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Characterization of Dual Extensions in the Category of Banach Spaces

Caracterización de Extensiones Duales en la Categoría de Espacios de Banach

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Abstract

The first part of this paper studies extensions in the category of Banach spaces (natural equivalence of functors). The second part proves a result which characterizes the duality of extensions.
Key words and phrases: Extensions and liftings, twisted sums, quasi-linear maps.

Resumen

La primera parte de este artículo estudia las extensiones en la categoría de los espacios de Banach (equivalencia natural de funtores). La segunda parte prueba un resultado que caracteriza la dualidad de tales extensiones.
Palabras y frases clave: Extensiones y subidas, sumas torcidas, aplicaciones casi-lineales.

Introduction

A *short exact sequence* in the category of Banach spaces is a diagram

$$0 \to Y \to X \to Z \to 0$$

where the image of each arrow is the kernel of the following.

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The open mapping theorem ensures that Y is a subspace of X and Z is the corresponding quotient.

Given Banach spaces Y and Z we define the *extensions* of Y by Z as the set

$$Ext(Y,Z) = \{[X] : 0 \to Y \to X \to Z \to 0 \text{ is exact}\}$$

where $[X_1] = [X_2] \Leftrightarrow$ there exists a bounded linear operator $T : X_1 \to X_2$ such that

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & X_1 & \to & Z & \to & 0 \\ & & \| & & T\downarrow & & \| & & \\ 0 & \to & Y & \to & X_2 & \to & Z & \to & 0 \end{array}$$

is commutative.

A map $F : Z \to Y$ is *quasi-linear* if it is homogeneous and there exists a constant $K > 0$ such that for all $z_1, z_2 \in Z$

$$\|F(z_1 + z_2) - (Fz_1 + Fz_2)\| \leq K(\|z_1\| + \|z_2\|).$$

Given a quasi-linear map $F : Z \to Y$, the F-*twisted sum* of Y and Z is defined as the quasi-Banach space

$$Y \oplus_F Z = \{(y,z) \in Y \times Z : \|(y,z)\|_F = \|y - Fz\| + \|z\| < +\infty\}$$

Before beginning we shall define

$$\begin{aligned} Lin(Z,Y) &= \{F : Z \to Y \text{ linear}\} \\ B(Z,Y) &= \{F : Z \to Y \text{ bounded}\} \\ \mathcal{L}(Z,Y) &= \{F : Z \to Y \text{ bounded and linear}\}. \end{aligned}$$

1 Three approaches

Definition 1.1. Let Y, Z be two Banach spaces. We define

$$\mathcal{Q}(Y,Z) = \{[F] : F : Z \to Y \text{ quasi-linear}\}$$

such that $[F_1] = [F_2] \Leftrightarrow d(F_1 - F_2, Lin(Z,Y)) < +\infty$.

Theorem 1.2. *Let Y, Z be two Banach spaces. There exists a bijection between $\mathcal{Q}(Y,Z)$ and $Ext(Y,Z)$.*

Proof. If

$$0 \to Y \xrightarrow{j} X \xrightarrow{p} Z \to 0$$

is an extension of Y by Z then we can consider a linear selection $L \in Lin(Z,X)$ and a bounded homogeneous selection $B \in B(Z,X)$ because the quotient map is open. Then $F = B - L \in \mathcal{Q}(Y,Z)$ since $p(B-L) = 0$, F is homogeneous and

$$\begin{aligned}\|F(z_1+z_2) - (Fz_1 + Fz_2)\| &= \|B(z_1+z_2) - (Bz_1 + Bz_2)\| \\ &\le \|B(z_1+z_2)\| + \|Bz_1\| + \|Bz_2\| \\ &\le \|B\|\|z_1+z_2\| + \|B\|(\|z_1\| + \|z_2\|) \\ &\le \|B\|(\|z_1\| + \|z_2\|) + \|B\|(\|z_1\| + \|z_2\|) \\ &\le 2\|B\|(\|z_1\| + \|z_2\|).\end{aligned}$$

To complete the proof let us show that

$$[Y \oplus_{F_1} Z] = [Y \oplus_{F_2} Z] \Leftrightarrow d(F_1 - F_2, Lin(Z,Y)) < +\infty.$$

$\Rightarrow)$ If $[Y \oplus_{F_1} Z] = [Y \oplus_{F_2} Z]$, there exists a bounded linear operator $T : Y \oplus_{F_1} Z \to Y \oplus_{F_2} Z$ such that

$$\begin{array}{ccccccccc} 0 & \to & Y & \to & Y \oplus_{F_1} Z & \to & Z & \to & 0 \\ & & \| & & T\downarrow & & \| & & \\ 0 & \to & Y & \to & Y \oplus_{F_2} Z & \to & Z & \to & 0 \end{array}$$

is commutative. In these conditions T must have the form $T(y,z) = (y + Lz, z)$, where $L \in Lin(Z,Y)$. Hence

$$\begin{aligned}\|F_1 z - F_2 z + Lz\| &= \|(F_1 z + Lz) - F_2 z\| \le \|(F_1 z + Lz, z)\|_{F_2} \\ &= \|T(F_1 z, z)\|_{F_2} \le \|T\|\|(F_1 z, z)\|_{F_1} = \|T\|\|z\|.\end{aligned}$$

$\Leftarrow)$ Supposing that $F_1 - F_2 = B - L$ with B bounded and L linear then $T(y,z) = (y + Lz - Bz, z)$ is a linear operator from $Y \oplus_{F_1} Z$ to $Y \oplus_{F_2} Z$. Let us prove that T is bounded:

$$\begin{aligned}\|T(y,z)\|_{F_2} &= \|(y + Lz - Bz, z)\|_{F_2} = \|y + Lz - Bz - F_2 z\| + \|z\| \\ &= \|y - F_1 z\| + \|z\| = \|(y,z)\|_{F_1}.\end{aligned}$$

$\square$

Corollary 1.3. *Let Y, Z be two Banach spaces and $[F] \in \mathcal{Q}(Y, Z)$. The following relationships are equivalent:*

(i) $0 \to Y \to Y \oplus_F Z \to Z \to 0$ *splits*

(ii) $[Y \oplus_F Z] = [Y \oplus Z]$

(iii) $d(F, Lin(Z, Y)) < +\infty$.

Definition 1.4. Let $0 \to Y \xrightarrow{j} X \xrightarrow{p} Z \to 0$ be a short exact sequence. A bounded linear operator h from Y to a Banach space E has an *extension* onto X if there is a bounded linear operator $\hat{h}$ from X to E such that $\hat{h}j = h$.

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] \arrow[d, "h"'] & X \arrow[r, "p"] \arrow[dl, "\hat{h}"] & Z \arrow[r] & 0 \\ & E & & & \end{tikzcd}$$

A bounded linear operator h from a Banach space E to Z has a *lifting* into X if there is a bounded linear operator $\hat{h}$ from E to X such that $p\hat{h} = h$.

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j"] & X \arrow[r, "p"] & Z \arrow[r] & 0 \\ & & & E \arrow[ul, "\hat{h}"] \arrow[u, "h"'] & \end{tikzcd}$$

Further information about extensions and liftings of operators can be found in [2].

Lemma 1.5. *Let X be a Banach space. There exists an index set I such that*

(i) $0 \to Ker\, p \to l_1(I) \xrightarrow{p} X \to 0$ *is exact. (Projective representation).*

(ii) $0 \to X \xrightarrow{j} l_\infty(I) \to l_\infty(I)/X \to 0$ *is exact. (Injective representation).*

Proof. Let I be such that $\overline{(x_i)_{i\in I}} = B_X$. Then

(i) $p(y) = \sum_{i\in I} y(i)x_i$ is surjective.

(ii) Let $(x_i^*)_{i\in I} \subset X^*$ be such that $x_i^*(x_i) = \|x_i\|\ \forall i \in I$ then $j(x) = (x_i^*(x))_{i\in I}$ is injective.

$\square$

We shall henceforth write $Ker\, p$ as K, $l_1(I)$ as l_1, and $l_\infty(I)$ as l_∞.

Definition 1.6. Let Y, Z be two Banach spaces. We define

$$\mathscr{E}(Y,Z) = \{[h] : h \in \mathcal{L}(K,Y)\}$$

such that $[h_1] = [h_2]$ if and only if $h_1 - h_2$ has an extension onto l_1, and

$$\mathscr{L}(Y,Z) = \{[h] : h \in \mathcal{L}(Z, l_\infty/Y)\}$$

such that $[h_1] = [h_2]$ if and only if $h_1 - h_2$ has a lifting into l_∞.

The following Lemma is frequently used in homological algebra (see [1]), and here it will allow us to prove the next theorem.

Lemma 1.7. *(Push-Out and Pull-Back universal properties.)*

(i) Let $h_1 : X \to X_1,\ h_2 : X \to X_2$ be two operators. Then

$$PO(h_1,h_2) =: X_1 \times X_2/_{\overline{\{(h_1x,h_2x)\ :\ x\in X\}}}$$

represents the covariant functor

$$E \in Ban \rightsquigarrow \left\{(\alpha,\beta) : \begin{tikzcd} X \arrow[r, "h_1"] \arrow[d, "h_2"'] & X_1 \arrow[d, "\alpha"'] \\ X_2 \arrow[r, "\beta"] & E \end{tikzcd} \text{ is commutative.}\right\}$$

(ii) Let $h_1 : X_1 \to X,\ h_2 : X_2 \to X$ be two operators. Then

$$PB(h_1,h_2) =: \{(x_1,x_2) \in X_1 \times X_2 : h_1x_1 = h_2x_2\}$$

represents the contravariant functor

$$E \in Ban \rightsquigarrow \left\{(\alpha,\beta) : \begin{tikzcd} X_1 \arrow[r, "h_1"] & X \\ E \arrow[u, "\alpha"] \arrow[r, "\beta"] & X_2 \arrow[u, "h_2"'] \end{tikzcd} \text{ is commutative.}\right\}$$

Theorem 1.8. *Let Y, Z be two Banach spaces. There exists a bijection between $\mathscr{E}(Y,Z)$, $\mathscr{L}(Y,Z)$ and $Ext(Y,Z)$.*

Proof. We shall prove that there exists a bijection between $\mathscr{E}(Y,Z)$ and $Ext(Y,Z)$. The other case is similar.

On the one hand, we have that $[l_1] \in Ext(K, Z)$. Hence using Theorem 1.2 there is a quasi-linear map $F : Z \to K$ such that $[l_1] = [K \oplus_F Z]$.

Given two bounded linear operators h_1, h_2 from K to Y, we define the natural quasi-linear maps $F_1 = h_1 F$ and $F_2 = h_2 F$ from Z to Y. We have to prove that $[Y \oplus_{F_1} Z] = [Y \oplus_{F_2} Z] \Leftrightarrow h_1 - h_2$ has an extension onto l_1. From Corollary 1.3 this is equivalent to proving that $[Y \oplus_{F_1 - F_2} Z] = [Y \oplus Z] \Leftrightarrow h_1 - h_2$ has an extension onto l_1.

$\Rightarrow$) Writing F for $F_1 - F_2$ and h for $h_1 - h_2$, the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "j"] \arrow[d, "h"'] & l_1 \arrow[r, "p"] \arrow[d, "H"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "i"] & Y \oplus_F Z \arrow[r, "q"] \arrow[l, bend left, "r"'] \arrow[d, equal] & Z \arrow[r] & 0 \\
 & & Y \oplus Z & &
\end{tikzcd}$$

This means that there exists a retract $r : Y \oplus Z \to Y$, hence $ri = I_Y$. Let us write $\hat{h} = rH$. Then $\hat{h}j = rHj = rih = h$ so that $\hat{h}$ is an extension of h.

$\Leftarrow$) Now we have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "j"] \arrow[d, "h"'] & l_1 \arrow[r, "p"] \arrow[d, "H"] \arrow[dl, "\hat{h}"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "i"] & Y \oplus_F Z \arrow[r, "q"] & Z \arrow[r] & 0
\end{tikzcd}$$

.

$Hj = ih$ so that $K \subset Ker(H - \hat{h})$. Hence $H - \hat{h}$ factors through p, and therefore there exists a bounded linear operator $s : Z \to Y \oplus_F Z$ such that $sp = H - \hat{h}$. Hence $qsp = qH - q\hat{h} = qH = p$, i.e. $qs = I_Z$. Thus $Y \oplus_F Z = Y \oplus Z$.

On the other hand, let $[h] \in \mathscr{E}(Y, Z)$ and let us consider the Push-Out of j and h, where j is the embedding of K in l_1. It only remains to prove that there exists a bounded linear operator $T : PO \to Y \oplus_F Z$ such that the following diagram is commutative, with F defined as at the beginning of this proof:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "j"] \arrow[d, "h"'] & l_1 \arrow[r, "p"] \arrow[d, "H"] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & PO(j, h) \arrow[r] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "i"] & Y \oplus_F Z \arrow[r, "q"] & Z \arrow[r] & 0.
\end{tikzcd}$$

Let b be a bounded selection of q. Hence we have a bounded linear operator $P : x \in l_1 \to bpx \in Y \oplus_F Z$, and the following diagram is commutative:

$$\begin{array}{ccc} K & \xrightarrow{j} & l_1 \\ h\downarrow & & \downarrow P \\ Y & \xrightarrow{i} & Y \oplus_F Z. \end{array}$$

Using the Push-Out universal property, there exists a unique bounded linear operator $t : PO \to K$ such that the third diagram is commutative. Considering $T = ith$, then T is the operator sought. $\square$

Corollary 1.9.

(i) *Let Z be a Banach space and let us consider its projective representation. Then K represents the covariant functor $Y \in Ban \rightsquigarrow Ext(Y, Z)$.*

(ii) *Let Y be a Banach space and let us consider its injective representation. Then l_∞/Y represents the contravariant functor $Z \in Ban \rightsquigarrow Ext(Y, Z)$.*

2 Main Result

Lemma 2.1. *Let Y, Z be two Banach spaces. Then*

$$Ext(Y, Z) = \{[Y \oplus Z]\} \Rightarrow Ext(Y_1, Z_1) = \{[Y_1 \oplus Z_1]\}$$

for all Y_1 complemented in Y, and for all Z_1 complemented in Z.

Proof. Let $[F] \in Ext(Y_1, Z_1)$. If $Ext(Y, Z) = \{[Y \oplus Z]\}$, this means that every short exact sequence that has Y as subspace and Z as quotient splits. Then there exists a retract $R : Y \oplus Z \to Y$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
 & & & & & \overset{\text{s}}{\curvearrowleft} & & & \\
0 \to & Y_1 & \hookrightarrow & Y_1 \oplus_F Z_1 & & \xrightarrow{q} & Z_1 & \to 0 \\
 & & \text{r} & & & \text{S} & & & \\
 & \| & \curvearrowleft & P\uparrow & & \curvearrowleft & p\uparrow\downarrow j & & \\
0 \to & Y_1 & \xrightarrow{i} & PB & & \xrightarrow{Q} & Z & \to 0 \\
 & & \text{R} & & & & & & \\
 & \phi\uparrow\downarrow\pi & \curvearrowleft & \downarrow\Pi & & & \| & & \\
0 \to & Y & \xrightarrow{I} & PBO & & \twoheadrightarrow & Z & \to 0. \\
 & & & \| & & & & & \\
 & & & Y \oplus Z & & & & &
\end{array}$$

If $r = \phi R\Pi$ then $ri = \phi R\Pi i = \phi RI\pi = \phi\pi = I_{Y_1}$. Hence, there is also a section S of Q. Thus $s = PSj$ is such that $qs = qPSj = pQSj = pj = I_{Z_1}$. Therefore $Ext(Y_1, Z_1) = \{[Y_1 \oplus Z_1]\}$.

□

Theorem 2.2. *Let Y, Z be two Banach spaces, then*

$$Ext(Y, Z) = \{[Y \oplus Z]\} \Rightarrow Ext(Z^*, Y^*) = \{[Z^* \oplus Y^*]\}$$

Proof. Let us consider the projective representation of Z

$$0 \to K \xrightarrow{j} l_1 \xrightarrow{p} Z \to 0$$

Let $h \in \mathcal{L}(K, Y)$. Then h has an extension onto l_1, and there exists $\hat{h} \in \mathcal{L}(l_1, Y)$ such that $h = \hat{h}j$. Therefore $h^{***} = j^{***}(\hat{h})^{***}$.

The following diagram is commutative:

$$\begin{array}{ccccccccc}
0 \to & K^{**} & \xrightarrow{j^{**}} & l_1^{**} & \xrightarrow{p^{**}} & Z^{**} & \to 0 \\
& \| & & \uparrow i & & \uparrow i_Z & \\
0 \to & K^{**} & \xrightarrow{j_1} & PB & \xrightarrow{p_1} & Z & \to 0.
\end{array}$$

If we consider now the following commutative diagram:

$$\begin{array}{ccccccc}
0 \to & Z^* & \xrightarrow{p^*} & l_1^* & \xrightarrow{j^*} & K^* & \to 0 \\
& \downarrow i_{Z^*} & & \downarrow i_{l_1^*} & & \downarrow i_{K^*} & \\
0 \to & Z^{***} & \xrightarrow{p^{***}} & l_1^{***} & \xrightarrow{j^{***}} & K^{***} & \to 0 \\
& \downarrow i_Z^* & & \downarrow i^* & & \| & \\
0 \to & Z^* & \xrightarrow{p_1^*} & PO & \xrightarrow{j_1^*} & K^{***} & \to 0 \\
& & & & & \uparrow h^{***} & \\
& & & & & Y^{***} &
\end{array}$$

then $PO = PB^*$.

Let $H = i^*(\hat{h})^{***}$. Since

$$Im\big(H\, \big|_{Y^*}\big) \subset Im\big(j_1^{*-1}\big|_{K^*}\big) = Im(i^* i_{l_1^*})$$

then $h^* = j^*((i^* i_{l_1^*})^{-1} \circ H\, \big|_{Y^*})$, so that $(i^* i_{l_1^*})^{-1} \circ H\, \big|_{Y^*}$ is the lifting we are looking for. □

In general the reciprocal is not true. For instance, the Lindenstrauss lifting principle ([3], Proposition 2.1.) states that $Ext(l_2, l_1) = \{[l_2 \oplus l_1]\}$.

If JL denotes the Johnson-Lindenstrauss space then c_0 is a subspace of JL and JL$/c_0 = l_2$ thus [JL] $\in Ext(c_0, l_2)$ and [JL] $\neq [c_0 \oplus l_2]$ because c_0 is not complemented in JL.

Adding a condition the reciprocal is true.

Theorem 2.3. *Let Y, Z be two Banach spaces such that Y is complemented in its bidual. Then*

$$Ext(Z^*, Y^*) = \{[Z^* \oplus Y^*]\} \Rightarrow Ext(Y, Z) = \{[Y \oplus Z]\}$$

Proof. Let $i_{K^*} : K^* \hookrightarrow K^{***}$, $i_{K^{**}} : K^{**} \hookrightarrow K^{****}$, $i_{l_1^{**}} : l_1^{**} \hookrightarrow l_1^{****}$, $i_{l_1^*} : l_1^* \hookrightarrow l_1^{***}$ be the canonical embeddings. Let $h \in \mathcal{L}(K, Y)$. Then h^* has a lifting into l_1^*, and we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to Z^* & \xrightarrow{p^*} & l_1^* & \xrightarrow{j^*} & K^* \to 0 \\ & & \hat{h^*} \nwarrow & & \uparrow h^* \\ & & & & Y^* \end{array}$$

where $h^* = j^* \hat{h^*}$ therefore $h^{**} = (\hat{h^*})^* j^{**}$ thus $h^{**} i_{K^*}^* = (\hat{h^*})^* j^{**} i_{K^*}^*$. Hence we have that

$$\begin{aligned} h^{**} =& h^{**} i_{K^*}^* i_{K^{**}} = (\hat{h^*})^* j^{**} i_{K^*}^* i_{K^{**}} \\ =& (\hat{h^*})^* i_{l_1^*}^* i_{l_1^{**}} j^{**}. \end{aligned}$$

In these conditions, $(\hat{h^*})^* i_{l_1^*}^* i_{l_1^{**}}$ is an extension of h^{**} onto l_1^{**}, so if $i_K : K \hookrightarrow K^{**}$, $i_{l_1} : l_1 \hookrightarrow l_1^{**}$, are the natural embeddings then $(\hat{h^*})^* i_{l_1^*}^* i_{l_1^{**}} i_{l_1}$ is an extension of $h^{**} i_K$ onto l_1.

Finally, using Lemma 2.1 the proof is complete. □

Corollary 2.4. *Let Y, Z be two Banach spaces such that Y is complemented in its bidual and $[F] \in \mathcal{Q}(Z, Y)$. If $F^* : Y^* \to Z^*$ is such that $F^* y^*(z) =: y^*(Fz)$, then $[F^*] \in \mathcal{Q}(Y^*, Z^*)$ and $d(F, Lin(Z, Y)) = d(F^*, Lin(Y^*, Z^*))$.*

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