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Some Values of Olson's Constant

Algunos Valores de la Constante de Olson

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Abstract

Let G be a finite abelian group of order n. Erdös conjectured that every subset A of G with cardinality $|A| \geq \sqrt{2n}$ contains a nonempty subset with zero sum. The *Olson Constant* $Ol(G)$ is defined as the smallest k such that every subset of G of cardinality k contains a nonempty 0-sum subset. Olson's Constant is analogous to Davenport's Constant $D(G)$, but no repetitions of elements of G are allowed. In this work we give the values of $Ol(G)$ for some groups G. Thus we have $Ol(Z_2^n) = n + 1$ and $Ol(Z_3^n) = 2n + 1$ for $n \geq 3$. With some restrictions on the subsets of Z_3^n we obtain an example of a graph with 8 vertices, 16 edges and minimal degree 3, which contains no cubic subgraph (this was accomplished with the aid of a computer). In addition we supply a table with the values of $Ol(G)$ for all abelian groups G with order ≤ 55.
Key words and phrases: zero-sum set, finite abelian group, sequences without repetition.

Resumen

Sea G un grupo abeliano finito de orden n. Erdös conjeturó que todo subconjunto A de G con cardinalidad $|A| \geq \sqrt{2n}$ contiene un subconjunto no vacío de suma cero. La *Constante de Olson* $Ol(G)$ se define como el menor k tal que todo subconjunto de G de cardinalidad k contiene un subconjunto no vacío de suma cero. La Constante de Olson es análoga a la Constante de Davenport $D(G)$, pero no se permiten repeticiones de elementos de G. En este trabajo damos los valores de $Ol(G)$ para algunos grupos G. Así tenemos $Ol(Z_2^n) = n+1$ y $Ol(Z_3^n) = 2n + 1$ para $n \geq 3$. Con algunas restricciones on los subconjuntos de Z_3^n obtenemos un ejemplo de un grafo con 8 vértices, 16 aristas y grado

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mínimo 3, que no contiene ningún subgrafo cúbico (esto se logró con la ayuda de un computador). Adicionalmente proporcionamos una tabla con los valores de $Ol(G)$ para todos los grupos abelianos G de orden ≤ 55.
Palabras y frases clave: conjuntos de suma cero, grupos abelianos finitos, secuencias sin repetición.

1 Introduction

Let G be a finite abelian group. *Davenport's Constant* $D(G)$ is defined as the smallest integer k such that every sequence S with $|S| = k$ contains a nonempty subsequence with zero sum. Thus $D(G) - 1$ is the maximal possible length of a sequence that does not contain nonempty subsequences with zero sum. H. Davenport pointed out the connection between this constant and Algebraic Number Theory, where it is used to measure the maximal number of ideal classes which can occur in the decomposition of an irreducible element of a prime ideal in the ring of integers of a number field (Midwestern Conference on Group Theory and Number Theory, Ohio State Univ., 1966). Davenport's Constant has been used to obtain results in Graph Theory (see below). It has also been used to show that there exist infinitely many Carmichael numbers [1]. The value of $D(G)$ has been determined for various types of groups, in particular for groups of rank ≤ 2 and for p-groups:

Theorem 1 ([6]). *Let G be a finite abelian group. If (a) G is a p-group or (b) G has rank ≤ 2 then $D(G) = 1 + \sum_{i=1}^{r}(n_i - 1)$, where the n_i's are the invariant factors of G.*

Results (a) and (b) were proved independently by Olson and Kruyswijk. For groups of rank ≥ 4 the above formula is no longer valid, in general. For groups of rank 3, the problem still remains open.

In this work we need the following corollary:

Corollary 1. $D(Z_2^n) = n + 1$, $D(Z_3^n) = 2n + 1$.

We'll now study injective sequences of elements of G (this is the same as considering subsets of G).

Definition 1. Let G be a finite abelian group. The *Olson Constant* $Ol(G)$ is the minimal k such that every subset $A \subset G$, with $|A| = k$, contains a nonempty subset with zero sum.

$Ol(G)$ is the analog of $D(G)$, with no repetitions of elements of G allowed. The name of this constant was proposed in 1994, during a seminar held at *Universidad Central de Venezuela* (Caracas), as a tribute to Olson and his work on this subject.

Problem 1. *Given a finite abelian group G, determine $Ol(G)$.*

One of the first results on this problem is due to Szmerédi [12], who proved a conjecture of Erdös and Heilbron, namely that a constant c exists such that if G is an abelian group of order n, $S \subset G$ and $|S| \geq c\sqrt{n}$, then zero is represented as a sum of distinct elements of S. Erdös conjectured that the above statement holds for $c = \sqrt{2}$ (see [5, p. 95] for related questions). Olson [11] showed that the conjecture holds with $c = 3$ (working with not neccesarily abelian groups, allowing rearrangements of the elements of S). When G is a cyclic group of prime order, Olson [10] showed that the constant c can be relaxed to $c = 2$. In this case not only zero but every element of the group can be obtained as a sum of elements of S. Further results in this direction may be found in [4]. The last results on Olson's constant are in [8], where Hamidoune and Zémor show that $Ol(Z_p) \leq \lceil \sqrt{2p} + 5\ln(p) \rceil$, for p prime. For arbitrary abelian groups G they proved that $Ol(G) \leq \lceil \sqrt{2|G|} + \varepsilon(|G|) \rceil$, where $\varepsilon(n) = \mathcal{O}(n^{1/3} \ln n)$.

2 Some Results

Proposition 1. $Ol(Z_2^n) = D(Z_2^n) = n + 1$.

Proof. This is a consequence of Corollary 1 and the fact that in Z_2^n a sequence with some repetition contains a nonempty 0-sum subset (since $x + x = 0$). □

In a conference in Lisbon, in 1995, Hamidoune [7] proposed the following:

Question 1. Let V be a vector space over Z_3 with dimension n. If $E \subset V$ and $|E| \geq 2n$, does E necessarily contain a nonempty subset with sum 0?

As we shall see later, an afirmative answer to Question 1 would readily imply Berge-Sauer's Conjecture. We thought at that moment that an affirmative answer was plausible since $D(Z_3^n) = 2n + 1$, and injective sequences are less abundant than arbitrary sequences; thus it seemed possible to relax $2n + 1$ to $2n$. Hence we refered to Question 1 as:

Conjecture 1. *Let $E \subset V$, where V is a vector space over Z_3 with dimension n. If $|E| \geq 2n$ then E contains a nonempty subset with zero sum.*

Later we wrote a computer program to check some problems about 0-sum and related questions. When Conjecture 1 was checked unfortunately it came out to be false. We found a counterexample in dimension 3:

Counterexample 1. *Let* $E = \{(100), (010), (110), (001), (101), (211)\} \subset Z_3^3$. *Then* E *does not contain any nonempty 0-sum set.*

After this Hamidoune asked us (in a private communication) what would happen if we permit 0-1 components only. At this respect we found:

Counterexample 2. *Let* $E = \{e_1, e_2, e_3, e_4, e_1 + e_2, e_1 + e_3, e_1 + e_4, e_1 + e_2 + e_3 + e_4\}$, *where* $\{e_1, e_2, e_3, e_4\}$ *is the canonical basis of* Z_3^4. *Then* E *does not contain any nonempty 0-sum set.*

Later Hamidoune asked for subsets of vectors with exactly two 1-components (and the remaining components equal to zero). And we found, with more work and some luck, a counterexample in dimension 9:

Counterexample 3. *Let* (ij) *denote the vector* $e_i + e_j$ *in* Z_3^9, $1 \le i, j \le 9$. *Let* $E = \{(12), (13), (14), (15), (16), (17), (18), (19), (23), (24), (35), (36), (37), (38), (45), (49), (69), (79)\} \subset Z_3^9$. *Then* $|E| = 18$ *and* E *does not contain any nonempty 0-sum set.*

Counterexample 3 was the first encountered, but actually there is an easier one in dimension eigth:

Counterexample 4. *Let* (ij) *denote the vector* $e_i + e_j$ *in* Z_3^8, $1 \le i, j \le 8$. *Let* $E = \{(12), (13), (14), (15), (16), (17), (18), (23), (24), (25), (26), (37), (47), (58), (68), (78)\} \subset Z_3^8$. *Then* $|E| = 16$ *and* E *does not contain any nonempty 0-sum set.*

From these counterexamples we can obtain exact values of Olson's constant and some of its variations for the groups Z_3^n.

Proposition 2. $Ol(Z_3) = 2$, $Ol(Z_3 \oplus Z_3) = 4$, $Ol(Z_3^n) = 2n+1$ *for all* $n \ge 3$.

Proof. The values of $Ol(Z_3^n)$ for $n = 1$ and $n = 2$ are given in Table 1. We know that $D(Z_3^n) = 2n + 1$ (Corollary 1) because Z_3^n is a p-group. Then $Ol(Z_3^n) \le D(Z_3^n) = 2n + 1$. But for $n = 3$, Counterexample 1 shows that $Ol(Z_3^3) \ge 7$. Then $Ol(Z_3^3) = 7$. Let $n \ge 3$ and suppose that $E \subset Z_3^n$, $|E| = 2n$ and E does not contain any nonempty 0-sum subset. Then $E \cup \{e_{n+1}, e_1 + e_{n+1}\}$ cannot contain any nonempty 0-sum subset. It follows that $Ol(Z_3^{n+1}) \ge 2(n + 1) + 1$. Hence $Ol(Z_3^{n+1}) = 2(n + 1) + 1$. □

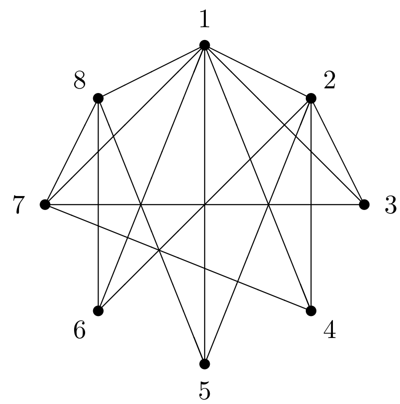

Figure 1: Graph corresponding to Counterexample 4.

Let us denote by $Ol_1(G)$ the analog of $Ol(G)$ when only 0-1 components are valid in the subsets, and $Ol_2(G)$ when only two 1's are valid and all other components are zero.

Remark 1. Of course $Ol_2(Z_3^n)$ is defined only for $n \geq 4$.

Obviously we have $Ol_2(Z_3^n) \leq Ol_1(Z_3^n) \leq Ol(Z_3^n)$.

Proposition 3. $Ol_1(Z_3^n) = 2n+1$ *for* $n \geq 4$.

Proof. Counterexample 1 shows that $Ol_1(Z_3^4) \geq 9$. Then $Ol_1(Z_3^4) = 9$, by Corollary 1. Suppose $n \geq 4$ and $E \subset Z_3^n$, $|E| = 2n$, such that E does not contain a nonempty 0-sum subset. Then $E \cup \{e_1 + e_{n+1}, e_{n+1}\}$ cannot contain a nonempty 0-sum subset. $\square$

With respect to $Ol_2(Z_3^n)$ we have $Ol_2(Z_3^4) = 6$, of course. With the aid of a computer we found the values $Ol_2(Z_3^5) = 9$, $Ol_2(Z_3^6) = 11$ and $Ol_2(Z_3^7) = 14$.

Proposition 4. $Ol_2(Z_3^n) = 2n+1$, *for all* $n \geq 8$.

Proof. By Corollary 1 we know that $Ol_2(Z_3^n) \leq 2n+1$. Counterexample 4 shows that $Ol_2(Z_3^8) \geq 17$. Suppose that E is a subset of Z_3^n with k elements, two 1's components and all other components equal to 0, and without nonempty 0-sum subsets. Then $E_1 = E \cup \{e_1 + e_{n+1}, e_2 + e_{n+1}\}$ cannot contain nonempty 0-sum subsets, and $|E_1| = k+2$. $\square$

Type of G	$Ol(G)$	Type of G	$Ol(G)$	Type of G	$Ol(G)$
2	2	22	7	38	9
3	2	23	7	39	9
4	3	24	7	40	9
2 2	3	2 12	7	2 20	9
5	3	2 2 6	7	2 2 10	9
6	4	25	8	41	9
7	4	5 5	7	42	10
8	4	26	8	43	9
2 4	4	27	8	44	10
2 2 2	4	3 9	7	2 22	10
9	5	3 3 3	7	45	10
3 3	4	28	8	3 15	10
10	5	2 14	8	46	10
11	5	29	8	47	10
12	5	30	8	48	10
2 6	5	31	8	2 24	10
13	5	32	8	2 2 12	10
14	6	2 16	8	2 2 2 6	9
15	6	2 2 8	8	4 12	10
16	6	2 2 2 4	7	49	10
2 8	6	2 2 2 2 2	6	7 7	10
2 2 4	6	4 8	8	50	10
2 2 2 2	5	2 4 4	8	5 10	10
4 4	6	33	8	51	10
17	6	34	9	52	11
18	6	35	9	2 26	11
3 6	6	36	9	53	10
19	6	2 18	9	54	11
20	7	3 12	9	3 18	11
2 10	7	6 6	9	3 3 6	10
21	7	37	9	55	11

Table 1: $Ol(G)$ for abelian groups.

Order	56	57	58	59	60	61	62	63	64
$Ol(G)$	11	11	11	11	11	11	12	11	12

Table 2: $Ol(G)$ for cyclic groups.

Later we realized that vectors with two 1-components can be considered as the edges of a symmetric simple graph with n nodes. The graph corresponding to Counterexample 4 is shown in Figure 1.

A 4-regular graph of n nodes has $2n$ edges. A 3-regular subgraph is obtained from a 0-sum subset of the $2n$ edges. Then Conjecture 1 implies the Berge-Sauer's conjecture. In 1984 Alon, Friedland and Kalai [2] used this fact to prove that if one adds an edge to a 4-regular graph (possibly with multiple edges), then the resultant graph contains a cubic graph. In any case the Berge-Sauer's conjecture was proved by Zhang in 1985 (cited by Locke, [9]). Another proof (by Taskinov) is cited in [7].

3 A table of *Ol(G)* for small values of $|G|$

A table with the values of $Ol(G)$ (actually of $Ol(G) - 1$) for cyclic groups G with $|G| \leq 50$, due to Devitt and Lam, is mentioned in [5] (as personal communication). A table with $Ol(G)$ and other related constants for abelian groups with orders up to 22, due to the author, appeared in [3]. Later we constructed Table 2, for cyclic groups with orders up to 64. Finally we constructed Table 1 for arbitrary abelian groups with orders up to 55. Let us recall that an abelian group G is determined (up to isomorphisms) by its invariant factors n_i, which are integers such that $1 < n_1|n_2|...|n_r$ and $G \approx Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_r}$. The ordered set $n_1, n_2, \ldots, n_r$ is called the *type* of G, and r is the *rank* of G.

As one can see the mentioned Erdös' conjecture seems sharp, specially for cyclic groups. We propose the following:

Conjecture 2. *If G is a finite abelian group of order n then $Ol(G) \leq Ol(Z_n)$.*

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