2β which saturates $X \cap S$ or $Y \cap S$; by Claim 5.13 the subgraph Γ of D induced by $V(D)\backslash V(C)$ is hamiltonian. As $|V(\Gamma)| = 2(n-\beta) > 2\beta = |S|$, then on a hamiltonian cycle of Γ, there exist arcs with both ends in $D\backslash S$; by Claim 5.9, if (xy) is such an arc, $d^+(x) + d^-(y) \geq 2n - \beta + 1$, then $d^+_\Gamma(x) + d^-_\Gamma(y) \geq \beta + 1$; by Lemma 3.3, D is hamiltonian.

Proposition 5.3 is proved.□

Remark 5.14. *For $\beta \geq k+1$, Theorem* 2.3 *is best possible in some sense because of the following examples* **:**

Example 1 :

Let $D = (X, Y, E)$ where $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2 \cup Y_3$ with $|X_1| = |Y_1| = \beta$, $|X_2| = \beta + k$, $|Y_2| = k+1$, $|Y_3| = \beta - 1$.

In D, there exist all the arcs between X_2 and Y, between X_1 and Y_3 and all the arcs from Y_2 to X_1 (no arc from X_1 to Y_2); $D \in \mathcal{D}(n,\beta,k)$, $|E| = F(n,\beta) - 1$ and D is not hamiltonian (there is no perfect matching from X_1 into Y).

Example 2 :

Same definition than **example 1**, with $|Y_2| = k$, $|Y_3| = \beta$; then $|E| = G(n,\beta) - 1$ and if $x \in X_1$, $y \in Y_3$, there is no hamiltonian path from x to y.

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