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Characterizations of Seminorms Given by Continuous Linear Functionals on Normed Linear Spaces and Applications to Gel'fand Measures

Caracterizaciones de Seminormas dadas por Funcionales Lineales Continuas en Espacios Normados y Aplicaciones a las medidas de Gel'fand

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Abstract

The investigations made in the papers [1] and [2] conducted us to consider the problem of characterizing operators of rank less than one in a Hilbert space. The purpose of this note is to show how this problem may be solved by characterizing the seminorms given (in a normed linear space) by continuous linear functionals. One of our results obtained in this respect may be described as follows: let $(E, \|.\|)$ be a normed linear space over the field $K = \mathbb{R}$ or $\mathbb{C}$, and set $U := \{\lambda \in K : |\lambda| = 1\}$. Then a seminorm q on E is of the type $q = |f|$, for some continuous linear functional f on E, if and only if there exists a closed subset M of E containing 0 such that

$$|q(x)| = |q(y)| \Longleftrightarrow x \in Uy + M \qquad (x, y \in E). \qquad (*)$$

In connection with this result, we give some characterizations for rank one orthogonal projections in Hilbert spaces and provide some applications to Gel'fand measures.

Key words and phrases: seminorms, Hahn-Banach theorem, Gel'fand pairs, Gel'fand measures, orthogonal projections and rank one operators in Hilbert spaces.

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Resumen

Las investigaciones realizadas en [1] y [2] nos condujeron a considerar el problema de caracterizar los operadores de rango menor que uno en un espacio de Hilbert. El propósito de esta nota es mostrar como este problema puede ser resuelto caracterizando las seminormas (en un espacio normado) dadas por funcionales lineales continuas. Uno de nuestros resultados obtenido al respecto puede ser descripto como sigue: Sea $(E, \|.\|)$ un espacio normado sobre $K = \mathbb{R}$ o $\mathbb{C}$, y pongamos $U := \{\lambda \in K : |\lambda| = 1\}$. Entonces una seminorma q sobre E es del tipo $q = |f|$, para alguna funcional lineal continua f sobre E, si y sólo si existe un subconjunto cerrado M de E que contiene al 0 y tal que

$$|q(x)| = |q(y)| \Longleftrightarrow x \in Uy + M \qquad (x, y \in E). \qquad (*)$$

En conexión con este resultado damos algunas caracterizaciones para las proyecciones ortogonales de rango uno en espacios de Hilbert, y aportamos algunas aplicaciones a las medidas de Gel'fand.
Palabras y frases clave: seminormas. teorema de Hahn-Banach, pares de Gel'fand, medidas de Gel'fand, proyecciones ortogonales y operadores de rango uno en espacios de Hilbert.

1 Introduction and statement of the main theorem

1.1. Throughout this paper E will be a normed linear space over the field $K = \mathbb{R}$ or $\mathbb{C}$, endowed with a norm $\|.\|$. We set $U := \{\lambda \in K : |\lambda| = 1\}$. We denote by S_E the set of vectors in E with norm one. The distance of a vector x to a subset N of E is denoted by $d(x, N)$, and the linear subspace spanned by N is denoted by Span (N). The main result of this note is the following theorem:

1.2. Theorem: *Let $q \neq 0$ be seminorm on the normed linear space E. Then the following assertions are equivalent:*

(1) There exists a closed subset M of E containing 0 such that

$$|q(x)| = |q(y)| \Longleftrightarrow x \in Uy + M \qquad (x, y \in E). \qquad (*)$$

(2) There exists a closed linear subspace M of E such that

$$|q(x)| = |q(y)| \Longleftrightarrow x \in Uy + M \qquad (x, y \in E). \qquad (*)$$

(3) There exists a continuous linear functional f on E such that $q = |f|$.
As a consequence, if q verifies (1) then it must be continuous.

This note is organized as follows. In the next section we give a proof of Theorem 1.2. In the third section we discuss some characterizations of rank one orthogonal projections in Hilbert spaces in connection to Theorem 1.2. In the last section we gather some characterizations of Gel'fand measures. We recall that the notion of Gel'fand measure was introduced in the papers [1] and [2] to generalize the concept of Gel'fand pair (see for example [4]).

2 Proof of Theorem 1.2

We only have to prove (2) $\Longrightarrow$ (3). If (2) is true then, by property (*), M coincides with the set $\{x \in E : q(x) = 0\}$. Therefore q is a continuous seminorm. Fix $x_0 \in E \setminus M$ with $\|x_0\| = 1$. For each $y \in E$ we can find $t \geq 0$ such that

$$|q(y)| = |q(tx_0)|.$$

By (*) we deduce that $y \in \mathrm{Span}\,(\{x_0\} \cup M)$. Hence, M is a closed hyperplane of E. By the Hahn-Banach theorem, we can find a continuous linear functional u on E such that

$$u(x_0) = 1, \quad \mathrm{Ker}(u) = M, \quad \text{and} \quad \|u\| = frac1d(x_0, M).$$

It follows that $|u(x)| = \|u\|\, d(x, M)$ for all $x \in E$. Next we shall prove that the seminorms q and $d(., M)$ are proportional. To this end, let $\sigma := \mathrm{Sup}\,\{q(x) : \|x\| = 1\}$. Then we have $0 < \sigma < \infty$ and for all $x \in E$ and all $m \in M$, we have the inequality

$$|q(x)| = |q(x - m)| \leq \sigma \, \|x - m\| \, ,$$

which yields $q(x) \leq \sigma d(x, M)$ for all $x \in E$. It remains to prove the inverse inequality. Let us fix a vector x in $E \setminus M$ and set $E_x := \{y \in E : q(y) = q(x)\}$. By assumptions, we have $E_x = Ux + M$. It is not hard to see that $d(x, M) = d(0, E_x)$.

Now, let B be the open ball having zero as centre and $d(x, M)$ as radius. Then, one can see easily that $B \cap E_x = \varnothing$. Let ϵ be any positive number verifying $0 < \epsilon < \sigma$. Then one can find a unit vector $z \in \mathrm{S}_E$ such that $\sigma - \epsilon \leq q(z)$. Let us put

$$\delta := \frac{q(x)}{q(z)}, \quad \text{and} \quad z_0 := \delta z.$$

We see that $z_0 \in E_x$ and therefore z_0 does not belong to the ball B, and then, the following inequalities hold

$$d(x,M) = d(0,E_x) \leq \|z_0\| = \frac{q(x)}{q(z)} \leq \frac{q(x)}{\sigma - \epsilon}.$$

These inequalities lead us to say that $\sigma d(x,M) \leq q(x)$. We conclude that $q = \sigma d(.,M)$. Therefore $q = |f|$, where f is the continuous linear functional given by $f = \sigma u/\|u\|$. □

3 On rank one orthogonal projections in Hilbert spaces

Let H be a Hilbert space. Let $<,>$ be its inner product and let $\|.\|$ be the associated norm. $\mathcal{L}(H)$ is the algebra of all bounded linear operators on H. The following theorem is connected to Theorem 1.2, and provides some characterizations for rank one orthogonal projections in $\mathcal{L}(H)$.

3.1. Theorem: *Let $A \in \mathcal{L}(H) \setminus \{0\}$ be a hermitian operator on H, and set $P := A^2/\|A^2\|$. Then the following assertions are equivalent:*

(1) P is an orthogonal projection of rank one in the Hilbert space H.

(2) For each integer $n \geq 1$, there exists a linear subspace M_n of H such that
$\{|< A^n(x) \mid x >| = |< A^n(y) \mid y >| \Longleftrightarrow x \in Uy + M_n \qquad (x, y \in H).$ *(*)*

(3) There exists an integer $m \geq 1$ and a linear subspace M_m of H such that
$|< A^m(x) \mid x >| = |< A^m(y) \mid y >| \Longleftrightarrow x \in Uy + M_m \qquad (x, y \in H).$ (*)

(4) There exists an integer $k \geq 1$ such that A^k is a rank one operator in H.

(5) $ASATA = ATASA$, for all operators $S, T \in \mathcal{L}(H)$ of rank one.

Proof. The assertions (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) are straightforward. Suppose that (3) is true. Then M_m must be equal to the set $\{x \in H :< A^m(x) \mid x >= 0\}$, so that M_m is closed. Fix $y \in M_m^\perp$ with $\|y\| = 1$. For each $z \in H$, we can find $t \geq 0$ such that $|< A^m(z) \mid z >| = |< A^m(ty) \mid ty >|$. It follows from (*) that $z \in \mathrm{Span}\,(\{y\} \cup M_m)$. We deduce that M_m has codimension one. Now for all $v \in M_m$, we have

$$\begin{aligned} |< A^m(y) \mid y >| &= |< A^m(y+v) \mid y+v >| \\ &= |< A^m(y) \mid y > +2\mathrm{Re}\,(< A^m(y) \mid v >)| \,. \end{aligned}$$

Hence $A^m(y) \in M_m^\perp = \mathrm{Span}\,(\{y\})$. We conclude that M_m coincides with the kernel of A^m. It follows that A^m is a rank one operator. Hence (3) $\Longrightarrow$ (4) is proved. It is clear that (4) $\Longrightarrow$ (5), and (5) $\Longrightarrow$ (1) are true. □

In connection with this result, we have the following

3.2 Proposition: *Let $A \in \mathcal{L}(H)$ be an idempotent operator on H such that*

(i) $\|A\| = 1$*, and*

(ii) $ASATA = ATASA$*, for all operators* $S, T \in \mathcal{L}(H)$ *of rank one.*

Then A is an orthogonal projection of rank one in the Hilbert space H.

Proof. One has only to prove that $A = A^*$. Condition (ii) implies that A is a rank one operator. Thus we get $A = E_{\xi,\eta}$, for some vectors $\xi, \eta \in H \setminus \{0\}$, where $E_{\xi,\eta}(v) :=< v \mid \eta > \xi$, $(\forall v \in H)$. Condition (i) implies that $1 = \|\xi\| \|\eta\|$. One can write in a unique manner $\eta = a\xi + \eta_0$, where $a = 1/\|\xi\|^2$ and η_0 is the vector verifying $< \xi \mid \eta_0 >= 0$. Therefore we obtain $1 = \|\xi\|^2 \|\eta\|^2 = \|\xi\|^2 [a^2 \|\xi\|^2 + \|\eta_0\|^2] = 1 + \|\xi\|^2 \|\eta_0\|^2$, which gives $\eta_0 = 0$, and $A = \frac{1}{\|\xi\|^2} E_{\xi,\xi} = A^*$. □

4 Applications to Harmonic Analysis

4.1 In all this section G is a topological locally compact group (not necessarily unimodular) endowed with a fixed left Haar measure dx, and modulus function Δ. The algebra of all regular and bounded measures on G will be denoted by $M_1(G)$. We denote by $L_1(G)$ the *-Banach algebra of (all class of complex) integrable functions on G. For any function f on G, we set $L_x f(y) := f(x^{-1}y)$ and $R_x f(y) := f(yx)$, $(x, y \in G)$.

4.2 Let $\mu \in M_1(G) \setminus \{0\}$ be a fixed measure such that $\mu = \mu * \mu = \mu^*$. We put $f^\mu := \mu * f * \mu$, $\forall f \in L_1(G)$, and set $L_1^\mu(G) := \mu * L_1(G) * \mu = \{f^\mu : f \in L_1(G)\}$. It is a closed subalgebra of the *-Banach algebra $L_1(G)$. We recall, (see [1] and [2]), that μ is a Gel'fand measure, if $L_1^\mu(G)$ is commutative (under the convolution). When $\mu = dk$ is the normalized Haar measure of a compact soubgroup k of G, then (G, K) is a Gel'fand pair, (see [4]), if and only if dk is a Gel'fand measure. We denote by $\hat{G}$ the set of all (classes of) continuous, unitary and irreducible representations of G (see for example [5] and [6]). We recall that for each representation $\pi \in \hat{G}$ in the Hilbert space H_π, the operator $\pi(\mu)$ is defined by

$$< \pi(\mu)v \mid w >:= \int_G < \pi(t)v \mid w > \, d\mu(t), \; (v, w \in H_\pi).$$

The previous sections could be used to provide new characterizations for Gel'fand measures. More precisely, we have the following

4.3 Theorem: *Let G be topological locally compact group, and let $\mu \in M_1(G)\setminus \{0\}$ be a fixed measure such that $\mu = \mu * \mu = \mu^*$. Then the following assertions are equivalent:*

(1) μ is a Gel'fand measure.

(2) $\pi(\mu)$ has rank ≤ 1, for all $\pi \in \hat{G}$.

(3) The linear space $\pi(L_1^\mu(G))$ has dimension ≤ 1, for all $\pi \in \hat{G}$.

(4) The linear space $\pi(M_1^\mu(G))$ has dimension ≤ 1, for all $\pi \in \hat{G}$.

(5) The algebra $M_1^\mu(G)$ is commutative.

(6) $[L(x)f]^\mu = \Delta(x^{-1})[R(x^{-1})f]^\mu$, for every $f \in L_1^\mu(G)$ and all $x \in G$.

(7) For every representation $\pi \in \hat{G}$, in the Hilbert space H_π, there exists a linear subspace M_π of H_π such that

$$|< \pi(\mu)(x) \mid x >| = |< \pi(\mu)(y) \mid y >| \Longleftrightarrow x \in Uy + M_\pi, \quad (x, y \in H_\pi) \text{ (*)}$$

(8) For every representation $\pi \in \hat{G}$, in the Hilbert space H_π, there exists a subset M_π containing zero such that

$$|< \pi(\mu)(x) \mid x >| = |< \pi(\mu)(y) \mid y >| \Longleftrightarrow x \in Uy + M_\pi, \quad (x, y \in H_\pi) \text{ (*)}$$

(9) $\pi(\mu)S\pi(\mu)T\pi(\mu) = \pi(\mu)T\pi(\mu)S\pi(\mu)$, for every representation $\pi \in \hat{G}$, in the Hilbert space H_π, and all operators $S, T \in \mathcal{L}(H_\pi)$ of rank one.

Proof. The equivalence between the assertions (1), (2), (3), (4) and (5) is proved in [1] and [2]. The equivalence between the assertions (7), (8) and (9) is a consequence from sections one, two and three. The equivalence between the assertions (1) and (6) is a consequence of a density argument and the following identities:

$$\begin{aligned} f * \mu * \delta_x * \mu &= \Delta(x^{-1})[R(x^{-1})(f * \mu)] * \mu, \\ \mu * \delta_x * \mu * f &= \mu * [L(x)(\mu * f)], \end{aligned}$$

valid for all $f \in L_1(G)$ and all $x \in G$, where δ_x designates the Dirac measure concentrated at the point x. □

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