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The Hermitian Morita Theorems

Los Teoremas Hermíticos de Morita

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Abstract

Similar to the Morita theorems proved in [1] and the relative version given by Van Oystaeyen and Verschoren in [9], we will prove in this note a (relative) hermitian version of the Morita theorems, i.e., we will describe which equivalences of the category of (relative) sesquilinear, resp. hermitian, modules are determined by a single object and viceversa. A first approach was made in [5], which includes some partial version of the Morita theorems in the hermitian context. As we will show in this note, the techniques developed in [6] permit us to present a complete solution to the problem of generalizing the Morita theorems to the hermitian case.
Key words and phrases: Morita theorems, hermitian forms.

Resumen

En esta nota probamos una versión hermítica (relativa) de los teoremas de Morita, análoga a la dada en [1] y a la versión relativa dada por Van Oystaeyen y Verschoren en [9]. Esto es, describimos qué equivalencias de la categoría de módulos sesquilineales, resp. hermíticos (relativos), están determinadas por un objeto único y viceversa. Una primera aproximación a la solución de este problema aparece ya en [5], en donde se incluye una versión parcial de los Teoremas de Morita en el contexto hermítico. Como demostramos en esta nota, las técnicas desarrolladas en [6] nos han permitido presentar una solución completa al problema

de la generalización, al contexto hermítico, de los Teoremas de Morita.
Palabras y frases clave: teoremas de Morita, formas hermíticas.

1 Generalities.

Throughout this paper, R is a commutative ring with unit and all rings are unitary R-algebras; the letters A,A', ... , will denote such R-algebras. Let us denote the category of left (resp. right) A-modules by A-**mod** (resp. **mod**-A) and the corresponding sets of morphisms by ${}_A[M, N]$ (resp $[M, N]_A$). Bimodules will always be defined over R.

An *algebra with involution* is a couple (A, α), where A is an R-algebra and $\alpha : A \to A$ an R-linear map satisfying $\alpha^2 = 1_A$ and $\alpha(a_1a_2) = \alpha(a_2)\alpha(a_1)$ for every a_1, $a_2 \in A$. We may define with respect to α a construction similar to the usual "restriction of scalars". However, since we use an involution instead of algebra morphisms, we have to switch sides. So, if M is a left (resp. right) A-module, then α induces a right (resp. left) A-module structure on M by putting $m \cdot a = \alpha(a)m$ (resp. $a \cdot m = m\alpha(a)$) for every $m \in M$ and $a \in A$. We denote this module by ${}^{\alpha}M$ (resp M^{α}). If (A', α') is a second R-algebra with involution and if M is an (A, A')-bimodule then the (A', A)-bimodule ${}^{\alpha'}M^{\alpha}$ is defined by putting $a' \cdot m \cdot a = \alpha(a)m\alpha'(a')$ for any $a \in A$, $a' \in A'$ and $m \in M$. If M is an A-bimodule, then we write $M_{\alpha} = {}^{\alpha}M^{\alpha}$.

Any left linear map $f \in {}_A[M, N]$ yields an obvious right linear map $f^{\alpha} \in [M^{\alpha}, N^{\alpha}]_A$. Actually, $(-)^{\alpha}$ and ${}^{\alpha}(-)$ define a category equivalence between A-**mod** and **mod**-A.

(1.1) Let us briefly recollect some definitions and properties of abstract localization. For a more detailed treatment, we refer to [2, 3, 4, 7, 8 et al]. We restrict to left A-modules, right A-modules being treated similarly.

A left exact subfunctor λ of the identity in A-**mod** such that $\lambda(M/\lambda M) = 0$ for any $M \in A$-**mod** will be called a *radical*. Any radical is completely determined by the couple $(\mathcal{T}_{\lambda}, \mathcal{F}_{\lambda})$, where the *torsion class* $\mathcal{T}_{\lambda}$ (resp. the *torsionfree class* $\mathcal{F}_{\lambda}$)) consists of λ-*torsion* (resp. λ-*torsionfree*) left A-modules, i.e. left A-modules M such that $\lambda M = 0$ (resp. $\lambda M = M$). On the other hand, the radical λ is also completely determined by the set $\mathcal{L}_{\lambda}$ of left A-ideals L such that A/L is λ-torsion. We call this set the *Gabriel filter associated to* λ. It is easy to see that $m \in \lambda M$ if and only if there exists some $L \in \mathcal{L}_{\lambda}$ such that $Lm = 0$.

A left A-module E is said to be λ-*injective*, if for any λ-isomorphism $f : M \to N$ in A-**mod**, i.e., a morphism with both λ-torsion kernel and cokernel, and any morphism $g : M \to E$ there exists a morphism $\overline{g} : N \to E$ extending G, i.e., with $g = \overline{g} \circ f$. If this morphism is always unique as such, then E is said to be λ-*closed*. This is also equivalent to E being λ-torsionfree and λ-injective. The full subcategory of A-**mod** consisting of the λ-closed left A-modules will be denoted by (A,λ)-**mod** and it is well known that the inclusion functor

$$i_\lambda : (A,\lambda)\text{-}\mathbf{mod} \hookrightarrow A\text{-}\mathbf{mod}$$

possesses an exact adjoint

$$a_\lambda : A\text{-}\mathbf{mod} \to (A,\lambda)\text{-}\mathbf{mod}$$

(the *reflector* of A-**mod** into (A,λ)-**mod**). The left exact functor

$$Q_\lambda = i_\lambda \circ a_\lambda : A\text{-}\mathbf{mod} \to A\text{-}\mathbf{mod}$$

is called the *localization functor at* λ and may be described in many different ways. For instance let E be an injective hull of $M/\lambda M$, then $Q_\lambda(M)$ consists of those $e \in E$ such that $Le \subseteq M/\lambda M$ for some $L \in \mathcal{L}_\lambda$. So, for any left A-module M, there exists a canonical λ-isomorphism

$$j_\lambda = j_{j,M} : M \to Q_\lambda(M),$$

which is the composition of the canonical morphism $M \to M/\lambda M$ and the inclusion $M/\lambda M \hookrightarrow Q_\lambda(M)$. If λ is a radical in A-**mod**, then $Q_\lambda(A)$ is canonically endowed with an R-algebra structure extending that of A. Moreover, if M is a left A-module (resp. an (A,A')-bimodule) then $Q_\lambda(M)$ possesses a natural left $Q_\lambda(A)$-module (resp. a $(Q_\lambda(A),A')$-bimodule) structure.

(1.2) Let us fix radicals λ and λ') in A-**mod** and A'-**mod** respectively. Then we say that an (A,A')-bimodule P is (λ,λ')-*flat* or *relatively flat (with respect to* (λ,λ')*)*, if for any left A'-linear map $f' : M' \to N'$ with λ'-torsion kernel, the left A-module $\mathrm{Ker}(P \otimes_{A'} f')$ is λ-torsion. It is easy to see that P is (λ,λ')-flat if and only if $Q_\lambda(P)$ is relatively flat, or equivalently if it satisfies each of the following conditions:

(1.2.1) for any injective left A'-linear map $i' : M' \hookrightarrow N'$, the left A-module $\mathrm{Ker}(P \otimes_{A'} i')$ is λ-torsion.

(1.2.2) for any λ'-torsion left A'-module T', the left A-module $P \otimes_{A'} T'$ is λ-torsion.

The next (technical) result will play a key-role in all that follows:

(1.3) Lemma. [6,9] *Let P be an (A, A')-bimodule and M' a left A'-module, then:*

(1.3.1) $Q\lambda(P \otimes_{A'} M' = Q\lambda(Q\lambda(P) \otimes_{A'} M'$;

(1.3.2) *if P is relatively flat, then $Q\lambda(P \otimes_{A'} M' = Q\lambda(P \otimes_{A'} Q_{\lambda'}(M'))$;*

(1.3.3) *if P is also relatively flat and λ-closed, then it has a canonical $(Q_\lambda(A), Q_{\lambda'}(A'))$-bimodule structure and for any left $Q_{\lambda'}(A')$-module M', the left A-modules $P \otimes_{A'} M'$, $P \otimes_{Q_{\lambda'}(A')} M'$ and $P \otimes_{Q_{\lambda'}(A')} Q_{\lambda'}(M')$ sre λ-isomorphic.*

Let M be an (A, A')-bimodule, M" a left A'-module, then we will write $M \widehat{\otimes}_{A'} M'$ for $Q_\lambda(M \otimes_{A'} M')$ and $m \widehat{\otimes}_{A'} m'$ for $j_\lambda(m \widehat{\otimes}_{A'} m')$ for any $m \in M$ and $m' \in M'$, where $j_\lambda : M \otimes_{A'} M' \to M \widehat{\otimes}_{A'} M'$ is the canonical localization map. So, the previous lemma allows us to write:

$$P \widehat{\otimes}_{A'} M' \widehat{\otimes}_{A''} M'' = (P \widehat{\otimes}_{A'} M') \widehat{\otimes}_{A''} M'' = P \widehat{\otimes}_{A'} (M' \widehat{\otimes}_{A''} M'')$$

whenever P is relatively flat.

(1.4) A λ-closed and (λ, λ')-flat (A, A')-bimodule P is said to be (λ, λ')-*invertible* or *relatively invertible (with respect to* (λ, λ')*)* , if there exists a λ'-closed and (λ', λ)-flat (A', A)-bimodule Q together with A-bimodule (resp. A-bimodule) isomorphisms

$$\varphi : P \widehat{\otimes}_{A'} Q \to Q_\lambda(A) \quad \text{resp.} \quad \psi : Q \otimes_A P \to Q_{\lambda'}(A').$$

Moreover, cf. [9], we may always assume the above isomorphisms to fit into the following commutative diagrams:

$$\begin{array}{ccc} P \widehat{\otimes}_{A'} Q \widehat{\otimes}_A P & \xrightarrow{P \widehat{\otimes}_{A'} \psi} & P \widehat{\otimes}_{A'} Q_{\lambda'}(A') \\ {\scriptstyle \varphi \widehat{\otimes}_A P} \downarrow & & \downarrow \\ Q_\lambda(A) \widehat{\otimes}_A P & \longrightarrow & P \end{array} \quad \text{resp.} \quad \begin{array}{ccc} Q \widehat{\otimes}_A P \widehat{\otimes}_{A'} Q & \xrightarrow{Q \widehat{\otimes}_A \varphi} & Q \widehat{\otimes}_A Q_\lambda(A) \\ {\scriptstyle \psi \widehat{\otimes}_{A'} Q} \downarrow & & \downarrow \\ Q_{\lambda'}(A') \widehat{\otimes}_{A'} Q & \longrightarrow & Q \end{array}$$

The module Q, which is obviously relatively invertible, is said to be an *inverse* for P, and is, as one easily verifies, isomorphic to ${}_A[P, Q_\lambda(A)]$. Moreover, the evaluation map $P \mathbin{\widehat{\otimes}}_{A'} ({}_A[P, Q_\lambda(A)]) :\longrightarrow Q_\lambda(A)$, may then be used as an isomorphism.

This leads us to the relative version of the Morita theorems, cf. [9]:

(1.5) Theorem. *Let λ (resp. λ') be a radical in A-**mod** (resp. A'-**mod**). Then there is a bijective correspondence between bimodule isomorphism classes of relatively invertible (A, A')-bimodules and isomorphism classes of category equivalences between the categories (A, λ)-**mod** and (A', λ')-**mod**.*

Note that the above correspondence is given by associating to any category equivalence $F : (A, \lambda)$-**mod** $\to (A', \lambda')$-**mod** , the (λ', λ)-invertible (A, A')-bimodule $F(Q_\lambda(A))$. Conversely, to any relatively invertible (A, A')-bimodule Q with inverse P, we associate the category equivalence

$$Q \mathbin{\widehat{\otimes}}_A - \cong {}_A[P, -] : (A, \lambda)\text{-}\mathbf{mod} \to (A', \lambda')\text{-}\mathbf{mod}.$$

Let as point out that $Q_{\lambda'}(A')$ and ${}_A[P, P]$ are isomorphic as left A'-bimodules.

(1.6) If λ is a radical in A-**mod** and $\alpha : A \to A$ an R-involution, then one easily verifies the set $\{\alpha(L) : L \in \mathcal{L}_\lambda\}$ to be a Gabriel filter of right A-ideals. We will write $\alpha(\lambda)$ for the associated radical (in **mod**-A) and $Q_{\alpha(\lambda)}$ for the localization functor at $\alpha(\lambda)$ in **mod**-A. The functors $(-)^\alpha$ and ${}^\alpha(-)$ define a category equivalence between the categories (A, λ)-**mod** and **mod**-$(A, \alpha(\lambda))$. Moreover, for any left A-module M, we have $Q_\lambda(M)^\alpha = Q_{\alpha\lambda}(M^\alpha)$ and if M is an (A, A')-bimodule, then ${}^{\alpha'}Q_\lambda(M)^\alpha = Q_{\alpha\lambda}({}^{\alpha'}M^\alpha)$, where α' is an R-involution on A'. In particular, if M is an A-bimodule, then $Q_\lambda(M)_\alpha = Q_{\alpha\lambda}(M_\alpha)$.

(1.7) A triple (A, α, λ) is called a *torsion triple*, if (A, α) is an R-algebra with involution and λ a radical in A-**mod** which satisfies the equivalent conditions:

(1.7.1) the R-involution $\alpha : A \to A$ extends (uniquely) to an R-involution $\widehat{\alpha} : Q_\lambda(A) \to Q\lambda(A)$;

(1.7.2) the R-algebras $Q_\lambda(A)$ and $Q_{\alpha(\lambda)}(A)$ are isomorphic over A;

(1.7.3) there exists a (λ, λ')-invertible (A, A')-bimodule P, for some algebra with involution (A', α') and radical λ' in A'-**mod**], with the property that

$P \cong {}^{\alpha}_{A}[P, Q_\lambda(A)]^{\alpha'}$ as (A, A')-bimodules.

Note that these conditions are trivially fulfilled whenever λ is induced by a radical in R-**mod**; for other examples we refer to [6,10].

2 Hermitically invertible modules.

(2.1) Let us fix a torsion triple (A, α, λ) and a λ-closed left A-module M. A map $h : M \times M \to Q_\lambda(A)$ which is biadditive and satisfies $h(a_1m_1, a_2m_2) = a_1h(m_1, m_2)\alpha(a_2)$ for every $a_1, a_2 \in A$ and $m_1, m_2 \in M$ is called a λ-*sesquilinear form*. If, moreover, $h(m_1, m_2) = \widehat{\alpha}(h(m_2, m_1))$, then h is called a λ-*hermitian form*. For any λ-sesquilinear form $h : M \times M \to Q_\lambda(A)$, define $h^a \in {}_A[M, {}^{\alpha}_{A}[M, Q_\lambda(A)]]$ by $h^a(m_2)(m_1) = h(m_1, m_2)$ for any $m_1, m_2 \in M$. This correspondence defines a bijection between the λ-sesquilinear forms on M and the left A-linear maps from M to ${}^{\alpha}_{A}[M, Q_\lambda(A)]$. If h^a is an isomorphism, then h is called *nonsingular*. If M is an (A, A')-bimodule and $h : M \times M \to Q_\lambda(A)$ a λ-sesquilinear form satisfying $h(m_1a', m_2) = h(m_1, m_2\alpha'(a'))$, for any $a' \in A'$ and $m_1, m_2 \in M$ then h is said to be A'-*compatible*. So, an A'-*compatible* λ-sesquilinear morphism $h : M \times M \to Q_\lambda(A)$ is essentially a bimodule morphism $M \widehat{\otimes}_{A'}{}^{\alpha'} M^{\alpha} \to Q_\lambda(A)$. Note also that this is equivalent to requiring that the map $h^a : M \to {}^{\alpha}_{A}[M, Q_\lambda(A)]^{\alpha'}$ is (A, A')-linear.

If M is a λ-closed left A-module and $h : M \times M \to Q_\lambda(A)$ a λ-sesquilinear form, then the couple (M, h) is called a λ-*sesquilinear module* or a *relative sesquilinear module*. If h is also λ=hermitian, then (M, h) is a λ-*hermitian module* or *relative hermitian module*. It is said to be A'-*compatible* (resp. nonsingular) whenever h is A'-compatible (resp. nonsingular).

(2.2) A *morphism* $f : (M, h) \to (N, k)$ between λ-sesquilinear left A-modules is a left A-linear map $f : M \to N$ such that $h = k \circ (f \times f)$, or, equivalently such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{h^a} & {}^{\alpha}_{A}[M, Q_\lambda(A)] \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle {}^{\alpha}_{A}[f, Q_\lambda(A)]} \\ N & \xrightarrow{k^a} & {}^{\alpha}_{A}[N, Q_\lambda(A)] \end{array}$$

commutes. We thus obtain categories $\mathcal{S}(A, \alpha, \lambda)$, resp. $\mathcal{H}(A, \alpha, \lambda)$, with objects the λ-sesquilinear left A-modules, resp. λ-hermitian left A-modules, and with obvious morphisms.

(2.3) Fix some torsion triples $(A, \alpha\lambda)$ and (A, α', λ'). A nonsingular λ-hermitian (A, A')-bimodule (P, h) is called *hermitically* (λ, λ')*-invertible* or *relatively hermitically invertible*, if P is (λ, λ')-invertible and h is A'-compatible. As one easily verifies, h is then also $Q_{\lambda'}(A')$-compatible.

As an easy example, let $p_{Q_\lambda(A)} : Q_\lambda(A) \times Q_\lambda(A) \to Q_\lambda(A)$ be defined by

$$p_{Q_\lambda(A)}(a_1, a_2) = a_1 \widehat{\alpha}(a_2),$$

for any $a_1, a_2 \in Q_\lambda(A)$. Then $(Q_\lambda(A), p_{Q_\lambda(A)})$ is a hermitically (λ, λ')-invertible A-bimodule.

If (P, h) is a relatively hermitically invertible (A, A')-bimodule, then we can make $Q = {}_A[P, Q_\lambda(A)]$ into a hermitically (λ', λ)-invertible (A', A)-bimodule by endowing it with the form $k : Q \times Q \to Q_{\lambda'}(A') \cong {}_A[P, P]$, defined by putting for any $q_1, q_2 \in Q$:

$$k(q_1, q_2) : P \to P : p \mapsto k(q_1, q_2)(p) = h(p, (h^a)^{-1}(q_1))(h^a)^{-1}(q_2).$$

The module (Q, k) is usually referred to as an “inverse” of (P, h).

(2.4) Let (M, h) be a relatively flat A'-compatible λ-sesquilinear (resp. λ-hermitian) (A, A')-bimodule and (M', h') a λ-sesquilinear (resp. λ-hermitian) left A'-module. Then we may define a λ-sesquilinear (resp. λ-hermitian) form

$$h \otimes_{A'} h' : M \otimes_{A'} M' \times M \otimes_{A'} M' \to Q_\lambda(A)$$

by

$$\begin{aligned} h \otimes_{A'} h'(m_1 \otimes_{A'} m_1', m_2 \otimes_{A'} m_2') &= h(m_1 h'(m_1', m_2'), m_2) \\ &= h(m_1, m_2 h'(m_1', m_2')), \end{aligned}$$

for any $m_1, m_2 \in M$ and $m_1', m_2' \in M'$. One easily verifies the tensor product thus defined to be associative, and the form $h \otimes_{A'} h'$ to be A''-compatible, whenever (M', h') is.

Since $Q_\lambda(A)$ is $\alpha(\lambda)$-closed and since $j_\lambda : M \widehat{\otimes}_{A'} M' \times M \widehat{\otimes}_{A'} M' \to Q_\lambda(A)$ is a λ-isomorphism, the form $h \otimes_{A'} h'$ defines a unique λ-sesquilinear (resp. λ-hermitian) form $h \widehat{\otimes}_{A'} h' : M \widehat{\otimes}_{A'} M' \times M \widehat{\otimes}_{A'} M' \to Q_\lambda(A)$ making the diagram

$$\begin{array}{ccc}
M \widehat{\otimes}_{A'} M' \times M \widehat{\otimes}_{A'} M' & & \\
\downarrow {\scriptstyle j_\lambda \times j_\lambda} & \searrow^{h \otimes_{A'} h'} & \\
 & & Q_\lambda(A) \\
 & \nearrow_{h \widehat{\otimes}_{A'} h'} & \\
M \widehat{\otimes}_{A'} M' \times M \widehat{\otimes}_{A'} M' & &
\end{array}$$

commutative, cf. [5]. It thus makes sense to define the *relative tensor product* $(M,h)\,\widehat{\otimes}_{A'}(M',h')$ to be the λ-sesquilinear (resp. λ-hermitian) left A-module $(M\,\widehat{\otimes}_{A'}M', h\,\widehat{\otimes}_{A'}h')$. An easy unicity argument shows this tensor product to be associative, whenever it is defined.

3 Morita theorems.

(3.1) Fix torsion triples (A,α,λ) and (A',α',λ'). Recall from [5,6] that any relatively hermitically invertible (A,A')-bimodule (P,h) determines an equivalence of categories

$$(P,h)\,\widehat{\otimes}_{A'} - : \mathcal{S}(A',\alpha',\lambda') \to \mathcal{S}(A,\alpha,\lambda)$$

and an equivalence

$$(P,h)\,\widehat{\otimes}_{A'} - : \mathcal{H}(A',\alpha',\lambda') \to \mathcal{H}(A,\alpha,\lambda)$$

Moreover, if (Q,k) is as in (2.3), then

$$(Q,k)\,\widehat{\otimes}_{A} - : \mathcal{S}(A,\alpha,\lambda) \to \mathcal{S}(A',\alpha',\lambda')$$

resp.

$$(Q,k)\,\widehat{\otimes}_{A} - : \mathcal{H}(A,\alpha,\lambda) \to \mathcal{H}(A',\alpha',\lambda')$$

is an inverse for $(P,h)\,\widehat{\otimes}_{A'} -$.

(3.2) In order to establish the complete Morita theorems, we need a notion of "good" category equivalence between categories of relative sesquilinear (resp. relative hermitian) modules: a category equivalence

$$F : \mathcal{S}(A,\alpha,\lambda) \to \mathcal{S}(A',\alpha',\lambda')$$

resp.

$$F : \mathcal{H}(A,\alpha,\lambda) \to \mathcal{H}(A',\alpha',\lambda')$$

is said to be *decent*, if it factorizes through a category equivalence

$$F : (A,\lambda)\text{-}\mathbf{mod} \to (A',\lambda')\text{-}\mathbf{mod}$$

note that we use the same simbol F, as no ambiguity may arise) i.e., if we have a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{S}(A,\alpha,\lambda) & \xrightarrow{F} & \mathcal{S}(A',\alpha',\lambda') \\ \downarrow & & \downarrow \\ (A,\lambda)\text{-}\mathbf{mod} & \xrightarrow{F} & (A',\lambda')\text{-}\mathbf{mod} \end{array}$$

where the vertical arrows are defined by forgetting the relative sesquilinear form (a similar condition holds for the category of relative hermitian modules) and if there exists an isomorphism $\eta : F({}^{\alpha}_{A}[(-), Q_\lambda(A)]) \cong {}^{\alpha'}_{A'}[F(-), Q_{\lambda'}(A')]$ such that for every λ-sesquilinear left A-module (M,l), we have a commutative diagram

$$\begin{array}{ccc} & F(M) & \\ {}^{F(l^\alpha)}\swarrow & & \searrow^{F(l)^\alpha} \\ F({}^{\alpha}_{A}[M, Q_\lambda(A)]) & \xrightarrow{\eta_M} & {}^{\alpha'}_{A'}[F(M), Q_{\lambda'}(A')] \end{array}$$

If (M,l) is a λ-sesquilinear (A,A'')-bimodule, then, by the naturality of η, we have that η_M is an (A',A'')-bimodule isomorphism. Moreover, we will only consider category equivalences between relatively sesquilinear modules which map relative hermitian modules to relative hermitian modules.

We will prove below that if G is an inverse for F, then G is decent as well.

Before we can show that the category equivalence induced by a relatively hermitically invertible bimodule is decent, we need the following lemma, whose proof is just a straightforward verification.

(3.3) Lemma. *Let U be a right A'-module, V a left A-module and W an (A,A')-bimodule, then the morphism*

$$\mu : [U,_A[V,W]]_{A'} \to {}_A[V,[U,W]_{A'}]$$

defined by $(\mu(f)(v))(u) = f(u)(v)$, for every $f \in [U,_A[V,W]]_{A'}$, $u \in U$ and $v \in V$, is an isomorphism. If V is an (A,A'')-bimodule, then μ is left A''-

linear and if U *is an* (A'', A')*-bimodule, then* μ *is right* A''*-linear.*

(3.4) Proposition (Morita I). *Fix torsion triples* (A, α, λ) *and* (A', α', λ'). *Then any relatively hermitically invertible* (A', A)*-bimodule* (Q, k) *defines a decent equivalence between the categories* $\mathcal{S}(A, \alpha, \lambda)$ *and mathhcal*$S(A', \alpha', \lambda')$ *and the categories* $\mathcal{H}(A, \alpha, \lambda)$ *and mathcal*$H(A', \alpha', \lambda')$.

Proof. Let (P, h) be an inverse for (Q, k). Define for every λ-closed left A-module M the isomorphism η_M as the composition of the following isomorphisms

$$\begin{aligned} Q \mathbin{\widehat{\otimes}}_A {}^{\alpha}_{A}[M, Q_\lambda(A)] &\cong {}_A[P, {}^{\alpha}_{A}[M, Q_\lambda(A)]] \cong {}_A[P, [M^\alpha, Q_\lambda(A)_\alpha]_A] \\ &\cong [M^\alpha, {}_A[P, Q_\lambda(A)_\alpha]]_A \cong [M^\alpha, {}_A[P, Q_\lambda(A)]]_A \\ &\cong [M^\alpha, Q]_A \cong {}^{\alpha'}_{A}[M, P] \\ &\cong {}^{\alpha'}_{A'}[Q \mathbin{\widehat{\otimes}}_A M, Q \mathbin{\widehat{\otimes}}_A P] \cong {}^{\alpha'}_{A'}[Q \mathbin{\widehat{\otimes}}_A M, Q_{\lambda'}(A')]. \end{aligned}$$

An easy verification shows that

$$\eta_M(q \widehat{\otimes}_A f)(q' \widehat{\otimes}_A m') = k(q' f(m'), q)$$

and that $\eta : Q \mathbin{\widehat{\otimes}}_A {}^{\alpha}_{A}[(-), Q_\lambda(A)] \cong {}^{\alpha'}_{A'}[Q \mathbin{\widehat{\otimes}}_A (-), Q_{\lambda'}(A')]$. So, for every $m \in M$ and $q \in Q$, we have that

$$\eta_M(q \widehat{\otimes}_A l^a(m)) = (k \widehat{\otimes}_A l)^a (q \widehat{\otimes}_A m),$$

i.e., $(Q, k) \widehat{\otimes}_A -$ is decent. By symmetry, $P \mathbin{\widehat{\otimes}}_{A'} -$ is decent as well. □

Conversely,

(3.5) Proposition (Morita II). *Let* (A, α, λ) *and* (A', α', λ') *be torsion triples. Then every decent category equivalence between* $\mathcal{S}(A, \alpha, \lambda)$ *and* $\mathcal{S}(A', \alpha', \lambda')$ *(resp.* $\mathcal{H}(A, \alpha, \lambda)$ *and mathcal*$H(A', \alpha', \lambda')$*) is induced by a relatively hermitically invertible* (A', A)*-bimodule.*

Proof. Let $F : \mathcal{H}(A, \alpha, \lambda) \to \mathcal{H}(A', \alpha', \lambda')$ be a decent category equivalence, then $(Q, k) = F(Q_\lambda(A), p_{Q_\lambda(A)})$ is a hermitically (λ', λ)-invertible (A', A)-bimodule. Let us now show that $F(-) = (Q, k) \widehat{\otimes}_A -$.

As $F = Q \mathbin{\widehat{\otimes}}_A - : (A, \lambda)$-**mod** $\to (A', \lambda')$-**mod**, we only have to verify that $F(l) = k \mathbin{\widehat{\otimes}}_A l$, for every λ-sesquilinear left A-module (M, l). Let

$$\eta : F({}^{\alpha}_{A}[(-), Q_\lambda(A)]) \xrightarrow{\sim} {}^{\alpha'}_{A'}[F(-), Q_{\lambda'}(A')],$$

then

$$\eta_{Q_\lambda(A)} = k^a : Q \to {}^{\alpha'}_{A'}[Q, Q_{\lambda'}(A')].$$

Let (M, l) be a λ-sesquilinear left A-module, then for every $m \in M$ we have a commutative diagram

$$\begin{array}{ccc} Q & \xrightarrow{Q\widehat{\otimes}_A {}^{\alpha}_{A}[l^a(M), Q_\lambda(A)]} & Q\widehat{\otimes}_A {}^{\alpha}_{A}[M, Q_\lambda(A)] \\ {\scriptstyle k^a}\downarrow & & \downarrow{\scriptstyle \eta_M} \\ {}^{\alpha'}_{A'}[Q\widehat{\otimes}_A, Q_{\lambda'}(A')] & \xrightarrow[{}^{\alpha'}_{A'}[Q\widehat{\otimes}_A l^a(m), Q_{\lambda'}(A')]]{} & {}^{\alpha'}_{A'}[Q\widehat{\otimes}_A M, Q_{\lambda'}(A')] \end{array}$$

after identifying

$$Q = Q\widehat{\otimes}_A {}^{\alpha}_{A}[Q_\lambda(A), Q_\lambda(A)]$$

and

$${}^{\alpha'}_{A'}[Q\widehat{\otimes}_A, Q_{\lambda'}(A')] = {}^{\alpha'}_{A'}[Q\widehat{\otimes}_A\, Q_\lambda(A), Q_{\lambda'}(A')].$$

So, since $F(l)^a = \eta_M \circ (Q \,\widehat{\otimes}_A\, l^a)$, we have for every $q, q' \in Q$ and $m, m' \in M$

$$\begin{array}{rcl} (F(l)^a(q \,\widehat{\otimes}_A\, m))(q' \,\widehat{\otimes}_A\, m') & = & ((\eta_M \circ (Q \,\widehat{\otimes}_A\, l^a))(q \,\widehat{\otimes}_A\, m))(q' \,\widehat{\otimes}_A\, m') \\ & = & \eta_M(q \,\widehat{\otimes}_A\, l^a(m))(q' \,\widehat{\otimes}_A\, m') \\ & = & ((\eta_M \circ (Q \,\widehat{\otimes}_A\, {}^{\alpha}_{A}[l^a(m), Q_\lambda(A)]))(q))(q' \,\widehat{\otimes}_A\, m') \\ & = & (({}^{\alpha'}_{A'}[Q \,\widehat{\otimes}_A\, l^a(m), Q_{\lambda'}(A')] \circ k^a)(q))(q' \,\widehat{\otimes}_A\, m') \\ & = & (k^a(q) \circ (Q \,\widehat{\otimes}_A\, l^a(m))))(q' \,\widehat{\otimes}_A\, m') \\ & = & k^a(q)(q' l(m', m)) \\ & = & ((k \,\widehat{\otimes}_A\, l)^a(q \,\widehat{\otimes}_A\, m))(q' \,\widehat{\otimes}_A\, m'), \end{array}$$

hence $F(l) = k \,\widehat{\otimes}_A\, l$, as claimed. □

(3.6) Corollary. With the same notations, if G is an inverse for F, then $G = (P, h) \,\widehat{\otimes}_{A'} -$, where (P, h) is an inverse for (Q, k).
In particular, G is also decent, as claimed before.

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