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NON-ISOMETRIC FUNCTIONAL CALCULUS FOR PAIRS OF COMMUTING CONTRACTIONS

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Abstract. In this note we study some properties of the class $C_0^{(2)}$ (introduced in [O1]) of pairs of commuting contractions (S,T) for which there exist a bounded analytic function h on the bidisk such that $h(S,T)=0$. We try to reproduce the basic results of the well-known C_0 theory for contractions (cf. [B]) and point at some of the difficulties in doing so. We define the annihilating ideal of a pair of non-unitary commuting contractions $(S,T)\in C_0^{(2)}$ as the collections of all functions h analytic on the bidisk for which $h(S,T)=0$. We prove some theorems relating algebraic properties of the annihilating ideal of a pair of non-unitary commuting contractions with the existence of common invariant subspaces. We pose some open problems and present some examples concerning this theory.

Resumen. En esta nota estudiamos algunas propiedades de la clase $C_0^{(2)}$ (definida en [O1]) compuesta por pares de contracciones (S,T), que conmutan entre si, para las cuales existe una función h analítica en el bidisco tal que $h(S,T)=0$. Tratamos de reproducir los resultados basicos de la conocida Teoría C_0 para contracciones (ver [B]) y destacamos algunas dificultades para lograrlo. Definimos el ideal aniquilador para un par de contracciones conmutantes completamente no unitarias $(S,T)\in C_0^{(2)}$ como la colección de funciones h analíticas en el bidisco para las cuales $h(S,T)=0$. Demostramos algunos resultados relacionando las propiedades algebraicas del ideal aniquilador de un par de contracciones conmutantes completamente no unitarias con la existencia de subespacios invariantes comunes. Planteamos algunos problemas abiertos y presentamos algunos ejemplos relacionados con esta Teoría.

1.- Introduction.

In [O1], see also [OP], we showed the existence of a pair (S,T) of commuting completely nonunitary contractions with a “fairly large” joint spectrum, acting on a separable, complex, infinite dimensional Hilbert space $\mathcal{H}$, such that there is a bounded analytic function h on the bidisk $\mathbb{D}^2$ with $h(S,T)=0$. Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and let $\mathbb{T}=\partial\mathbb{D}$. In this note we further study the class of pairs of operators for which the functional calculus developed in [BDØ] is not an isometry (cf. [O1], [O2], and [OP]). More precisely, we are concerned with those pairs for which the functional calculus has a kernel. In the one variable case, there is an extensive theory for the class $C_0(\mathcal{H})$ of completely nonunitary contractions $T\in\mathcal{L}(\mathcal{H})$ such that there is a function m in the algebra of bounded analytic functions on the disk $H^\infty(\mathbb{D})$ with $m(T)=0$ (cf. [B] for a comprehensive treatise on this subject). We may define the analogous class $C_0^{(2)}(\mathcal{H})$ to consist of those pairs (T_1,T_2) of commuting, completely nonunitary contractions on $\mathcal{H}$ with $\sigma(T_j)\supset\mathbb{T}$, $j=1,2$, such that there is a function h in the class of bounded analytic functions

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX on the bidisk $H^\infty(\mathbb{D}^2)$ with $h(T_1, T_2) = 0$. The assumption on the spectrum of T_j, $j = 1, 2$, is natural since we are interested in finding common invariant subspaces, so having conditions under which each of the members of the pair have (nontrivial) invariant subspaces (cf. [BCP]) will permit us to avoid both extremely hard and trivial situations.

The note is organized as follows: In Section 1 we present some preliminary results and our basic notation. In Section 2 we define and study the annihilating ideal associated with a pair of contractions. In Section 3 we study a common invariant subspace problem. Finally, in Section 4 we make some concluding remarks and pose some open problems. We will assume the reader is familiar with basic results of multivariable operator theory as presented in [C]. The following is Theorem 4.4 of [BDØ] (see also [O2]):

Theorem 1.1. *If S and T are commuting completely nonunitary contractions in $\mathcal{L}(\mathcal{H})$, then there is an algebra homomorphism $\Phi : H^\infty(\mathbb{D}^2) \to \mathcal{L}(\mathcal{H})$ with the following properties:*

1. *$\Phi(1) = I_{\mathcal{H}}$, $\Phi(w_1) = S$, $\Phi(w_2) = T$, where w_1 and w_2 denote the coordinate functions.*
2. *$\|\Phi(h)\| \le \|h\|_\infty$, for all $h \in H^\infty(\mathbb{D}^2)$.*
3. *Φ is weak* continuous. (i.e., continuous when both $H^\infty(\mathbb{D}^2)$ and $\mathcal{L}(\mathcal{H})$ are given the corresponding weak* topologies).*

2.- The annihilating ideal

If $(T_1, T_2) \in C_0^{(2)}(\mathcal{H})$, then

$$J_{T_1,T_2} = \mathrm{Ker}(\Phi_{T_1,T_2}) = \{h \in H^\infty(\mathbb{D}^2) \ : \ h(T_1, T_2) = 0\}$$

is a nontrivial closed ideal in the Banach algebra $H^\infty(\mathbb{T}^2)$, which we shall call the *annihilating ideal* of the pair (T_1, T_2). If I and J are ideals of some ring $\mathbf{R}$, we denote by $I \cdot J$ the set of products $\{mn \ : \ m \in I,\, n \in J\}$.

We now study the relation between the ideal J_{T_1,T_2} (for $(T_1, T_2) \in C_0^{(2)}(\mathcal{H})$) and common invariant subspaces of (T_1, T_2). The following is an extension to the case of two variables of Proposition 6.1 of [NF].

Proposition 2.1. *Let $(T_1, T_2) \in C_0^{(2)}(\mathcal{H})$ and let $\mathcal{H}_1$ be a common invariant subspace of (T_1, T_2). Let*

$$T_1 = \begin{pmatrix} \widetilde{T}_1 & * \\ 0 & \widehat{T}_1 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \widetilde{T}_2 & * \\ 0 & \widehat{T}_2 \end{pmatrix},$$

be the triangularization of T_1 and T_2 corresponding to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ (where $\mathcal{H}_2 = \mathcal{H} \ominus \mathcal{H}_1$). Then $(\widetilde{T}_1, \widetilde{T}_2) \in C_0^{(2)}(\mathcal{H}_1)$, $(\widehat{T}_1, \widehat{T}_2) \in C_0^{(2)}(\mathcal{H}_2)$,

$$\begin{aligned} &J_{\widetilde{T}_1,\widetilde{T}_2} \supseteq J_{T_1,T_2}, \\ &J_{\widehat{T}_1,\widehat{T}_2} \supseteq J_{T_1,T_2}, \ \textit{and} \\ &J_{\widetilde{T}_1,\widetilde{T}_2} \cdot J_{\widehat{T}_1,\widehat{T}_2} \subseteq J_{T_1,T_2}. \end{aligned}$$

Proof. We have, for $n \in \mathbb{N}$,

$$\widetilde{T}_j^n = T_j^n|_{\mathcal{H}_1}, \quad \widehat{T}_j^n = P_{\mathcal{H}_2} T_j^n|_{\mathcal{H}_2}, \quad j = 1, 2.$$

Thus,

$$\begin{aligned} \widetilde{T}_1^n \widetilde{T}_2^m &= T_1^n T_2^m|_{\mathcal{H}_1}, \\ \widehat{T}_1^n \widehat{T}_2^m &= P_{\mathcal{H}_2} T_1^n T_2^m|_{\mathcal{H}_2}, \quad n, m \in \mathbb{N}. \end{aligned} \tag{1}$$

Since T_j is completely nonunitary so are $\widetilde{T}_j$ and $\widehat{T}_j$ $(j = 1, 2)$. By taking weak*-limits of polynomials we can deduce from (1) that for $h \in H^\infty(\mathbb{T}^2)$

$$h(\widetilde{T}_1, \widetilde{T}_2) = h(T_1, T_2)|_{\mathcal{H}_1},$$

and

$$h(\widehat{T}_1, \widehat{T}_2) = P_{\mathcal{H}_2} h(T_1, T_2)|_{\mathcal{H}_2}.$$

Taking $h \in J_{T_1,T_2}$ we see that

$$h(\widetilde{T}_1, \widetilde{T}_2) = 0 \quad \text{and} \quad h(\widehat{T}_1, \widehat{T}_2) = 0.$$

Thus,

$$J_{\widetilde{T}_1,\widetilde{T}_2} \supseteq J_{T_1,T_2} \quad \text{and} \quad J_{\widehat{T}_1,\widehat{T}_2} \supseteq J_{T_1,T_2}.$$

Now take $h_1 \in J_{\widetilde{T}_1,\widetilde{T}_2}$, $h_2 \in J_{\widehat{T}_1,\widehat{T}_2}$, and let $v_1 \in \mathcal{H}_1$. Then

$$\begin{aligned} h_1 h_2(T_1, T_2) v_1 &= h_1 h_2(\widetilde{T}_1, \widetilde{T}_2) v_1 \\ &= h_1(\widetilde{T}_1, \widetilde{T}_2) h_2(\widetilde{T}_1, \widetilde{T}_2) v_1 = 0. \end{aligned}$$

For $v_2 \in \mathcal{H}_2$ we have

$$P_{\mathcal{H}_2} h_2(T_1, T_2) v_2 = h_2(\widehat{T}_1, \widehat{T}_2) v_2 = 0,$$

so $u = h_2(T_1, T_2) v_2$ is orthogonal to $\mathcal{H}_2$, and thus $u \in \mathcal{H}_1$. Hence,

$$h_1 h_2(T_1, T_2) v_2 = h_1(T_1, T_2) u = h_1(\widetilde{T}_1, \widetilde{T}_2) u = 0.$$

Therefore, $h_1 h_2 \in J_{T_1,T_2}$, so $J_{\widetilde{T}_1,\widetilde{T}_2} \cdot J_{\widehat{T}_1,\widehat{T}_2} \subseteq J_{T_1,T_2}$, and the theorem is proved. □

In the single contraction case the annihilating ideal of a $C_0(\mathcal{H})$ operator is generated by an inner function (and is, thus, principal). We now study the structure of the ideal J_{T_1,T_2} for a pair $(T_1, T_2) \in C_0^{(2)}(\mathcal{H})$. Note that if T_1 is a scalar multiple of T_2, then we are actually working in a "one variable" setting. This case will be excluded in most of our results.

Theorem 2.2. *Let $(T_1, T_2) \in C_0^{(2)}(\mathcal{H})$ be a pair of nonzero operators such that T_1 is not a scalar multiple of T_2. Then J_{T_1,T_2} is not maximal in $H^\infty(\mathbb{T}^2)$.*

Proof. Assume on the contrary that J_{T_1,T_2} is maximal. Then by the Gelfand-Mazur Theorem, the Banach algebra $H^\infty(\mathbb{T}^2)/J_{T_1,T_2}$ is isomorphic to the field $\mathbb{C}$ of complex numbers. ¿From the algebra homomorphism $\Phi_{T_1,T_2} : H^\infty(\mathbb{T}^2) \to \mathcal{A}_{T_1,T_2}$, we get an algebra isomorphism

$$\widetilde{\Phi} : \mathbb{C} = H^\infty(\mathbb{T}^2)/J_{T_1,T_2} \to \mathrm{Im}(\Phi)$$

defined by $\widetilde{\Phi}(h + J_{T_1,T_2}) = \Phi_{T_1,T_2}(h)$.

Since for each λ in $\mathbb{C}$ the constant function with value λ is not in J_{T_1,T_2} and since $\Phi(1) = I_{\mathcal{H}}$, we have $\widetilde{\Phi}(\lambda + J_{T_1,T_2}) = \lambda I$. Hence the image of $\widetilde{\Phi}$ is $\{\lambda I \ : \ \lambda \in \mathbb{C}\}$ and thus one-dimensional, which is a contradiction since T_1 is not a scalar multiple of T_2. $\square$

3.- A common invariant subspace

Let $(T_1, T_2) \in C_0^{(2)}(\mathcal{H})$ and assume that neither T_1 nor T_2 is a scalar multiple of the other. By the Theorem 2.2, we can find an ideal J with $H^\infty(\mathbb{T}^2) \supsetneq J \supsetneq J_{T_1,T_2}$. Define a subspace of $\mathcal{H}$ by

$$\mathcal{H}_J = \{v \in \mathcal{H} \ : \ h(T_1, T_2)v = 0,\, h \in J\}.$$

Now $\mathcal{H}_J$ is a (possibly trivial) common invariant subspace for (T_1, T_2), and also $\mathcal{H}_J \neq \mathcal{H}$ (since $J \neq J_{T_1,T_2}$). Can $\mathcal{H}_J = (0)$? Unfortunately the answer is yes, as shown in the following example:

Example 3.1. *Let $T \notin C_0(\mathcal{H})$ be a one-to-one, completely nonunitary contraction with $\sigma(T) \supseteq \mathbb{T}$. Let $g \in H^\infty(\mathbb{D})$ be a map of $\mathbb{D}$ onto itself and assume g is not a scalar multiple of the position function. The pair $(T, g(T)) \in C_0^{(2)}(\mathcal{H})$, since $h(T, g(T)) = 0$ for $h(w_1, w_2) = w_2 - g(w_1)$. Let J be the ideal generated by $J_{T,g(T)}$ and the coordinate function w_1. We have $H^\infty(\mathbb{T}^2) \supsetneq J \supsetneq J_{T,g(T)}$, but $\mathcal{H}_J =$ Ker$(T) = (0)$.* ♦

Note that in the previous example the pair in question has a nontrivial common invariant subspace.

Problem 3.2. *Under what conditions do we have $\mathcal{H}_J \neq (0)$?*

We now further study the structure of J_{T_1,T_2}. The next theorem indicates that, at least with respect to common invariant subspaces, the situation is not so completely desperate.

Proposition 3.3. *Let $(T_1, T_2) \in C_0^{(2)}(\mathcal{H})$. Then either J_{T_1,T_2} is a prime ideal or the pair (T_1, T_2) has a (nontrivial) common hyperinvariant subspace.*

Proof. if J_{T_1,T_2} is not prime we can find $m_1, m_2 \notin J_{T_1,T_2}$ with $m_1 m_2 \in J_{T_1,T_2}$; but then,

$$m_1(T_1, T_2) m_2(T_1, T_2) = 0 \tag{1}$$

and $m_j(T_1,T_2)\neq 0$ $(j=1,2)$. Since $m_1(T_1,T_2)$ and $m_2(T_1,T_2)$ commute, (1) implies that either $\mathrm{Ker}(m_1(T_1,T_2))$ or $\mathrm{Ker}(m_2(T_1,T_2))$ is a nontrivial common hyperinvariant subspace for (T_1,T_2). $\square$

One example in which we can apply the above proposition is the following: take $(T_1,T_2)\in C_0^{(2)}(\mathcal{H})$ and assume that $T_1^n=T_2^n$. Then the function $w_1^n-w_2^n$ belongs to the annihilating ideal of (T_1,T_2). Let $1\neq\xi\in\mathbb{C}$ be an n-th root of unity. Then ξ satisfies

$$\xi^{n-1}+\xi^{n-2}+\cdots+\xi+1=0. \tag{2}$$

By writing, for each $k=0,1,\ldots,n-1$

$$w_1^n-w_2^n=(w_1-\xi^k w_2)(w_1^{n-1}+\xi^k w_1^{n-1}w_2+\cdots+\xi^{k(n-2)}w_1w_2^{n-2}+\xi^{k(n-1)}w_2^{n-1}).$$

We see that either $T_1=\xi^k T_2$, for some $k=0,1,\ldots,n-1$ (in which case the main theorem in [BCP] would imply the existence of a common invariant subspace), or J_{T_1,T_2} is not a prime ideal (in which case Proposition 3.3 would imply the existence of a common invariant subspace), or we have that, for every $k=0,1,\ldots,n-1$,

$$T_1^{n-1}+\xi^k T_1^{n-1}T_2+\cdots+\xi^{k(n-2)}T_1T_2^{n-2}+\xi^{k(n-1)}T_2^{n-1}=0.$$

Adding all these equations and using (2) we get that $nT_1=0$, which is a contradiction. Thus, from the equation $T_1^n=T_2^n$ we can deduce the existence of (nontrivial) common invariant subspaces.

Note that similar situations can be more complicated; for example, if T_1 and T_2 are related by the equation $T_1^2=T_2^3$, then the above argument would fail. The author doesn't know whether we can conclude the existence of common invariant subspaces in this situation.

4.- Some open problems

One consequence of Proposition 3.3 is that if J_{T_1,T_2} is prime, then $\mathcal{H}_J$, constructed above, is trivial. To see this, let $(T_1,T_2)\in C_0^{(2)}(\mathcal{H})$ and assume J_{T_1,T_2} is prime. Let

$$T_j=\begin{pmatrix} S_j & X_j \\ 0 & S'_J\end{pmatrix}$$

be the triangularization of T_j with respect to the decomposition $\mathcal{H}=\mathcal{H}_J\oplus\mathcal{H}'_J$ $(\mathcal{H}'_J=\mathcal{H}\ominus\mathcal{H}_J)$, for some ideal J with $H^\infty(\mathbb{T}^2)\supsetneq J\supsetneq J_{T_1,T_2}$ $(j=1,2)$. By Proposition 4.1.1 we have

$$J_{S_1,S_2}\supseteq J_{T_1,T_2},\quad J_{S'_1,S'_2}\supseteq J_{T_1,T_2},$$

and $J_{S_1,S_2}\cdot J_{S'_1,S'_2}\subseteq J_{T_1,T_2}$. Since J_{T_1,T_2} is prime we must have either $J_{S_1,S_2}=J_{T_1,T_2}$ or $J_{S'_1,S'_2}=J_{T_1,T_2}$. We see this by applying the following standard argument: assume $J_{S'_1,S'_2}\neq J_{T_1,T_2}$. If $h\in J_{S'_1,S'_2}\setminus J_{T_1,T_2}$, then $hJ_{S_1,S_2}\subset J_{T_1,T_2}$ and, thus, for all $m\in J_{S_1,S_2}$ we have $hm\in J_{T_1,T_2}$, which implies $m\in J_{T_1,T_2}$. This proves $J_{S_1,S_2}=J_{T_1,T_2}$. Since $\mathcal{H}_J\neq\mathcal{H}$ we must have $(S'_1,S'_2)=(T_1,T_2)$.

We conclude this note with what we consider to be the main obstacle for a "good" $C_0^{(2)}(\mathcal{H})$ theory. The success of the single contraction $C_0(\mathcal{H})$ theory is due in part to the fact that the annihilating ideal is principal. In our case this does not hold, as the following example shows:

Example 4.1. *Take T to be a nonzero operator in $C_0(\mathcal{H})$ with $\sigma(T) \supset \mathbb{T}$ (cf. [NF, Corollary 5.3]) and let m_T be the minimal (inner) function of T. Consider the pair (T, T). We have that $m_T(w_1)$, $m_T(w_2)$ and $w_1 - w_2$ are all in $J_{T,T}$, but clearly they are independent. Hence, $J_{T,T}$ is not principal.* ♦

Of course, in the above situation we know that common invariant subspace spaces exist (cf. [BCP]). Furthermore, as pointed out before we are working, in this case, in a "one variable" setting. It would be interesting to find a "real" two variable example. It is natural to ask the following questions:

Problem 4.2. *Under what conditions is J_{T_1,T_2} principal?*

Note that while it is known that there is a nonfinitely generated ideal J in $H^\infty(\mathbb{D}^2)$ (c.f., [R], Theorem 4.4.2), there is no guarantee that there exist a pair (T_1, T_2) with $J_{(T_1,T_2)} = J$. We can thus formulate the following problem.

Problem 4.3. *Is J_{T_1,T_2} finitely generated?*

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