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A Proof of the Schauder-Tychonoff Theorem

Una Demostración del Teorema de Schauder-Tijonov

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Abstract

We present a new proof of the version of the Shauder-Tychonov theorem provided by Coppel in *Stability and Asymptotic Behavior of Differential Equations*, Heath Mathematical Monographs, Boston (1965). Our alternative proof mainly relies on the Schauder fixed point theorem.
Key words and phrases: fixed point theorems, discontinuous functions.

Resumen

En este trabajo se presenta una nueva demostración de la versión del teorema del punto fijo de Schauder-Tijonov dada por Coppel en *Stability and Asymptotic Behavior of Differential Equations*, Heath Mathematical Monographs, Boston (1965). Nuestra demostración alternativa se apoya directamente en el teorema del punto fijo de Schauder.
Palabras y frases clave: teoremas de punto fijo, funciones discontinuas.

1 Introduction

What is an elementary proof in Mathematics? The answer depends on its starting point. The proof of the Schauder-Tychonoff theorem given by Dunford-Schwartz [4] could be considered as elemental if we take into account

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that their proof is based on the Zorn lemma. Just five lines. However, the application of this maximal principle requires the existence of an invariant closed subset K_1 of a compact set K which belongs to a locally convex linear topological space with respect to a continuous map $T : K \to K$ [Lemma V.10.4 in [4]]. This starting point could be unlikely considered as an elementary fact. The Schauder-Tychonoff theorem is an important tool in the investigation of the solutions of ordinary differential equations (ODE) on the interval $[0, \infty)$ [3]. This fact explains the need of appropriate versions for concrete applications in ODE. In this course of ideas a proof of this theorem was given by Coppel [3] based on the Brouwer fixed point theorem [5]. The Brouwer theorem is a well known result with proofs of different levels, the simplest one, to our knowledge, is given in [5]. In this sense we dare to say that the proof given by Coppel is elementary.

Perhaps the most important principle of Analysis is the Schauder fixed point theorem, a well known result expounded in the texts of Functional Analysis with an immense amount of applications. We will give an "elementary" proof of the Schauder-Tychonoff theorem relying on that result [7] (see also [9]). Our treatment follows the scope of Coppel [3], that is, we will provide a proof which is useful in applications to ODE.

In order to start, we recall the version of the Schauder theorem that we use in this paper [10].

Theorem 1.. (**Schauder, 1930 [8]**) *Let E be a normed space, $A \subset E$ convex and non-empty, and $C \subset A$ compact. Then every continuous map $T : A \to C$ has at least one fixed point.*

Some notations are necessary. Let $J = [0, \infty)$. We will denote by $\mathbb{E}$ a Banach space with norm $|\cdot|_{\mathbb{E}}$. We also consider the set $C(J, \mathbb{E}) := \{f : J \to \mathbb{E} \mid f \text{ is continuous}\}$, which in the case $\mathbb{E} = \mathbb{R}$, will be shortly denoted by $C(J)$.

Definition 1.. *A family $\mathcal{F}$ in $C(J, \mathbb{E})$ is equicontinuous at $t \in J$ if given $\varepsilon > 0$, there exists $\delta = \delta(t, \varepsilon) > 0$ such that, if $t, s \in J$ satisfy $|t - s| < \delta$, then $|f(t) - f(s)| < \varepsilon$ for all $f \in \mathcal{F}$.*

In 1935, the Soviet mathematician A. Tychonoff gave a generalization of the Schauder theorem for locally convex vector spaces [12]. This result is usually termed the Schauder-Tychonoff theorem.

Theorem 2.. (**Coppel, 1965 [3]**) *Let F be the set formed by those functions $x \in C(J, \mathbb{R}^n)$ such that*

$$|x(t)| \leq \mu(t) \quad \textit{for all } t \in J,$$

where $\mu(t)$ is a fixed positive continuous function. Let $T : F \to F$ satisfy the following properties

(**i**) *T is continuous, in the sense that if $x_n \in F$ $(n = 1, 2, \ldots)$ and $x_n \to x$ uniformly on every compact subinterval of J, then $T[x_n] \to T[x]$ uniformly on every compact subinterval of J.*

(**ii**) *The functions in the image set $T[F]$ are equicontinuous and bounded at every point of J.*

Then the map T has at least one fixed point in F.

In this paper we provide the following:

[R1] A new proof of theorem 2 based on the Schauder theorem.

[R2] We show that μ might be a discontinuous function. In addition, we replace the requirement on functions μ by a new condition that can be easily verified.

[R3] In the statement of the theorem 2, the hypothesis "Let F be the set formed by those functions $x \in C(J, \mathbb{R}^n)$" can be replaced by a more general one.

[R4] We give a version of the Schauder-Tychonoff theorem for a space of discontinuous functions.

2 The Schauder-Tychonoff theorem

For each $n = 1, 2, 3, \ldots$, we define the function

$$\varphi_n(t) = \begin{cases} 1, & \text{if } t \in [0, n]; \\ n + 1 - t, & \text{if } t \in (n, n+1]; \\ 0, & \text{if } t \in (n+1, \infty). \end{cases}$$

We denote $\mathcal{S} = \{\varphi_n(t) : n = 1, 2, 3, \ldots\}$.

Definition 2. *We shall say that a set $F \subset C(J, \mathbb{E})$ has the property* (CS) *iff*

$$f \in F,\ \varphi \in \mathcal{S} \text{ imply } \varphi f \in F.$$

The class of spaces with the property (CS) comprises functional spaces which are of interest in ODE theory: $B(J)$, the space of bounded functions on J; $B(\infty)$, the space of functions with limit at $t = \infty$; $L^p(J)$, $1 \leq p \leq \infty$, the Lebesgue space of integrable functions on J; $BV(J)$, the space of functions of bounded variation on J, etc. On the other hand, important spaces used in analysis do not fulfill the (CS) property, for example the space of almost periodic functions, or even simpler, the space of periodic functions on the real line.

For a bounded function $f : J \to \mathbb{E}$ we define the number

$$|f|_\infty = \sup_{t \in J} |f(t)|_{\mathbb{E}}.$$

We will denote by $BC(J, \mathbb{E})$ the space of bounded functions in $C(J, \mathbb{E})$. The space $BC(J, \mathbb{E})$ equipped with the norm $|\cdot|_\infty$ is a Banach space.

Theorem 3.. *Let F be a non-empty and convex subset of $C(J, \mathbb{E})$ with the following properties:*

[A] *The set F has the property (CS).*

[B] *The set F is closed in the following sense: if $f_n \in F$, $n = 1, 2, \ldots$ and $f_n \to f$ uniformly on every compact subinterval of J, then $f \in F$.*

Let T be a map from F into F with the following properties:

[$ST1$] $\sup\{|T[f](0)|_{\mathbb{E}} : f \in F\} < \infty$.

[$ST2$] *T is continuous, in the sense that if $f_n \in F$, $n = 1, 2, \ldots$ and $f_n \to f$ uniformly on every compact subinterval of J, then $T[f_n] \to T[f]$ uniformly on every compact subinterval of J.*

[$ST3$] *The family $T[F]$ is equicontinuous at every point of J.*

Then the map T has a least one fixed point in F.

Proof. We begin with a simple remark. Conditions [ST1] and [ST3] imply that for every $r > 0$ the following condition holds

$$\sup_{f \in \mathcal{F},\ t \in [0,r]} |T[f](t)|_{\mathbb{E}} < \infty.$$

The last follows from the Heine-Borel property of the interval $[0, r]$.

To start the proof, let us contemplate the set

$$F_1 = \{f \in F : f(t) = 0, t \geq 2\}.$$

The set F_1 is not empty by property [A]. Moreover, F_1 is a convex set in $BC(J,\mathbb{E})$. Condition [B] implies that F_1 is a closed set in $BC(J,\mathbb{E})$. Let us define $T_1 : F_1 \to F_1$ by

$$T_1[f] = \varphi_1 T[f].$$

Let $\{g_k\}_{k=1}^{\infty}$ be a sequence contained in $T_1[F_1]$. Using conditions [ST1], [ST3], and applying the Ascoli-Arzelá theorem, we see that there exists a subsequence $\{g_{k_i}\}_{i=1}^{\infty}$ of $\{g_k\}_{k=1}^{\infty}$ converging in $BC(J,\mathbb{E})$. From this fact the set $C_1 = \overline{T_1[F_1]}$ ($\overline{A}$ denotes the closure of the set A in the topology of $BC(J,\mathbb{E})$) results to be a compact set in $BC(J,\mathbb{E})$. Therefore, $T_1 : F_1 \to C_1 \subset F_1$. From Theorem 1, there exists $g_1 \in F_1$, a fixed point of the operator T_1: $g_1 = T_1[g_1] = \varphi_1 T[g_1]$, and whence

$$g_1 = T[g_1] \text{ on } [0,1].$$

For $n = 2$, let us define

$$F_2 = \{f \in F : \ f|_{[0,1]} = g_1, \ f(t) = 0, \ t \geq 3\}.$$

The set F_2 is nonempty since $g_1 \in F_2$. Repeating the same arguments as above we verify that F_2 is a convex and closed set in $BC(J,\mathbb{E})$. The operator $T_2 : F_2 \to F_2$ defined by $T_2[f] = \varphi_2 T[f]$ is compact, that is, $C_2 = \overline{T_2[F_2]}$ is compact. Therefore, there exists a $g_2 \in F_2$ such that $g_2 = T_2[g_2] = \varphi_2 T[g_2]$ on J. In particular,

$$g_2 = T[g_2] \text{ on } [0,2] \text{ and } g_1 = g_2 \text{ on } [0,1].$$

This procedure can be repeated by induction to guarantee the existence of a function $g_n \in F_n = \{f \in F : \ f|_{[0,n-1]} = g_{n-1}, f(t) = 0, t \geq n+1\}$, such that $g_n = T[g_n]$ on $[0,n]$. Let us define $f : J \to \mathbb{E}$ by $f(t) = g_n(t)$ if $t \in [0,n]$. Then, condition [B] implies $f \in F$ and $T[f] = f$. $\square$

We have required the more general condition [A] in theorem 2 instead of condition $|x(t)| \leq \mu(t)$, $t \in J$. In order to use the Brouwer fixed point theorem, the proof given by Coppel requires the continuity of the function μ and $\mathbb{E} = \mathbb{R}^n$. Note that the set F defined in theorem 2 satisfies the property (CS). Consequently, theorem 2 is a corollary of theorem 3.

A second remark concerns to the definition of condition (CS) given by means of the sequence $\mathcal{S}$. The functions $\varphi \in \mathcal{S}$ can be replaced by smooth functions, a manoeuvre that could be important in the treatment of spaces of differentiable functions.

3 An example from the theory of epidemics

Let us consider the integral equation

$$x(t) = a(t) + \int_{t-1}^{t} f(s, x(s))ds, \quad t \geq 1 \tag{1}$$

modeling the spread of an infectious disease [1]. In this equation $x(t)$ is the number of infectious individuals of the population at time t and $f(t, x(t))$ models the instantaneous rate of infected individuals per unit of time. The integral $\int_{t-1}^{t} f(s, x(s))ds$ represents the number of infectious individuals within the period $[t-1, t]$. The function $a(t)$ may have different interpretations: people under vaccination, emigrations, etc. This model is frequently used to study the problem of finding a steady-state solution, which can explain, once the infection enters the population, whether the disease persist or dies out.

We are interested in the asymptotic behavior of the solution of equation (1) at $t = \infty$. Problems on the whole real line $\mathbb{R}$ has been considered by many authors [1, 11]. We will assume the history of the model (1) is given on the interval $[0, 1]$ by a function $\varphi : [0, 1] \to J$ which is assumed to be continuous. Thus we are interested in the behavior of the solution to the initial value problem

$$\begin{cases} x(t) = a(t) + \int_{t-1}^{t} f(s, x(s))ds, \quad t \geq 1 \\ x(t) = \varphi(t), \quad t \in [0, 1] \end{cases} \tag{2}$$

at $t = \infty$. In general, problem (2) has no solution. In what follows we will assume that

$$\varphi(1) = a(1) + \int_{0}^{1} f(s, \varphi(s))ds. \tag{3}$$

If the matching condition (3) is satisfied, then the continuous solution of (2) exists on J. This can be achieved by a step by step procedure similar to that of the initial value problems of equations with time lag, but such a method does not give a description of the solutions at $t = \infty$. We will study the problem (2) on the following stage: let $\alpha = (\alpha_n)_{n=1}^{\infty}$ be a sequence of real numbers satisfying $\alpha_n \geq 1$ for all n. We will denote by $\mathbb{E}_\alpha$ the linear space of continuous functions $x : [0, +\infty) \to \mathbb{R}$ satisfying

$$||x||_\alpha = \sup_{n \geq 1} \max_{[n-1,n]} \alpha_n |x(t)| < \infty.$$

It is clear that $(\mathbb{E}_\alpha, ||x||_\alpha)$ is a complete normed space. Further on, we will assume that the function $a(t)$ in the right hand side of equation (1) is contained in $\mathbb{E}_\alpha$. Regarding the function $f : [0, +\infty) \times J \to J$, we will assume its continuity and

$$0 \leq f(t,x) \leq b(t)x^\gamma, \;\; 0 \leq \gamma < 1, \;\; b(t) \geq 0, x \geq 0 \tag{4}$$

where $b(t)$ is continuous. Let

$$\mathcal{F} := \{x : [0, +\infty) \to J | x = \varphi \text{ on } [0,1] \text{ and } ||x||_\alpha \leq \rho\}.$$

It is clear that $\mathcal{F}$ satisfies property [A]. Let us assume that a sequence $(f_n)_{n=1}^\infty \subset \mathcal{F}$ converges uniformly to a function f on every interval $[0,T]$, with $T > 0$. We will show that $f \in \mathcal{F}$. Let $n \in \{1, 2, \ldots\}$. Then

$$\alpha_n |f_k(t)| \leq \rho \text{ if } t \in [n-1, n].$$

If $k \to \infty$ we have

$$\alpha_n |f(t)| \leq \rho \text{ if } t \in [n-1, n].$$

Thus the continuous function $f \in \mathbb{E}_\alpha$. We have proved that conditions [A] and [B] of theorem 3 are fulfilled. Let us now introduce the operator

$$\begin{cases} T[x](t) = a(t) + \displaystyle\int_{t-1}^t f(s, x(s))ds, \;\; t \geq 1 \\ \\ T[x](t) = \varphi(t), \;\; t \in [0,1]. \end{cases} \tag{5}$$

For any $x \in \mathcal{F}$ we have

$$\alpha_n T[x](t) \leq \alpha_n a(t) + \int_{t-1}^t \alpha_n b(s) x^\gamma(s) ds, \;\; t \in [n-1, n], \;\; n \geq 2$$

$$\alpha_1 T[x](t) = \alpha_1 \varphi(t) \leq \alpha_1 \max_{[0,1]} \varphi(t), t \in [0,1].$$

Therefore,

$$\begin{aligned} 0 \leq \max_{[n-1,n]} \alpha_n T[x](t) &\leq ||a||_\alpha + \int_{n-2}^{n-1} \alpha_n b(s) x^\gamma(s) ds + \int_{n-1}^{n} \alpha_n b(s) x^\gamma(s) ds \\ &\leq ||a||_\alpha + \left[\int_{n-2}^{n-1} \alpha_n b(s) ds + \int_{n-1}^{n} \alpha_n b(s) ds \right] ||x||_\alpha^\gamma \\ &\leq ||a||_\alpha + (||b||_{<\alpha>} + ||b||_\alpha) ||x||_\alpha^\gamma, \;\; n \geq 2, \end{aligned}$$

where $<\alpha>$ is the sequence defined by $<\alpha>(n)=\alpha_{n+1}$. Since $0<\gamma<1$, we will have $||T[x]||_\alpha \le \rho$ if $||x||_\alpha \le \rho$, for a large value of ρ. Condition [ST1] is certainly satisfied, and [ST2] follows from the continuity of the function $f(t,x(t))$. From the inequality

$$\left|\int_{t-1}^{t} f(u,x(u))du-\int_{s-1}^{s} f(u,x(u))du\right| \le \int_{s}^{t} |f(u+1,x(u+1))-f(u,x(u))|du$$

and the requirement (4) we obtain that the family $\{T[x]\}_{x\in\mathcal{F}}$ is equicontinuous at every $t\ge 0$. Theorem 3 implies that the initial value problem (5) has a solution x belonging to the space $\mathbb{E}_\alpha$ if $a\in\mathbb{E}_\alpha$ and $b\in\mathbb{E}_\alpha\cap\mathbb{E}_{<\alpha>}$.

4 The case $\mathbb{E}=\mathbb{R}$

The proof of theorem 3 yields another version of the Schauder-Tychonoff theorem which is useful in applications (see the problem of existence of positive periodic solutions for an epidemic model in Arino et al. [1]).

Definition 3.. *A function $\mu : J\to\mathbb{R}$ is called locally bounded iff*

$$\sup\{|\mu(t)| : t\in[0,N]\}<\infty$$

for all $N=1,2,3,\dots$

Corollary 1.. *Let F be a convex subset of $C(J,\mathbb{R})$ with the following properties:*

[A] There exists locally bounded functions $\nu,\mu : J\to J$ such that

$$f\in F,\ t\in J \Longrightarrow \nu(t)\le f(t)\le\mu(t).$$

[B] For some $h\in F$, the set $G := F-h=\{f-h : f\in F\}$ has the property (CS).

[C] The set F is closed in the following sense: if $f_n\in F$, $n=1,2,\dots$ and $f_n\to f$ uniformly on every compact subinterval of J, then $f\in F$.

Additionally to the previous, let us consider a map T from F into F with the following properties:

[ST22] T is continuous, in the sense that if $f_n\in F$, $n=1,2,\dots$ and $f_n\to f$ uniformly on every compact subinterval of J, then $T[f_n]\to T[f]$ uniformly on every compact subinterval of J.

[$ST33$] *The family $T[F]$ is equicontinuous at every point of J.*

Then the map T has a least one fixed point in F.

Proof. We sketch the proof. We have $\nu - h \leq 0$, $\mu - h \geq 0$, and any $g \in G$ satisfies

$$\nu - h \leq g \leq \mu - h.$$

Further, we replace the operator T by $\tilde{T} : G \to G$, where $\tilde{T}(g) := T(f) - h$, if $f \in F$ and $g = f - h$. The family G satisfies conditions [A] and [B] of theorem 3. Since $\tilde{T} : G \to G$, then $g \in G$ implies $\nu - h \leq \tilde{T}[g] \leq \mu - h$. This means that condition [ST1] of theorem 3 is fulfilled. Conditions [ST2] and [ST3] for the operator $\tilde{T}$ follows from [ST22] and [ST33]. Hence, from theorem 3 we obtain a point $g \in G$ such that $g = \tilde{T}(g)$. If $g = f - h$ we obtain $f - h = \tilde{T}(f - h) = T(f) - h$ implying $f = T(f)$, $f \in F$. □

5 Discontinuous functions

An application concerns to the theory of impulsive equations, where solutions with instantaneous jumps are used [2]. The following terminology is pertinent in that context. A function $f : J \to \mathbb{E}$ will be termed right continuous iff for every $t \in J$ we have that $\lim_{h\to 0^-} f(t+h)$ exists and $f(t) = \lim_{h\to 0^+} f(t+h)$. Throughout we will write $f(t^-) = \lim_{h\to 0^-} f(t+h)$. We will denote by $C_+(J,\mathbb{E})$ the space of right continuous functions $f : J \to \mathbb{E}$. The space $BC_+(J,\mathbb{E}) = \{f \in C_+(J,\mathbb{E}) : f \text{ is bounded}\}$, equipped with the norm $|f|_\infty = \sup\{|f(t)|_{\mathbb{E}} : t \in J\}$, is a complete normed space. We will say that a set of functions $\mathcal{F} \subset C_+(J,\mathbb{E})$ has the property $(\mathrm{CS})_+$ iff

$$f \in \mathcal{F}, \text{ and } \varphi \in \mathcal{S} \text{ imply } \varphi f \in \mathcal{F}.$$

For each $f \in C_+(J,\mathbb{E})$ we define $\mathcal{D}(f) := \{t \in J : f \text{ is discontinuous at } t\}$. For every $\sigma > 0$ we will denote by $C_+^\sigma(I,\mathbb{E})$ the subset of $BC_+(J,\mathbb{E})$ such that if $f \in C_+^\sigma(J,\mathbb{E})$ and $t_1, t_2 \in \mathcal{D}(f)$, then $|t_1 - t_2| \geq \sigma$. The metric space $(C_+^\sigma(J,\mathbb{E}), d)$, where $d(f,g) = |f - g|_\infty$, is complete.

Definition 4. (See [6].) *A family $\mathcal{F} \subset C_+^\sigma(J,\mathbb{E})$ is right equicontinuous iff for any $\varepsilon > 0$ and any $t \in J$, there exists a $\delta = \delta(t,\varepsilon) > 0$ such that*

(i) For all $f \in \mathcal{F}$ and $s \in [t, t+\delta) \cap J$ we have $|f(t) - f(s)|_{\mathbb{E}} < \varepsilon$.

(ii) For all $f \in \mathcal{F}$ and $s \in (t-\delta, t) \cap J$ we have $|f(t^-) - f(s)|_{\mathbb{E}} < \varepsilon$.

Meneses and Naulin [6] have obtained the following Ascoli-Arzelá theorem for the set $C_+^\sigma(I,\mathbb{E}) := \{f : I \to \mathbb{R} | f$ is right continuous on $I\}$, where $I = [a,b]$.

Theorem 4.. *Let $\mathcal{A} \subset C_+^\sigma(I,\mathbb{E})$ be a bounded family:*

$$\sup\{|f|_\infty : f \in \mathcal{A}\} < \infty.$$

If $\mathcal{A}$ is right equicontinuous, then from every sequence $\{x_n\} \subset \mathcal{A}$ it is possible to extract a subsequence $\{x_{n_i}\}$ converging uniformly on the interval I.

Following the tracks of the proof of theorem 3 we get the following Schauder-Tychonoff theorem for discontinuous functions.

Theorem 5.. *Let $\mathcal{F}$ be a non empty and convex set of functions in $C_+(J,\mathbb{E})$ with the properties*

[A] *The set $\mathcal{F}$ has the property* $(\mathrm{CS})_+$.

[B] *The set $\mathcal{F}$ is closed in the following sense: if $f_n \in F$, $n = 1,2,\ldots$ and $f_n \to f$ uniformly on every compact subinterval of J, then $f \in F$.*

Let also T be a map from $\mathcal{F}$ into $\mathcal{F}$ with the following properties:

[$ST1$] *There exists a $\sigma > 0$ such that $T[\mathcal{F}] \subset C_+^\sigma(J,\mathbb{E})$.*

[$ST2$] $\sup\{|T[f]|_\infty : f \in \mathcal{F}\} < \infty$.

[$ST3$] *T is continuous, in the sense that if $f_n \in \mathcal{F}$, $n = 1,2,\ldots$ and $f_n \to f$ uniformly on every compact subinterval of J, then $T[f_n] \to T[f]$ uniformly on every compact subinterval of J.*

[$ST4$] *The family $T[\mathcal{F}]$ is right equicontinuous at every point of J.*

Then the map T has a least one fixed point in $\mathcal{F}$.

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