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The Erdös-Ginzburg-Ziv Theorem in Abelian non-Cyclic Groups

El Teorema de Erdös-Ginzburg-Ziv en Grupos Abelianos no Cíclicos

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Abstract

A theorem by Caro states that every sequence of elements in an abelian non cyclic group of order n, not of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_{2m}$, with length $\frac{4n}{3} + 1$ contains an n-subsequence (subsequence of length n) with a zero-sum. In this paper, we obtain a more precise result by showing that in an abelian non cyclic group, not of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_{2m}$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_{3m}$, every sequence of length $\frac{5n}{4} + 2$ contains an n-subsequence with a zero-sum.
Keywords and phrases: abelian groups, Erdös-Ginzburg-Ziv Theorem, Davenport constant.

Resumen

Un teorema de Caro establece que cualquier secuencia de elementos de un grupo abeliano G de orden n, tal que $G \notin \{\mathbb{Z}_n, \mathbb{Z}_2 \oplus \mathbb{Z}_{2m}\}$, con longitud $\frac{4n}{3} + 1$ contiene una n-subsecuencia (subsecuencia de longitud n) con suma cero. En este artículo obtenemos un resultado más preciso al mostrar que si $G \notin \{\mathbb{Z}_n, \mathbb{Z}_2 \oplus \mathbb{Z}_{2m}, \mathbb{Z}_3 \oplus \mathbb{Z}_{3m}\}$, cualquier secuencia de elementos de G de longitud $\frac{5n}{4} + 2$ contiene una n-subsecuencia con

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suma cero.
Palabras y frases claves: grupos abelianos, Teorema de Erdös-Ginzburg-Ziv, constante de Davenport.

1 Introduction

Let G be an abelian group of order n. The Davenport constant of G, denoted by $D(G)$, is the minimal d such that every sequence of elements of G with length d contains a nonempty subsequence with a zero-sum. Let $ZS(G)$ be the smallest integer t such that every sequence of t elements of G contains an n-subsequence with a zero-sum. The Erdös-Ginzburg-Ziv Theorem [5] states that $ZS(G) \leq 2n-1$. In [1], Alon, Bialostocki and Caro show that $ZS(G) \leq \frac{3n}{2}$ for every abelian non-cyclic group G of order n. Moreover they stated that the equality holds only for the groups of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_{2m}$. In [4] Caro generalizes this result by showing that $ZS(G) \leq \frac{4n}{3}+1$ for every abelian non-cyclic group G, of order n and not of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_{2m}$. Moreover the equality holds only for the groups of the form $\mathbb{Z}_3 \oplus \mathbb{Z}_{3m}$. Let G be an abelian group. Gao proves in [6, 7] the fundamental relation $ZS(G) = |G| + D(G) - 1$.

Our result is the following:

Let G be an abelian non cyclic group of order n, not of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_{2m}$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_{3m}$, then $ZS(G) \leq \frac{5n}{4} + 2$. Furthermore equality holds only for the groups of the form $\mathbb{Z}_4 \oplus \mathbb{Z}_{4m}$.

Gao Theorem is our main tool. We shall use some estimates of $D(G)$ and prove a few lemmas in this direction. In particular we prove that $D(G) \leq \frac{n}{4}+3$ for every non cyclic abelian group G of order n not of the form $\mathbb{Z}_2 \oplus \mathbb{Z}_{2m}$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_{3m}$. Moreover, equality holds only for the groups of the form $\mathbb{Z}_4 \oplus \mathbb{Z}_{4m}$.

Our methods are much more elementary than the methods used by Caro in [4]. In particular we will not use the Baker-Schmidt Theorem.

2 The Davenport constant

In this section we begin by summarizing some results on the Davenport constant. Some new bounds are given.

It is well known that every finite abelian group G is a directed sum of cyclic groups, say $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ with $n_1 \mid n_2 \mid \cdots \mid n_r$. The *rank* of G

denoted by $r = r(G)$ is the number of non zero cyclic groups in the directed sum of G.

We use the following results:

Theorem 1 ([2, 9]). *Let G be an abelian $p-$group (p prime) of the form $G = \mathbb{Z}_{p^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_k}}$. Then $D(G) = 1 + \sum_{i=1}^{k}(p^{\alpha_i} - 1)$.*

Theorem 2 ([2, 9]). $D(\mathbb{Z}_n \oplus \mathbb{Z}_{nm}) = n + nm - 1$.

Theorems 1 and 2 were shown independently by Olson and Kruyswijk.

Lemma 1 ([4, 6]). *For H and K finite abelian groups, we have*

$$D(H \oplus K) \leq (D(H) - 1)|K| + D(K).$$

Let us introduce a few definitions and one lemma from an unpublished manuscript by Hamidoune.

Let G be a finite abelian group. Let $D_k(G)$ be the smallest integer t such that every sequence with length t contains k disjoint subsequences, each one with a zero-sum.

Let $D^s(G)$ be the smallest number t (possibly ∞) such that every sequence with length t contains a subsequence with length less or equal to s and a zero-sum.

Lemma 2 ([8]). *Suppose $D_j(H) + s \geq D^s(H)$. Then*

$$D(H \oplus K) \leq s(D(K) - j) + D_j(H).$$

Proof. By looking to the first coordinate, one may form $D(K) - j$ subsequences, each of length $\leq s$, and the sum of the first coordinates is zero in each of the subsequences. The remaining elements contain j disjoint subsequences each one with a zero-sum, by the definition of $D_j(H)$. Looking to the second coordinate, it can be formed a collection of the $D(K)$-sums where the sum of the second coordinate is zero. □

In the following lemma, $\exp(G)$ is the smallest r such that $ra = 0$ for all a in G.

Lemma 3. *Let G be an abelian non-cyclic group. Then*

$$D^{D(G)-1}(G) = D(G) + 1.$$

Proof. Let $S = a_1, \ldots, a_{D(G)+1}$ be a sequence of $D(G) + 1$ elements in G. Let T be an arbitrary subsequence of S with $|T| = D(G)$, then T contains a nonempty zero-sum subsequence of length less than $D(G)$ and we are done, or T is a zero-sum sequence. Therefore S contains a nonempty zero-sum subsequence of length less than $D(G)$ and we are done, or every subsequence T of S with $|T| = D(G)$ is a zero-sum sequence and hence $a_1 = \cdots = a_{D(G)+1}$, thus every subsequence of length $\exp(G)(= D(G))$ is zero-sum. This proves the upper bound.

To prove the lower bound, let $b_1, \ldots, b_{D(G)-1}$ be a sequence of $D(G) - 1$ elements in G which contains no nonempty zero-sum subsequence. Set $W = b_1, \ldots, b_{D(G)-1}, -(b_1 + \cdots + b_{D(G)-1})$. Clearly $|W| = D(G)$ and W contains no nonempty zero-sum subsequence of length less than $D(G)$. This proves that $D^{D(G)-1} = D(G) + 1$. □

Lemma 4. *Let K be an abelian group. Then we have*

$$D(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus K) \leq 2D(K) + 3.$$

Proof. Set $L = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. It may be seen easily that $D_2(L) = 7$. Let μ be a sequence of elements of L with length 7. Clearly μ has two disjoint subsequences, with a zero-sum each, if it is assumed the value 0 or if there is one repeated value x, since $2x = 0$. Moreover, among the 5 remaining elements there is a subsequence with a zero-sum. It only remains to consider the case where μ assumes the values $L \setminus 0$. It may be checked easily that μ has two disjoint subsequences, each one with a zero-sum. On the otherside clearly $D_2(L) = 8$. By Lemma 2, $D(L \oplus K) \leq 2(D(K) - 2) + D_2(L) \leq 2D(K) - 4 + 7 = 2D(K) + 3$. □

We need the following lemma:

Lemma 5. *Let K be an abelian group. The following relation holds: $D(\mathbb{Z}_{2^n} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus K) \leq 2^n D(K) + 2$ for $n \geq 2$.*

Proof. Set $L = \mathbb{Z}_{2^n} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Set $t = 2^n$. Let us prove that $D^t(L) \leq 2t + 2$. Let $\mu = \{x_i, 1 \leq i \leq 2t + 2\}$ be a sequence of elements of L. Consider the sequence of elements of $\mathbb{Z}_t \oplus L$, $\mu' = \{(1, x_i); 1 \leq i \leq 2t + 2\}$. By Theorem 1, there exists $T \subset [1, 2t + 2]$ with $\sum_{i \in T}(1, x_i) = 0$ and $|T| \geq 1$. It follows that $|T| \in \{t, 2t\}$, since the first coordinate must vanish. It would be enough to consider the case $|T| = 2t$. Take $T' \subset T$ such that $|T'| = 2t - 1$. Now by Theorem 1, there exists $S \subset T'$, such that $\sum_{i \in S} x_i = 0$ and $|T'| \geq |S| \geq 1$. It follows that $\sum_{i \in T \setminus S} x_i = 0$. Now one of the non empty sets S and $S \setminus T$ has cardinality less or equal to t.

By Lemma 2, $D(L \oplus K) \leq t(D(K) - 1) + D(L) \leq tD(K) - t + t + 2 = tD(K) + 2$. □

We prove the next lemmas:

Lemma 6. *Let K be an abelian group. We have the following relation:*

$$D(\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus K) \leq 6D(K) + 1.$$

Proof. Set $L = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Since $D(L) = 7$ (by Theorem 1), then by Lemma 3 we have $D^6(L) = 8$.

By Lemma 2, $D(L \oplus K) \leq 6(D(K) - 1) + D(L) \leq 6D(K) - 6 + 7 = 6D(K) + 1$. □

Lemma 7. *Let P be a p- group with rank 3 such that $D(P) > \frac{|P|}{4}$. Then $P \in \{\mathbb{Z}_{2^n} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3\}$.*

Proof. Set $P = S \oplus T \oplus R$. Put $s = |S|$, $|T| = t$ and $|R| = r$. Asume $s \geq t \geq r$. By Theorem 1 we have

$$\frac{1}{4} \leq \frac{1}{sr} + \frac{1}{tr} + \frac{1}{st} - \frac{2}{|P|} \leq \frac{3p-2}{p^3}.$$

It follows that $p \leq 3$. Let us now show that $t = p$. Suppose the contrary. We have

$$\frac{1}{4} < \frac{1}{sr} + \frac{1}{tr} + \frac{1}{st} - \frac{2}{|P|} \leq \frac{2p^2+p-2}{p^5} \leq \frac{1}{4},$$

a contradiction. The result follows now for $p = 2$. Suppose $p = 3$. Let us also show that $s \leq p^2 = 9$. Otherwise we have:

$$\frac{1}{4} < \frac{1}{sr} + \frac{1}{tr} + \frac{1}{st} - \frac{2}{|P|} \leq \frac{p^2+2p-2}{p^4} \leq \frac{13}{81},$$

a contradiction. □

3 The main result

Proposition 1. *Let G be an abelian group of order n, not in $\{\mathbb{Z}_n, \mathbb{Z}_2 \oplus \mathbb{Z}_{2m}, \mathbb{Z}_3 \oplus \mathbb{Z}_{3m}\}$. Then $D(G) \leq \frac{n}{4} + 3$. Moreover equality holds only for the groups of the form $\mathbb{Z}_4 \oplus \mathbb{Z}_{4m}$.*

Proof. We shall prove only the first part; the second one follows using exactly the same arguments.

Set $G = G_1 \oplus \cdots \oplus G_s$ where each G_i is a p_i−group. We consider two cases:

Case 1: $r(G_i) \leq 2$ for all i.

It is well known that we can write $G = \mathbb{Z}_v \oplus \mathbb{Z}_{mv}$. Then by Theorem 2 $D(G) = v + mv - 1$. The expression

$$\frac{4(D(G)-3)}{|G|} = \frac{4[v(1+m)-1-3]}{v^2 m}$$

is a decreasing function with respect to $m \geq 1$ and $v \geq 2$. Therefore

$$\frac{4(D(G)-3)}{|G|} \leq 1, \text{ for } v \geq 4.$$

For $v = 2$, $G = \mathbb{Z}_2 \oplus \mathbb{Z}_{2m}$. In the case $v = 3$, $G = \mathbb{Z}_3 \oplus \mathbb{Z}_{3m}$.

Case 2: $r(G_i) \geq 3$ for some $(1 \leq i \leq s)$.

In this case we can write $G = P \oplus H$, where P is a p−group with rank 3. When $P \notin \{\mathbb{Z}_{2^n} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3\}$ we have

$$\frac{D(G)}{|G|} \leq \frac{D(P)|H|}{|P||H|} = \frac{D(P)}{|P|} \leq \frac{1}{4}.$$

Otherwise the result holds using Lemma 5, Lemma 6 and Lemma 7. □

Corollary 1. *Let G be an abelian group of order n not in $\{\mathbb{Z}_n, \mathbb{Z}_2 \oplus \mathbb{Z}_{2m}, \mathbb{Z}_3 \oplus \mathbb{Z}_{3m}\}$. Then $ZS(G) \leq \frac{5n}{4} + 2$. Moreover equality holds only for the groups of the form $\mathbb{Z}_4 \oplus \mathbb{Z}_{4m}$.*

Proof. Directly apply Proposition 1 and the Gao Theorem. □

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