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Maximum line-free set geometry in $\mathbb{Z}_3^d$

Geometría del máximo conjunto libre de líneas en $\mathbb{Z}_3^d$

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Abstract

Let $\mathbb{Z}_3^d$ be the affine space of dimension d on the $\mathbb{Z}_3$ field. The main goal of this paper is the geometric representation of the maximum line-free set E in $\mathbb{Z}_3^d$. Therefore the complement T of E is the minimum size set in $\mathbb{Z}_3^d$ intersecting all the affine lines in $\mathbb{Z}_3^d$. We describe, from an affine point of view, the geometric structure for $d = 2, 3$ of set E producing the maximal cardinality. The case $d = 4$ is partly obtained from these structures.
Key words and phrases: affine space; line; plane; hyperplane; line-free; zero-sum.

Resumen

Sea $\mathbb{Z}_3^d$ el espacio afín de dimensión d sobre el cuerpo $\mathbb{Z}_3$. El principal objetivo de este artículo es la representación geométrica del conjunto E de máxima cardinalidad sin líneas en $\mathbb{Z}_3^d$. En consecuencia el complemento T de E es un conjunto en $\mathbb{Z}_3^d$ de mínima cardinalidad el cual

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intercepta todas las líneas afines en $\mathbb{Z}_3^d$. Describimos desde una perspectiva afín, las estructuras geométricas de E para $d = 2, 3$ and 4.
Palabras y frases clave: espacio afín; línea; plano; hiperplano; suma-cero.

1 Introduction

Let $\mathbb{Z}_3^d$ be the affine space of dimension d over the field $\mathbb{Z}_3$. In [1] Cruthirds, Mattics, Isaacs and Quinn show that the maximum sizes of line-free sets in $\mathbb{Z}_3^2$ and $\mathbb{Z}_3^3$ are 4 and 9. Moreover they claim that there is a line in any 21 points of $\mathbb{Z}_3^4$. In [5] Pellegrino shows that 20 is the size of the maximum line-free set in $\mathbb{Z}_3^4$. We have taken a different approach in the sense that we specify the geometric structure of the maximum line-free set E that produces $d = 2, 3$ and 4. The size of the maximum line-free set is 45 for $d = 5$ and it was discovered a few years ago [3]. Characterize the maximum line-free sets in $\mathbb{Z}_3^d$ is an open problem that has received a lot of attention over the years. The article [2] by Davis and Maclagan covers some of these issues and contains other useful references. The context of our work are the sequences with zero-sum. Notice that a 3-subset in $\mathbb{Z}_3^d$ has zero-sum if and only if it is a line of $\mathbb{Z}_3^d$. The Erdős-Ginzburg-Ziv theorem [4] states that any $(2n-1)$-sequence in an abelian group of order n, contains an $n-$subsequence with zero-sum. In a geometric formulation this theorem is equivalent to state that every $2n - 1$ points of one-dimensional lattice contain a subsequence of n points, which has its center of gravity on the lattice.

Our main goal is to show that:

- the number of line-free points in $\mathbb{Z}_q^3$ ($q \geq 3$ prime power) is at least $(q-1)^3 + 1$.
- the maximum number of line-free points in $\mathbb{Z}_3^2$ is 4. Moreover a set of 4 points in $\mathbb{Z}_3^2$ is line-free if and only if they form a parallelogram.
- the maximum number of line-free points in $\mathbb{Z}_3^3$ is 9. Additionally 9 points in $\mathbb{Z}_3^3$ are line-free if and only if they are distributed on a cube: 4 points are placed in the vertices of one face. Other 4 points are placed in the middle points of the sides of its parallel face. Finally the last point is placed in the center of the cube. This distribution will be denoted by the following figures A, B, C in $\mathbb{Z}_3^2$:

$A = \begin{array}{ccc} \circ & \cdot & \circ \\ \cdot & \cdot & \cdot \\ \circ & \cdot & \circ \end{array}$, $B = \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot \\ \cdot & \cdot & \cdot \end{array}$, $C = \begin{array}{ccc} \cdot & \circ & \cdot \\ \circ & \cdot & \circ \\ \cdot & \circ & \cdot \end{array}$, showing the positions of sections of E by parallel planes, with respect to the canvas $\begin{array}{ccc} 22 & 02 & 12 \\ 20 & 00 & 10 \\ 21 & 01 & 11 \end{array}$, where the points inside and outside the figure are represented by $\circ$ and by $\cdot$ respectively.

- In $\mathbb{Z}_3^4 = \mathbb{Z}_3^2 \times \mathbb{Z}_3^2$, it is possible to distribute on 3^2 parallel planes (whose position is given by a 2-vector with components 0, 1 or 2) 20 points in order to build a line-free set E. This set is described by its intersection with each of these planes, where A, B or C are placed. Each three aligned planes (i.e. its position vector constitute a line in $\mathbb{Z}_3^2$) constitute a 3-dimensional affine subspace of $\mathbb{Z}_3^4$, then they must contain at most 9 points. The maximum number of line-free points in $\mathbb{Z}_3^4$ is 20. Moreover we conjecture that the above distribution is the only way to build a line-free set on 20 points in $\mathbb{Z}_3^4$.

In what follows, some elements of affine geometry used in this work are described.

1.1 Elements of affine geometry and orbits

Let F_q^d be an affine space of dimension $d \geq 1$ on a finite field with q elements F_q. Each line in F_q^d has q points and the total number of points of F_q^d is q^d. Moreover each point lies on $\frac{q^d-1}{q-1}$ lines, hence the affine space F_q^d contains $q^{d-1}\frac{q^d-1}{q-1}$ lines. Finally any affine space of dimension e with $0 \leq e \leq d-1$ lies on $\frac{q^{d-e}-1}{q-1}$ affine spaces of dimension $e+1$. Let $a, b \in \mathbb{Z}_3^d$, we denote by $m(a,b)$ the middle point of a and b and $b-a$ the vector v such that the translate of a by v is b.

Definition 1. *A* 4*-subset* $\{p_1, p_2, p_3, p_4\}$ *of* $\mathbb{Z}_3^d$ *(with* $d \geq 2$*) is a* parallelogram *if there exists an ordering of these points such that one of the equivalent equalities holds:*

- $m(p_1, p_3) = m(p_2, p_4)$
- $p_2 - p_1 = p_3 - p_4$
- $p_4 - p_1 = p_3 - p_2$

Then the pairs of points $\{p_1, p_3\}$ and $\{p_2, p_4\}$ are the diagonals and the other pairs the sides of the parallelogram.

We shall use the following theorem:

Theorem 1. *Let G be a group which acts on a set X, and let x be any chosen elements on X. Then we have the equation:*

$$|\theta(x)| \times |G_x| = |G|$$

where $\theta(x) = \{gx : g \in G\}$ is the orbit of x and $G_x = \{g \in G : gx = x\}$ is the subgroup of G called the stabilizer of x.

Besides this introduction, this paper contains two main sections. Section 2 discusses general results on the field F_q. In section 3, the structure of the maximum line-free set in $\mathbb{Z}_3^d$, $d = 2, 3, 4$ is given.

2 The problem in F_q^d

Theorem 2. *Let $f_q(d)$ be the maximum cardinality of the line-free subsets in F_q^d. Then we have $f_q(d+e) \geq f_q(d)f_q(e)$.*

Proof. It is sufficient to consider line-free subsets C and D in F_q^d and F_q^e respectively. Their cartesian product $C \times D$ is also line-free in $F_q^d \times F_q^e$ that is isomorphic to F_q^{e+d}. □

Theorem 3. *The expression $\sqrt[d]{f_q(d)}$ has a limit l_q when $d \to \infty$. Moreover $q - 1 \leq l_q \leq q$.*

Proof. Set $u_d = \sqrt[d]{f_q(d)}$. Since $q - 1 \leq u_d < q$ we know that u_d has a superior limit and an inferior limit. We have for $n \geq 1$ and $0 \leq a \leq b$ the inequality $f_q(nb + a) \geq f_q(b)^n f_q(a)$. Hence $u_{nb+a} \geq u_b^{\frac{nb}{nb+a}} \geq u_b^{\frac{n}{n+1}}$ and then $\liminf(u_d) \geq u_b$. Therefore $\liminf(u_d) \geq \limsup(u_d)$ so that the sequence u_d has a limit. Finally since $f_q(1) = q - 1$ then $q - 1 \leq l_q$. The inequality $l_q \leq q$ is obvious. □

Theorem 4. *Let $q \geq 3$ be a prime power. Then $f_q(3) \geq (q - 1)^3 + 1$.*

Proof. Consider F_q^3 as a product $G \times D$ with G the affine plane on F_q and D the affine line on F_q; both are endowed with a point 0.

Let T be the subset of $G \times D$ defined by the union of $G \setminus \{0\} \times 0$ (the horizontal plane with the point 0 removed) and the $q - 1$ sets (unions of two lines in the "horizontal" plane $G \times \{a_i\}$, meeting at $(0, a_i)$) defined as $(B_i \bigcup C_i) \times \{a_i\}$, where the a_i's are all the elements of $D \setminus \{0\}$ and (B_i, C_i)'s are pairs of lines of G, having distinct directions and passing through 0, such that all $q+1$ directions are obtained at least once (this can be done since $q + 1 \leq 2(q-1)$ for $q \geq 3$).

We will show that this set T intersects all the lines in $G \times D$; therefore its complement is line-free. Since T has $(q^2 - 1) + (q-1)(2q-1) = 3q^2 - 3q$ elements, its complement has $(q-1)^3 + 1$.

First, the horizontal lines. Each line of $G \times \{0\}$ contains q or $q-1$ points of E and each line in $G \times \{a\}$ for $a \neq 0$ intersects 1 or 2 points of T.

Then the non-horizontal lines meet the horizontal plane $G \times \{0\}$. If the intersection is not $(0,0)$, it is a point of T. Otherwise, the line belongs to a vertical plane $P = F \times D$, with F a line of G going through 0.

Now such a plane P intersects T along the vertical line $L_1 = \{0\} \times D$ without $(0,0)$, the horizontal line $F \times \{0\}$ without $(0,0)$ and along at least another horizontal line. Hence every line in it intersects T. □

Notice that in $\mathbb{Z}_2^d$ any set with 2 points is a line. Then the minimal cardinality of a set intersecting all the affine lines in $\mathbb{Z}_2^d$ is $2^d - 1$.

3 The problem in $\mathbb{Z}_3^d$ for $d = 2, 3, 4$

Theorem 5. *In $\mathbb{Z}_3^2$ a set of 4 points is line-free if and only if it is a parallelogram. Moreover the maximum number of line-free points in $\mathbb{Z}_3^2$ is 4.*

Proof. Let $\{a, b, c\}$ be a line-free set of 3 points in $\mathbb{Z}_3^2$. There is, up to affine isomorphism, only one way of placing them in $\mathbb{Z}_3^2$. That is to say, if $\{a', b', c'\}$ is a line-free set on 3 points, there exists a bijective affine application mapping a to a', b to b' and c to c'. Therefore the other 6 points of the 9 points constituting $\mathbb{Z}_3^2$ can be split in two categories:

- the middle points $m(a, b)$, $m(b, c)$ and $m(c, a)$, which are not suitable because they form lines with a, b, c.
- each one of the other 3 points added to $\{a, b, c\}$ builds a parallelogram, that is of course line-free.

Hence, each line-free set on 4 points is a parallelogram.

It is now easy to see that each one of the 3 remaining points is either the middle point of one of the 4 sides of the parallelogram or the common middle point of the two diagonals. This observation proves that there is no line-free set on 5 points in $\mathbb{Z}_3^2$. □

Theorem 6. *The maximum line-free set E in $\mathbb{Z}_3^3$ is 9. Moreover, a set of 9 points $(x, y, z)$is line-free if and only if they are distributed, up to affine isomorphism, as follows:* 222, 212, 112, 122, 021, 201, 101, 011, 000. *These points are distributed on three parallel planes $\Pi_{z=2}, \Pi_{z=0}, \Pi_{z=1}$, such that $E \cap \Pi_{z=2} = A, E \cap \Pi_{z=0} = B, E \cap \Pi_{z=1} = C$.*

Proof. It is easy to check that this set is line-free.

Now let us prove its unicity and maximality (up to the action of the affine group).

Let $E \subseteq \mathbb{Z}_3^3$ with $|E| \geq 2$ and E line-free. Let L be a line with 2 points of E. If each one of the 4 planes through L has at most 3 points of E, then E has at most $4(3-2)+2 = 6$ points.

Now assume $|E| \geq 8$ and E line-free. There exists at least a line L having two points of E and for each such line there exist at least two planes P such that $E \cap P$ has 4 points, and thus is a parallelogram.

Choosing now L as a side of a parallelogram, two cases appear:

- L is a side of the parallelograms $P_1 \cap E$ and $P_2 \cap E$.

  By an appropriate choice of origin and basis in $\mathbb{Z}_3^3$, we can assume that the 6 points of E in P_1 and P_2 have coordinates 021, 011 (points of L), 222, 212, 122, 112. The points that are not yet forbidden are 020, 000, 010, 221, 201, 211, 121, 101, 111. Surely 0 or 1 points among 010, 211, 111 are in E, since at most 4 points of E have 1 as second coordinate.

  - If one of these 3 points is in E, we may assume by symmetry that it is 010. Then 201, 101 are also forbidden.

    If 000 is in E, then 020, 221, 121 are also forbidden. We have then 8 points only in E.

    If 000 is not in E, then E does not meet the plane $y = 0$, it has at most 8 points.

  - Now, we assume that none of these 3 points is in E; we can assume by symmetry that also none of 020, 221, 121 is in E, and then E may contain the 3 points 000, 201, 101. The set obtained in this way has thus 9 points.

- L is a side of the parallelogram $P_1 \cap E$ and a diagonal of $P_2 \cap E$. Then we may assume that the points are 021, 011 (points of L), 122, 112, 101, 201.

  Then the following points are not yet forbidden 222, 202, 212, 100, 022, 002, 012, 000.

Since we do not want to have again a line to be a side of two parallelograms, we are left with only 202, 002, 000, 100. Moreover, the vertices in the pairs $(000, 202)$, $(000, 100)$, $(100, 002)$, $(202, 002)$ cannot be used together without creating two parallelograms with a common side. The pair $(000, 202)$ is aligned with 101 and the pair $(100, 002)$ is aligned with 201.

So this leads to no new line-free configuration with at least 8 points. We can now conclude that only a configuration, up to the action of the affine group, is line-free and has 9 points.

□

In the following remark, we conclude that the affine bijections preserving that configuration, denoted by E, give 1 orbit on it. Therefore each one of these 9 line-free points in E play the same role. Moreover we show that G has at least 144 elements. Hence there exist 144/9 affine bijections preserving E and fixing any chosen element of E.

Remark 1. *In Theorem 6, the 6 points which are a bijection with 6 points of the given 9 line-free points, are not whatever. They constitute the two parallelograms $P_1 \cap E$ and $P_2 \cap E$ having L as a common side. The origin and the base taken in the geometric reasoning in this theorem are the following: the origin is a point of the side L and the base is constituted by the three sides of the two parallelograms passing through the selected origin, being one of them the common side L.*

Now we show that the group G of affine bijections of $\mathbb{Z}_3^3$ preserving E has at least 144 elements. This group G acts on E and on the complete space.

Consider the following two affine applications on E:

- $x, y, z \Longrightarrow 2x + 2y, x + 2y, 2z$ *of order 8, preserves E, leaving point* 000 *fixed and cyclically permuting the other 8 points.*
- *the one that sends points* 000 *on* 222, 011 *on* 011, 201 *on* 000, 021 *on* 101, *preserves also E.*

The first application forms two orbits, one of which is the point 000 *and the other contains the other 8 elements in E. The second applications send* 000 *on the other orbit.*

According to the above reasoning, we observe that the complete group of affine bijections preserving E *defined only one orbit on* E*. Therefore by Theorem 1, since* E *has nine elements, the number of elements in the group* G *is nine times that of* $H = G_{000}$*, subgroup of* G *preserving* 000*. This group* H *forms only one orbit on the elements of* $E \setminus 000$*. Hence by Theorem 1, its number of elements is 8 times that of* $K = H_{011}$*, subgroup of* G *preserving* E *and fixing* 000 *and* 011 *(*011 *was chosen arbitrarily in* $E \setminus 000$*).*

This group K*, being formed of affine bijections, will send the planes passing through the line defined by* 000 *and* 011*. Since it preserves* E*, it will send the intersections of the superposed planes with* E*. Now, these intersections are* 000 011 101 112*,* 000 011 201 212 *,* 000 011 021 *,* 000 011 122 222*.*

It is observed that the third plane will not change, since it has only 3 points of E *and it is the only one. Moreover, point* 021 *is fixed by* K*. The fourth plane will not change, since* $\{000, 101\}$ *is a diagonal of the parallelogram on this fourth plane, and a side of the parallelogram for the two first plans.*

In consequence, an element k *of group* K *can exchange points* 122 *and* 222 *or can leave them fixed. In the second case, it fixes (at least) 4 non coplanar points, hence it is the identity (because* k *is affine). If* k *exchange these two points, we can observe that it is the affine symmetry with respect to the plane defined by the 3 points* 000 011 021 *that changes* 122 *and* 222*, because there is only one of such affine symmetry, and* k *coincides with this symmetry on (at least) 4 non coplanar points.*

Hence, there are only 2 elements in K*: the identity and one symmetry, so that* H *has at least* $8 \times 2 = 16$ *elements and* G *has at least* $9 \times 16 = 144$ *elements.*

Remark 2. *From Theorem 6 and more precisely the fact that* G *has only one orbit, we can conclude that there exist three line-free figures of 8 points, up to affine isomorphism: the one that can be extended to a line-free figure on 9 points* A, B, C *and the following two non isomorphic figures, with 8 line-free points and that cannot be extended to 9 line-free points:*

- *the usual cube (up to affine isomorphism)* $A \, \emptyset \, A$*.*
- *the cube with a shifted edge (up to affine isomorphism)* $A \, \emptyset A'$ *where*

$$A' = \begin{array}{ccc} \circ & \cdot & \circ \\ \cdot & \cdot & \circ \\ \circ & \cdot & \cdot \end{array},$$

Theorem 7. *The maximum number of line-free points* (x, y, z, w) *in* $\mathbb{Z}_3^4$ *is 20 and a way to build a line-free set* E *with 20 points in* $\mathbb{Z}_3^4$ *assimilated to* $\mathbb{Z}_3^2 \times \mathbb{Z}_3^2$ *is the following:*

$$\begin{array}{ccc} A & B & C \\ B & \emptyset & B \\ C & B & A \end{array}$$

where the 9 symbols show the intersections of E *with the 9 planes* $\mathbb{Z}_3^2 \times (z, w)$*. For example ,* $E \cap \mathbb{Z}_3^2 \times (2, 2) = A$*,* $E \cap \mathbb{Z}_3^2 \times (0, 0) = \emptyset$*.*

Proof. Let $E \subseteq \mathbb{Z}_3^4$ with $|E| > 2$ line-free. Let L be a line with 2 points of E. If every plane through L has at most 3 points of E, then the 13 planes through L show that E has at most 15 points.

Thus we assume $|E| \geq 16$. There exists a plane P containing at least 4 points of E. By Theorem 6 the four 3-dimensional affine subspaces containing P have at most 9 points, then E has at most $4 + (9 - 4)4 = 24$ points. Hence $|E| \leq 24$.

At most 4 of the 9 parallel planes to P have 4 points of E. Assuming, E contains 5 parallel planes, say $\Pi_{z=a_i, w=b_i}$, $1 \leq i \leq 5$. By Theorem 5 in the subset $\{(a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5)\}$ of $\mathbb{Z}_3^2$ there is a line L, for example $L = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$. It is easy to show that the 3 planes $\Pi_{z=a_i, w=b_i}$, $1 \leq i \leq 3$ constitute a 3-dimensional affine subspace with 12 points of E. Therefore by Theorem 6 there exists a line in E. Contradiction.

Now we consider the following four cases according to the number of parallel planes to P containing 4 points of E.

4 planes. By Theorems 5 and 6, they must constitute a parallelogram where the vertices are of type A or type C. By a simple inspection according to the way of combining A and C, we can see that the maximal cardinality is 20, obtained in the following way: $\begin{array}{ccc} A & B & C \\ B & \emptyset & B \\ C & B & A \end{array}$

The expressions X, Y, Z, in the following two cases, represent sets with at most 2 points of $\mathbb{Z}_3^2$. Moreover, owing to the affine bijection that sends A to C, we can reduce the types of feasible planes with 2 points of E to H and V and their translates where $H = \begin{array}{ccc} \cdot & \cdot & \cdot \\ \circ & \cdot & \circ \\ \cdot & \cdot & \cdot \end{array}$ and $V = \begin{array}{ccc} \cdot & \circ & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot \end{array}$

To avoid lines in E, three aligned planes must satisfy:

- if $P \cap E$ is C the intersections of the two other ones with E must be a translate of V and a translate of H.
- if $P \cap E$ is A the intersections of the two other ones with E must be two translates of V or two translates of H.

3 planes. In this case no distribution is possible with 20 points. Indeed, by the following inspection we observe that it is not possible to have a line-free set on 20 points in $\mathbb{Z}_3^4$ with the distribution: $\begin{matrix} A & B & C \\ B & \emptyset & X_1 \\ C & Y_2 & Z_2 \end{matrix}$. Notice that the presence of two C forces $\emptyset$ in the matrix central position. If Y_2 is a translate of V then the presence of C forces that Z_2 is a translate of H and X_1 is a translate of V. But a forbidden CVB appears then as BX_1C. If Y_2 is a translate of H then the presence of C forces that Z_2 is a translate of V and X_1 is a translate of H. But a forbidden CHB appears then as BX_1C.

2 planes. Let us consider the possibility of having a line-free set on 21 line-free points in $\mathbb{Z}_3^4$ with the following distribution $\begin{matrix} A & B & C \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{matrix}$. If X_1 is a translate of H then the presence of A forces that X_2 is a translate of H. Then the presence of C and X_i's forces Y_1 and Y_2 to be translates of V, then the presence of A and the Y_i's implies that the Z_i's have to be translates of V. But a forbidden CVV appears then as CZ_1Z_2.

Starting with X_1 a translate of V, the same argument leads also to forbid CHH to be the third column. Therefore, this case excludes line-free sets with more than 20 points.

1 plane. Since there exists only one plane P with 4 points among the planes with the same direction, then by Theorem 6 the four 3-dimensional affine subspaces containing P have at most 8 points. Therefore E has at most 20 points.

□

Conjecture 1. *The only way, from an affine point of view, to build a line-free set on 20 points in $\mathbb{Z}_3^4$ is:* $\begin{matrix} A & B & C \\ B & \emptyset & B \\ C & B & A \end{matrix}$

Problem 1. *Try to use a similar idea of Theorem 7 to distribute geometrically* 45 *line-free points in* $\mathbb{Z}_3^5$. *In general, the characterization of the geometric structure from an affine point of view, of the maximum line-free set in* $\mathbb{Z}_3^d$, $d \geq 5$, *is an open problem.*

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