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Operator Decomposition of Continuous Mappings

Descomposición por Operadores de Funciones Continuas

Ennis Rosas (erosas@cumana.sucre.udo.edu.ve)
Departamento de Matemáticas
Universidad de Oriente. Cumaná. Venezuela

Jorge Vielma (vielma@ciens.ula.ve)
Departamento de Matemáticas. Facultad de Ciencias
Universidad de los Andes. Mérida. Venezuela

Abstract

In this paper we introduce the concepts of (α, β) weakly continuous mapping and closed (α, β) continuous mapping. We prove that a map f is (α, id) weakly continuous if and only if f is (α, β) weakly continuous and (α, β^*) weakly continuous, where β and β^* are mutually dual operators. The concept of (α, β) weakly continuity generalizes the concepts of weakly continuity in the sense of N. Levine and expansion continuity in the sense of J. Tong.
Key words and phrases: weakly continuity, mutually dual operators, expansion continuity.

Resumen

En este trabajo se introducen los conceptos de *función* (α, β) *débilmente continua* y de *función* (α, β) *cerrada continua* y se prueba que una función f es $(\alpha$, id$)$ débilmente continua si y sólo si f es (α, β) débilmente continua y (α, β^*) débilmente continua, donde β and β^* son operadores mutuamente duales. El concepto de (α, β) continuidad débil generaliza el concepto de continuidad débil en el sentido de N. Levine y el de continuidad expansiva en el sentido de J. Tong.
Palabras y frases clave: continuidad débil, operadores mutuamente duales, continuidad expansiva.

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In [1], Kasahara introduced the concept of operator associated with a topology Γ of a space X as a map α from Γ to $P(X)$ such that $U \subset \alpha(U)$ for every $U \in \Gamma$. In this paper we modify his definition by allowing the operator α to be defined in $P(X)$, as follows:

Definition 1. Let (X, Γ) be a topological space and $\alpha : P(X) \to P(X)$ a function. We say that α is an *operator* on Γ if $U \subset \alpha(U)$ for every $U \in \Gamma$.

Note : In [2], this kind of operator is called an *expansion* of X, when α is defined on Γ.

Definition 2. (See [1,3]) Let (X, Γ) be a topological space and α an operator on Γ. A subset A of X is said to be *α-open* if for each $x \in A$ there exists a Γ-open neighborhood U of x such that $\alpha(U) \subset A$. A subset B of X is *α-closed* if its complement $X - B$ is α-open.

Note that α-open sets are also open sets in (X, Γ).

Definition 3. Let (X, Γ) and (Y, Ψ) be topological spaces and α and β operators on Γ and Ψ respectively. We say that a map $f : X \to Y$ is (α, β) *weakly continuous* if and only if $\alpha(f^{-1}(V)) \subseteq \text{Int } f^{-1}(\beta(V))$ for every Ψ-open set V in Y.

Note: the above definition should not be confused with the definition of (α, β) continuous map in the sense of H. Ogata [5].

Remarks 1.

1. Observe that when $\alpha = id$ and $\beta = id$ then f is (α, β) weakly continuous if and only if f is continuous in the usual sense.

2. If $\alpha = id$, $\beta =$closure operator and f is (α, β) weakly continuous, then f is weakly continuous in the sense of N. Levine [2].

3. If $\alpha =$ id and β is any operator, then f is (α, β) weakly continuous if and only if f is expansion continuous in the sense of J. Tong [4].

4. There are operators α and β which are too restrictive to allow the existence of (α, β) weakly continuous maps, for example, if we choose $\alpha(V) = (Fr(V))^c$ and $\beta =$ id, then no map f is (α, β) weakly continuous. In order to prove this affirmation, let's suppose that there is an (α, β) weakly continuous map $f : X \to Y$. Then we would obtain that $X - Fr(f^{-1}(V)) \subset Int(f^{-1}(V))$, which in general is false.

5. If we ask the operator α to satisfy the additional condition: $\alpha(\phi) = \phi$ then the constant maps are always (α, β) weakly continuous, for any operator β.

Definition 4. Let (X, Γ) and (Y, Ψ) be topological spaces and α and β operators on Γ and Ψ respectively. A map $f : X \to Y$ is said to be *closed* (α, β) *continuous* if $f^{-1}((\beta(V)^c))$ is an α-closed set in (X, Γ) for each $V \in \Psi$.

Definition 5. Let (X, Γ) be a topological space. A pair of operators α and β on Γ are *mutually dual* if $\alpha(V) \cap \beta(V) = V$ for every $V \in \Gamma$.

We observe that the above definition generalizes Definition 6 of [4].

Example 1. The identity operator and the closure operator are mutually dual operators.

Definition 6. Let (X, Γ) be a topological space. An operator α on Γ is said to be *subadditive* if for every collection of open sets $\{U_\beta : \beta \in B\}$, $\alpha(\cup_{\beta\in B} U_\beta) \subseteq \cup_{\beta\in B}(\alpha(U_\beta))$.

As an example of a subadditive operator we can take the closure operator.

Lemma 1. *Let (X, Γ) be a topological space and α a subadditive operator on Γ. Then for every α-open set U we have that $\alpha(U) = U$.*

Proof. Let W be an open set. Then for every $x \in W$ there exists an open set U_x such that $U_x \subseteq \alpha(U_x) \subset W$. Therefore $\cup U_x \subseteq \cup\alpha(U_x) \subset W$, so $\cup U_x \subseteq \alpha(\cup(U_x) \subset W$. Therefore, $W \subseteq \alpha(W) \subset W$ and so $\alpha(W) = W$. □

Theorem 1. *Let (X, Γ) and (Y, Ψ) be two topological spaces and α an operator on Γ. If β and β^* are mutually dual operators on Ψ then a map $f : X \to Y$ is (α, id) weakly continuous if and only if f is both (α, β) and (α, β^*) weakly continuous.*

Proof. Suppose f is $(\alpha$,id) weakly continuous. Then for every V $\in \Psi$ we have $\alpha(f^{-1}(\mathrm{V})) \subseteq \mathrm{Int}(f^{-1}(\mathrm{V})) \subseteq \mathrm{Int}(f^{-1}(\beta(\mathrm{V}))) \cap \mathrm{Int}(f^{-1}(\beta^*(\mathrm{V})))$. This implies that f is (α, β) and (α, β^*) weakly continuous.

Conversely, suppose f is (α, β) and (α,β^*) weakly continuous. Then for every $V \in \Psi$ we have $\alpha(f^{-1}(V)) \subseteq Int(f^{-1}(\beta(V)))$ and $\alpha(f^{-1}(V)) \subseteq \mathrm{Int}(f^{-1}(\beta^*(V)))$. Thus $\alpha(f^{-1}(V)) \subseteq Int(f^{-1}(\beta(V))) \cap Int(f^{-1}(\beta^*(V))) = Int(f^{-1}(\beta(V) \cap \beta^*(V)))$.

Since β and $\beta^*$are mutually dual, we get that $\alpha(f^{-1}(V)) \subseteq \mathrm{Int}(f^{-1}(V))$, which implies that f is (α, id) weakly continuous. □

Corollary 1. *([2]) Let (X,Γ) and (Y,Ψ) be two topological spaces. Then a map $f : X \to Y$ is weakly continuous if and only if $f^{-1}(V) \subseteq Int(f^{-1}(cl(V)))$ for each open set V in Ψ.*

Proof. According to Remark 1.2, f is weakly continuous if and only if f is (id, cl) weakly continuous. Now since the identity operator and the closure operator are mutually dual, the conclusion follows from Theorem 8. □

Corollary 2. *([4]) Let (X,Γ) and (Y,Ψ) be two topological spaces and let β and β^* be mutually dual operators on Ψ. Then a map $f : X \to Y$ is continuous if and only if f is β expansion continuous and β^* expansion continuous.*

Proof. According to Remark 1.1, f is continuous if and only if f is (id,id) weakly continuous. Also, Remark 1.3 assures that f is β and β^* expansion continuous if and only if f is (id,β) weakly continuous and (id,β^*) weakly continuous, then the conclution follows from theorem 8. □

Theorem 2. *Let (X,Γ) and (Y,Ψ) be topological spaces and let α and β be operators on Γ and Ψ respectively. If α is subadditive and monotone then every closed (α,β) continuous map is (α,β) weakly continuous.*

Proof. Let $f : X \to Y$ be a closed map and let $V \in \Psi$. We know that $f^{-1}((\beta(V)^c)^c$ is an α-closed set in (X,Γ), then $(f^{-1}((\beta(V))^c))^c$ is an α-open subset of (X,Γ). Now since $f^{-1}(\beta(V))$ is open in (X,Γ), then $f^{-1}(\beta(V)) = \text{Int}(f^{-1}(\beta(V))$, then $(f^{-1}(V) \subset \text{Int}(f^{-1}(\beta(V)))$. Since α $(\text{Int}(f^{-1}(\beta(V))) = \text{Int}(f^{-1}(\beta(V))$ and α is monotone, then $\alpha((f^{-1}(V)) \subset \text{Int}(f^{-1}(\beta(V)))$. □

Finally we would like to point out that Corollary 1 in [4] is not true, for if $X = \{a,b,c\}$, $\Gamma = \{\phi,\{b\},X\}$, $\alpha : P(X) \to P(X)$ defined as follows: $\alpha(\phi) = \phi$, $\alpha(X) = X$, $\alpha(\{b\}) = \{b,c\}$, $\alpha(\{a,b\}) = \{a,b\}$, $\alpha(\{a,c\}) = \{a,c\}$, $\alpha(\{b,c\}) = \{b,c\}$ and $\alpha(\{c\}) = \{c\}$, $\beta : P(X) \to P(X)$ defined as follows: $\beta(\phi) = \phi$, $\beta(X) = X$, $\beta(\{b\}) = \{a,b\}$, $\beta(\{a,b\}) = \{a,b\}$, $\beta(\{a,c\}) = \{a,c\}$, $\beta(\{b,c\}) = \{b,c\}$ and $\beta(\{c\}) = \{c\}$, then α and β are mutually dual and $id : (X,\Gamma) \to (X,\Gamma)$ is continuous but it is not closed (α,β) continuous.

In fact, $(\text{id})^{-1}(\alpha(\{b\})^c) = (id)^{-1}((\{b,c\})^c) = (id)^{-1}(\{a\})) = \{a\}$ which is not Γ-closed in X.

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