Divulgaciones Matemáticas Vol. 9 No. 2(2001), pp. 137–147

Remarks on Intuitionistic Modal Logics

Observaciones sobre Lógicas Modales Intuicionistas

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Abstract

This paper is devoted to study an extension of intuitionistic modal logic introduced by Fischer-Servi [6] by means of Lemmon-Scott axiom. We shall prove that this logic is canonical.
Key words and phrases: modal logic, intuitionistic logic, intuitionistic modal Logic.

Resumen

Este trabajo se dedica a estudiar una extensión de la lógica modal intuicionista introducida por Fischer-Servi [6] por medio del axioma de Lemmon-Scott. Se prueba que esta lógica es canónica.
Palabras y frases clave: lógica modal, lógica intuicionista, lógica modal intuicionista,

1 Introduction

Edwald [5], Fischer-Servi [6] and Plotkin and Stirling [10] (see also [1] and [11]) introduced independently an intuitionistic modal logic, called **IK**, with two modal operators $\Box$ and $\Diamond$. The relational semantic for **IK** is represented by triples of type $\langle X, \leq, R\rangle$ where $\leq$ is a quasi-ordering on X and R is an accessibility relation, such that $(\leq^{-1} \circ R) \subseteq (R\circ \leq^{-1})$ and $(R\circ \leq) \subseteq (\leq \circ R)$. Fischer-Servi studies several extensions for **IK**, by means of axioms like $\Box\varphi \to \varphi$, and their duals $\varphi \to \Diamond\varphi$, but she does not study extensions with only one axiom, for example $\Box\varphi \to \varphi$, or $\varphi \to \Diamond\varphi$. Since the modal

Recibido 2000/04/28. Revisado 2001/05/04. Aceptado 2001/05/07.
MSC (2000): Primary 03B45.

operators are independent, in the sense that $\Box$ is not defined in terms of $\Diamond$, and reciprocally, $\Diamond$ is not defined in terms of $\Box$, we can give extensions of **IK** such as **IK** $+ \{\varphi \to \Diamond\varphi\}$ such that they are complete. Recently, in [12] F. Wolter and M. Zakharyaschev, studied some intuitionistic modal logics weaker than **IK** and show that some extensions of these logics by means of the axioms $\Diamond^m\Box^n p \to \Box^k\Diamond^l p$ are canonical. On the other hand, there exists a general modal schema discovered by Lemmon and Scott that contains, as a particular instance, many of the best known modal formulas. This formula was characterized by R. Goldblatt by means of a first-order condition. The purpose of the present work is to study an extension of the logic **IK** by means of a similar formula. We shall give a of first-order condition for this formula.

In the next section, the preliminaries, we shall recall the basic notions of the logic **IK**. Section 3 deals with the Kripke semantics for the extensions of **IK** by means of the Lemmon-Scott axiom. Section 4 is devoted to the proof that this logic is canonical.

2 Preliminaries

The language of propositional modal logic that we assume in the paper has the connectives $\{\wedge, \vee, \to, \Box, \Diamond\}$ and has in addition one propositional constant $\bot$. The set of propositional variables is denoted by Var. The negation $\neg$ and the constant $\top$ are defined by $\neg p = p \to \bot$ and $\top = \neg\bot$, respectively. Fm will denote the set of formulas.

The intuitionistic modal logic **IK** is the logic with the following sets of axioms and the following rules:

1. Any axiomatization of the Intuitionistic Propositional Calculus (IPC).
2. $(\Box\varphi \wedge \Box\psi) \to \Box\varphi \wedge \Box\psi$
3. $\Diamond(\varphi \vee \psi) \to \Diamond\varphi \vee \Diamond\psi$
4. $\Box\top$
5. $\neg\Diamond\bot$
6. $\Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi)$
7. $(\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi)$
8. $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

9. $\dfrac{\varphi \to \psi}{\Box\varphi \to \Box\psi}$

10. $\dfrac{\varphi \to \psi}{\Diamond\varphi \to \Diamond\psi}$

The Kripke semantics for **IK** is represented by the relational structures $\mathcal{F} = \langle X, \leq, R\rangle$ where $\leq$ is a quasi-ordering on X, that is, a binary reflexive and transitive relation on X, R is a binary relation on X, and the following two conditions are held:

(1) $(R \circ \leq) \subseteq (\leq \circ R)$

(2) $(\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$,

where $\circ$ denotes the composition between binary relations.

Let $\mathcal{F} = \langle X, \leq, R\rangle$ be a frame. For $Y \subseteq X$, we put $[Y) = \{x \in X : y \leq x$, for some $y \in Y\}$ and $(Y] = \{x \in X : x \leq y$, for some $y \in Y\}$. A subset Y of X is *increasing* if $Y = [Y)$ and is decreasing if $Y = (Y]$. The sets of all increasing sets of X will be denoted by $\mathcal{P}_i(X)$. We define two relations that will be very important in the rest of this work. Let $R_\Box = \leq \circ R$ and $R_\Diamond = R \circ \leq^{-1}$. These relations are fundamental in the analysis of extensions of **IK**. For $x \in X$, we denote $R(x) = \{y \in X : (x, y) \in R\}$. For $Y \subseteq X$, we write $Y^c = X - Y$.

A *valuation* on a frame $\mathcal{F}$ is a function $V : Var \to \mathcal{P}_i(X)$. All valuation V can be extended recursively to Fm by means of the following clauses:

1. $V(\bot) = \emptyset$,
2. $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$,
3. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$,
4. $V(\varphi \to \psi) = \{x \in X : [x) \cap V(\varphi) \subseteq V(\psi)\}$,
5. $V(\Box\varphi) = \{x \in X : R_\Box(x) \subseteq V(\varphi)\} = \Box_{R_\Box}(V(\varphi))$, and
6. $V(\Diamond\varphi) = \{x \in X : R(x) \cap V(\varphi) \neq \emptyset\} = \Diamond_R(V(\varphi))$.

We note that $V(\Diamond\varphi) = \{x \in X : R_\Diamond(x) \cap V(\varphi) \neq \emptyset\}$. Indeed, suppose that $R_\Diamond(x) \cap V(\varphi) \neq \emptyset$. Then there exist $y, z \in X$ such that $y \in R(x)$, $z \leq y$ and $z \in V(\varphi)$. Since $V(\varphi) \in \mathcal{P}_i(X)$, $y \in V(\varphi)$. Thus $R(x) \cap V(\varphi) \neq \emptyset$. The other direction follows by the reflexivity of $\leq$.

We define the semantic notions of truth and validity in a model and validity in a frame for formulas.

Given a model $\langle \mathcal{F}, V \rangle$ and a point $x \in X$ we say that a formula φ is *true* at x in $\langle \mathcal{F}, V \rangle$, in symbols $\langle \mathcal{F}, V \rangle \vDash_x \varphi$, if $x \in V(\varphi)$. A formula φ is *valid in a model* $\langle \mathcal{F}, V \rangle$, in symbols $\langle \mathcal{F}, V \rangle \models \varphi$, if it is true at every point in X. A formula φ is *valid in a frame* $\mathcal{F}$, in symbols $\mathcal{F} \models \varphi$, if for any valuation V on $\mathcal{F}$, φ is valid in the model $\langle \mathcal{F}, V \rangle$.

Let $\mathcal{I}$ be any modal logic that is an extension of **IK**. We will denote by $\mathrm{Fr}(\mathcal{I})$ the class of all frames where every formula of $\mathcal{I}$ is valid. Now let F be a class of frames. $Th(\mathsf{F})$ denotes the class of all formulas that are valid in every frame in F. A modal logic $\mathcal{I}$ is *characterized* by a class F of frames, or it is *complete* relative to a class F of frames, F-*complete* for short, if $Th(\mathsf{F}) = \mathcal{I}$.

Let us use the following notation. Let $\varphi \in Fm$. Then we shall write $\Box^0\varphi = \varphi$, $\Diamond^0\varphi = \varphi$, $\Box^{n+1}\varphi = \Box\Box^n\varphi$ and $\Diamond^{n+1}\varphi = \Diamond\Diamond^n\varphi$.

Let R be a relation on a set X. Let us define R^n recursively by: R^0 is the identity on X and $R^{n+1} = R^n \circ R$.

Lemma 1. *Let $\mathcal{F} = \langle X, \leq, R \rangle$ be a frame. Then*

1. $\leq^{-1} \circ R^n \subseteq R^n \circ \leq^{-1}$.
2. $R^n \circ \leq \subseteq \leq R^n$.
3. $R^n_\Diamond = R^n \circ \leq^{-1}$.
4. $R^n_\Box = \leq \circ R^n$.

Proof. 1. By induction on n. Suppose that 1. is valid for n and let $x, y, z \in X$ such that $x \leq^{-1} y$ and $(y, z) \in R^{n+1}$. Then there exists $z_1 \in X$ such that $(y, z_1) \in R^n$ and $(z_1, z) \in R$. By inductive hypothesis we get $(x, z_1) \in \leq^{-1} \circ R^n \subseteq R^n \circ \leq^{-1}$. It follows that there exists $w \in X$ such that $(x, w) \in R^n$ and $z_1 \leq w$. Since $(z_1, z) \in R$, we have $(w, z) \in \leq^{-1} \circ R \subseteq R \circ \leq^{-1}$. Then there exists $k \in X$ such that $(w, k) \in R$ and $z \leq w$. Since $(x, w) \in R^n$, then $(x, w) \in R^{n+1} \circ \leq^{-1}$.

The proof of 2. is similar, and 3. and 4. follow from 1. and 2., respectively. □

Lemma 2. *Let $\mathcal{F} = \langle X, \leq, R \rangle$ be a frame. Then for any $x \in X$, $R^n_\Box(x)$, $(R^n_\Diamond(x))^c \in \mathcal{P}_i(X)$.*

Proof. Let $a \leq b$ and $(x, a) \in R^n_\Box$. Then there exists $c \in X$ such that $x \leq c$ and $(c, a) \in R^n$. Then $(c, b) \in R^n \circ \leq$, and by 2. of Lemma 1, there exists $w \in X$ such that $c \leq w$ and $(w, b) \in R^n$. Since $x \leq c \leq w$, we get $(x, b) \in R^n_\Box$.

The proof of $(R^n_\Diamond(x))^c \in \mathcal{P}_i(X)$ is similar. □

Corollary 3. *Let $\mathcal{F} = \langle X, \leq, R\rangle$ be a frame. Let $x \in X$. Let V, V' be the functions defined by:*

1. $V(p) = R^n_\Box(x)$ *and*

2. $V'(p) = (R^n_\Diamond(x))^c$.

each variable p and an $n \geq 0$. Then V. and V' are valuations.

Proof. It is immediate by Lemma 2. □

3 Lemon-Scott axiom

In this section we extend the modal logic **IK** with the *Lemmon-Scott* axiom (LS). This is a natural generalization of the Lemmon-Scott axiom of classical modal logic, which has been characterized by R. Goldblatt in [7]. We shall adapt the techniques given by Goldblatt to our case. It is known that the LS axiom cover many known modal formulas, as for example the axiom $\Diamond^m\Box^n p \to \Box^k\Diamond^l p$.

We shall say that a formula α is positive if it can be constructed using no connectives other than $\vee, \wedge, \Box, \Diamond$. Let $\alpha(p_1, p_2, \ldots .p_n)$ be a positive formula, where $p_1, p_2, \ldots .p_n$ are the variables occurring in α. The formula obtained by uniformly substitutions, for each $t \leq i \leq k$, the formula ψ_i for p_i in α is the formula $\alpha(\psi_1, \psi_2, \ldots .\psi_n)$.

Let $\alpha(p_1, p_2, \ldots .p_n)$ be a positive formula and let us consider $\vec{n} = (n_1, \ldots, n_k)$ and $\vec{m} = (m_1, \ldots, m_k)$, where $n_i,\ m_i \in \mathbb{N}$. Let $\mathcal{F} = \langle X, \leq, R\rangle$ be a frame and let us consider $\vec{t} = (t_1, \ldots, t_k)$, with $t_i \in X$.

Let $x \in X$. We shall define a first-order condition $R_\alpha(x, \vec{t}, \vec{n})$ on the frame $\mathcal{F}$ by recursion as follows:

$$R_{p_i}(x, \vec{t}, \vec{n}) \quad \Leftrightarrow \quad (t_i, x) \in R^{n_i}_\Box, \quad i \leq k, \quad p_i \in Var,$$

$$R_{\alpha\wedge\beta}(x, \vec{t}, \vec{n}) \quad \Leftrightarrow \quad R_\alpha(x, \vec{t}, \vec{n}) \ \wedge R_\beta(x, \vec{t}, \vec{n})$$

$$R_{\alpha\vee\beta}(x, \vec{t}, \vec{n}) \quad \Leftrightarrow \quad R_\alpha(x, \vec{t}, \vec{n}) \ \vee \ R_\beta(x, \vec{t}, \vec{n})$$

$$R_{\Box\alpha}(x, \vec{t}, \vec{n}) \quad \Leftrightarrow \quad \forall y((x, y) \in R_\Box \Rightarrow R_\alpha(y, \vec{t}, \vec{n}))$$

$$R_{\Diamond\alpha}(x, \vec{t}, \vec{n}) \quad \Leftrightarrow \quad \exists y((x, y) \in R \ \wedge \ R_\alpha(y, \vec{t}, \vec{n}))$$

The first-order condition of Lemmon-Scott is:

$$R(\alpha,\vec{n},\vec{m}) : \forall x \forall t_1 \ldots \forall t_k (xR^{m_1}t_1 \wedge xR^{m_2}t_2 \wedge \ldots xR^{m_k}t_k \Rightarrow R_\alpha(x,\vec{t},\vec{n})).$$

We note that when the relation $\leq$ is the equality, the first-order condition $R(\alpha,\vec{n},\vec{m})$ is the first-order condition given in [7].

Let $\alpha(p_1, p_2, \ldots, p_k)$ be a positive formula. Then the *Lemmon-Scott* axiom is the formula

$$ILS(\alpha_n^m) : \qquad \Diamond^{m_1}\Box^{n_1}p_1 \wedge \Diamond^{m_2}\Box^{n_2}p_2 \wedge \ldots \wedge \Diamond^{m_k}\Box^{n_k}p_k \to \alpha(p_1, p_2, \ldots, p_k).$$

Proposition 4. *Let $\mathcal{F}$ be a frame. Then $\mathcal{F} \vDash ILS(\alpha_n^m)$ if and only if $R(\alpha,\vec{n},\vec{m})$ is valid en $\mathcal{F}$.*

Proof. Assume that $\mathcal{F} \vDash ILS(\alpha_n^m)$. Let $x \in X$ and $\vec{t} = (t_1, \ldots, t_k) \in X^k$ such that $(x, t_i) \in R^{m_i}$, $i \leq k$. Let us consider the function V defined by $V(p_i) = R_\Box^{n_i}(t_i)$. By Corollary 3, V is a valuation. Since $t_i \in V(\Box^{n_i}p_i)$, we get $x \in \bigcap_{i=1}^{k} V(\Diamond^{m_i}\Box^{n_i}p_i)$. Then by assumption, $x \in V(\alpha(p_1, p_2, \ldots, p_k))$. Now, by induction on the complexity of α we shall prove that $R_\alpha(x,\vec{t},\vec{n})$ is valid in $\mathcal{F}$.

- Let $\alpha = p_i$. Then, $x \in V(\alpha(p_1, p_2, \ldots, p_k)) = V(p_i) = R_\Box^{n_i}(t_i)$. So, $(t_i, x) \in R_\Box^{n_i}$, for $i \leq k$.
- Let $\alpha = \Diamond\varphi$. Then, $x \in V(\Diamond\varphi(p_1, p_2, \ldots, p_k)) = \Diamond_R V(\varphi(p_1, p_2, \ldots, p_k))$. It follows that there exists $y \in X$ such that $(x, y) \in R$ and $y \in V(\varphi(p_1, p_2, \ldots, p_k))$. By inductive hypothesis, $(x, y) \in R$ and $R_\varphi(y,\vec{t},\vec{n})$. Thus, $R_{\Diamond\varphi}(x,\vec{t},\vec{n})$ is valid in $\mathcal{F}$. The other cases are similar and left to the reader.

Assume that $R(\alpha,\vec{n},\vec{m})$ is valid in $\mathcal{F}$. Let V be a valuation on $\mathcal{F}$ and let $x \in X$ such that $x \in \bigcap_{i=1}^{k} V(\Diamond^{m_i}\Box^{n_i}p_i)$. Then for each $i \leq k$, there exists $t_i \in X$ such that $(x, t_i) \in R^{m_i}$ and $t_i \in V(\Box^{n_i}p_i)$. By induction on the complexity of $\alpha(p_1, p_2, \ldots, p_k)$ we prove that $x \in V(\alpha(p_1, p_2, \ldots, p_k))$.

- Let $\alpha = p_i$. Since, $R_{p_i}(x,\vec{t},\vec{n})$ is $(t_i, x) \in R_\Box^{n_i}$, for $i \leq k$. Since $t_i \in V(\Box^{n_i}p_i)$, we have $x \in V(p_i)$.
- Let $\alpha = \Box\varphi$. Let $(x, y) \in R_\Box$. Since, $R_{\Box\varphi}(x,\vec{t},\vec{n})$ is $\forall y((x, y) \in R_\Box \Rightarrow R_\varphi(y,\vec{t},\vec{n}))$, and as $t_i \in V(\Box^{n_i}p_i)$, then by inductive hypothesis we have that for all $y \in R_\Box(x)$, $y \in V(\varphi)$. Therefore, $x \in V(\Box\varphi)$.

- Let $\alpha = \Diamond\varphi$. Since, $R_{\Diamond\varphi}(x, \vec{t}, \vec{n})$ is $\exists y((x, y) \in R \wedge R_{\varphi}(y, \vec{t}, \vec{n}))$, then by inductive hypothesis there exists $y \in R(x)$ and $y \in V(\varphi)$. Therefore, $x \in V(\Diamond\varphi)$.

The cases $\alpha = \varphi \vee \psi$ and $\alpha = \varphi \wedge \psi$ are similar and left to the reader. □

4 Completeness

The completeness of the logic $\mathbf{IK} + \{ILS(\alpha_n^m)\}$ will be prove by means of the canonical model. First, we shall recall some notions.

Let us fix a modal logic $\mathcal{I}$ that is an extension of **IK**. A set of formulas is a *theory* of $\mathcal{I}$, or an $\mathcal{I}$-theory, if it is closed under the deducibility relation $\vdash_{\mathcal{I}}$. A theory is *consistent* if it is not the set of all formulas. Equivalently, if the formula $\perp$ does not belong to it. A *prime theory* of $\mathcal{I}$, or a prime $\mathcal{I}$-theory, is a consistent $\mathcal{I}$-theory P with the following property: if $(\varphi \vee \psi) \in \Gamma$, then $\varphi \in P$ or $\psi \in \Gamma$.

Proposition 5. *Let Γ be a consistent theory and let Δ be a set of formulas closed under disjunctions (i.e. if $\varphi, \psi \in \Delta$ then $\varphi \vee \psi \in \Delta$) and such that $\Gamma \cap \Delta = \emptyset$. Then there is a prime theory P such that $\Gamma \subseteq P$ and $P \cap \Delta = \emptyset$.*

Proof. See [6]. □

Let us denote by X_c the set of all prime $\mathcal{I}$-theories. We define the relation $R_c \subseteq X_c \times X_c$ as follows:

$$(P, Q) \in R_c \Leftrightarrow \Box^{-1}(P) \subseteq Q \subseteq \Diamond^{-1}(P),$$

where $\Box^{-1}(P) = \{\varphi : \Box\varphi \in P\}$ and $\Diamond^{-1}(P) = \{\varphi : \Diamond\varphi \in P\}$. In [6] it was shown that the structure $\mathcal{F}_c = \langle X_c, \subseteq, R_c \rangle$ is indeed a frame. It will be called the *canonical frame* for $\mathcal{I}$.

Let Q be a prime $\mathcal{I}$-theory and let us consider the sets $Q^c = \{\varphi : \varphi \notin Q\}$ and $\Box(Q^c) = \{\Box\varphi : \varphi \in Q^c\}$. Then the set $\Box(Q^c)$ is closed under disjunctions. To see this, we note first that if $\Box\varphi \vdash_{\mathcal{I}} \Box\psi$ and $\psi \in Q^c$, then $\Box\varphi \in \Box(Q^c)$, because $\Box\varphi \vdash_{\mathcal{I}} \Box\psi \Leftrightarrow \Box\varphi \wedge \Box\psi \dashv\vdash_{\mathcal{I}} \Box(\varphi \wedge \psi) \dashv\vdash_{\mathcal{I}} \Box\varphi$ and as $\psi \notin Q$, $\varphi \wedge \psi \notin Q$. So, if $\psi, \varphi \notin Q$ then $\psi \vee \varphi \notin Q$, and since $\Box\varphi \vee \Box\psi \vdash_{\mathcal{I}} \Box(\varphi \vee \psi)$, we get $\Box\varphi \vee \Box\psi \in \Box(Q^c)$.

The results of the following theorem is establish in [6] but we shall give a simplified proof for completeness.

Proposition 6. *Let $P, Q \in X_c$. Then*

1. $\Box^{-1}(P) \subseteq Q$ *if and only if* $(P, Q) \in R_{\Box}$.
2. $Q \subseteq \Diamond^{-1}(P)$ *if and only if* $(P, Q) \in R_{\Diamond}$.
3. $\Box\varphi \notin P$ *if and only if there exists* $Q \in X_c$ *such that* $(P, Q) \in R_{\Box}$ *and* $\varphi \notin Q$.
4. $\Diamond\varphi \in P$*if and only if there exists* $Q \in X_c$ *such that* $(P, Q) \in R_{\Diamond}$ *and* $\varphi \in Q$.
5. $(R_c \circ \subseteq) \subseteq (\subseteq \circ R_c)$.
6. $(\subseteq^{-1} \circ R_c) \subseteq (R_c \circ \subseteq^{-1})$.
7. $R_c = R_{\Box} \cap R_{\Diamond}$.

Proof. 1. Let $P, Q \in X_c$ such that $\Box^{-1}(P) \subseteq Q$. Let us consider the theory $T = \{\varphi : P \cup \Diamond Q \vdash_{\mathcal{I}} \varphi\}$. We prove that

$$T \cap \Box(Q^c) = \emptyset.$$

Suppose the contrary. Then there exists $\varphi \in P$, $\psi \in Q$ and $\alpha \notin Q$ such that $\varphi \wedge \Diamond\psi \vdash_{\mathcal{I}} \Box\alpha$. Since $\varphi \vdash_{\mathcal{I}} \Diamond\psi \to \Box\alpha$ and $\Diamond\psi \to \Box\alpha \vdash_{\mathcal{I}} \Box(\psi \to \alpha)$, we get $\Box(\psi \to \alpha) \in P$. It follows that $\psi \to \alpha \in Q$, which is a contradiction. Then $T \cap \Box(Q^c) = \emptyset$. By Proposition 5, there exists $D \in X_c$ such that $P \subseteq D$, $Q \subseteq \Diamond^{-1}(D)$ and $\Box^{-1}(D) \subseteq Q$. Therefore, $(P, Q) \in R_{\Box}$.

The other direction is immediate.

3. Let us suppose that $\Box\varphi \notin P$. Let T_{φ} be the closure under disjunctions of the set $\{\varphi\}$. Then $\Box^{-1}(P) \cap T_{\varphi} = \emptyset$. By Proposition 5, there exists a prime theory Q such that $\Box^{-1}(P) \subseteq Q$ and $\varphi \notin Q$. By 1. above we get the desired result.

5. Let $P, Q, D \in X_c$ such that $(P, D) \in R_c$ and $D \subseteq Q$. Then $\Box^{-1}(P) \subseteq Q$. By 1. above we have $(P, Q) \in R_{\Box} = \subseteq \circ R_c$.

The proof of 2., 4., and 5. are similar. The proof of 6. follows from 1. and 2. $\Box$

Define the *canonical model* for $\mathcal{I}$ as the model $\langle \mathcal{F}_c, V_c \rangle$ on the canonical frame $\mathcal{F}_c$, where V_c is the valuation defined by $V_c(p) = \{P \in X_c : p \in P\}$, for any variable p. It is clear that V_c is a valuation since the sets $\{P \in X_c : p \in P\}$ are increasing.

Proposition 7. $\langle \mathcal{F}_c, V_c \rangle \vDash_P \varphi \;\; \Leftrightarrow \;\; \varphi \in P$.

Proof. See [6]. $\Box$

Corollary 8. *The modal logic* **IK** *is canonical and hence frame complete.*

Lemma 9. *Let $\mathcal{F}_c$ be the canonical frame of $\mathcal{I}$. Then for every $P, Q \in X_c$,*

$$(P, Q) \in R_{\Box}^{n} \Leftrightarrow \{\varphi : \Box^{n}\varphi \in P\} \subseteq Q. \tag{1}$$

Proof. The proof is by induction on n. The case $n = 1$ follows from Proposition 6. Suppose that (1) is valid for n. If $(P, Q) \in R_{\Box}^{n+1}$, then it is immediate to check that $\left\{\varphi : \Box^{n+1}\varphi \in P\right\} \subseteq Q$. Suppose that

$$\left\{\varphi : \Box^{n+1}\varphi \in P\right\} \subseteq Q.$$

Let us consider the set $\Box^{-1}(P)$ and let Δ be the closure under disjunctions of the set $\{\Box^{n}\psi : \psi \notin Q\}$. We prove that

$$\Box^{-1}(P) \cap \Delta = \emptyset.$$

Suppose the contrary. Then there exists $\Box\alpha \in P$ and there exists $\psi \in \Delta$ such that $\vdash_{\mathcal{I}} \alpha \to \Box^{n}\psi$. Then $\vdash_{\mathcal{I}} \Box\alpha \to \Box^{n+1}\psi$. Since P is a theory, $\Box^{n+1}\psi \in P$. It follows, $\psi \in Q$, which is a contradiction. Then, by Proposition 5, there exists $D \in X_c$ such that $\Box^{-1}(P) \subseteq D$ and $\Delta \cap D = \emptyset$. It follows that $(P, Q) \in R_{\Box}$, and by inductive hypothesis, it follows that $(D, Q) \in R_{\Box}^{n}$, i.e., $(P, Q) \in R_{\Box}^{n+1}$. □

Now, we prove that the intuitionistic modal logic $\mathbf{IK} + \{ILS(\alpha_n^m)\}$ is canonical. Let $\mathcal{I}$ be a intuitionistic modal logic such that $\mathbf{IK} + \{ILS(\alpha_n^m)\} \subseteq \mathcal{I}$.

Proposition 10. *Let $\mathcal{F}_c$ be the canonical frame of $\mathcal{I}$. Let $\alpha(p_1, p_2, \ldots, p_n)$ be a positive formula. Let $Q \in X_c$ and $\vec{P} = (P_1, P_2, \ldots, P_k) \in X_c^k$. Then*

$$R_{\alpha}(Q, \vec{P}, \vec{n}) \text{ is valid in } \mathcal{F}_c \text{ iff } \{\alpha(\psi_1, \psi_2, \ldots, \psi_k) : \Box^{n_i}\psi_i \in P_i, i \leq k\} \subseteq Q.$$

Proof. The proof is by induction on the complexity of α. We give the proof for the case $k = 1$. The case $\alpha = p$ follows by the Lemma 9.

Let $\alpha = \Diamond\varphi$. Suppose that $\{\Diamond\varphi(\psi) : \Box^{n}\psi \in P\} \subseteq Q$. We prove that $R_{\Diamond\varphi}(Q, P, n)$ Let us consider the set $\Gamma = \{\varphi(\psi) : \Box^{n}\psi \in P\}$ and let us consider the theory $T(\Gamma)$ generated by Γ. We prove that

$$T(\Gamma) \cap \Diamond^{-1}(Q)^{c} = \emptyset.$$

If we suppose the contrary, then there exists $\varphi(\psi_1), \varphi(\psi_2), \ldots, \varphi(\psi_n) \in \Gamma$ and $\beta \notin \Diamond^{-1}(Q)$ such that $\vdash_{\mathcal{I}} \varphi(\psi_1) \wedge \varphi(\psi_2) \wedge \ldots \wedge \varphi(\psi_n) \to \beta$. Then,

$$\vdash_{\mathcal{I}} \Diamond(\varphi(\psi_1) \wedge \varphi(\psi_2) \wedge \ldots \wedge \varphi(\psi_n)) \to \Diamond\beta.$$

Thus, $\Diamond(\varphi(\psi_1)\wedge\varphi(\psi_2)\wedge\ldots\wedge\varphi(\psi_n))\notin Q$. But since $\varphi(\psi_1),\varphi(\psi_2),\ldots,\varphi(\psi_n)\in\Gamma$, $\Box^n(\psi_1\wedge\psi_2\wedge\ldots\wedge\psi_n)\in P$. It follows $\Diamond(\varphi(\psi_1)\wedge\varphi(\psi_2)\wedge\ldots\wedge\varphi(\psi_n))\in Q$, which is a contradiction. Therefore, there is a prime theory D such that $\Gamma\subseteq D$ and $D\subseteq\Diamond^{-1}(Q)$. Since $(Q,D)\in R_\Diamond=R_c\circ\subseteq^{-1}$, then there exists $K\in X_c$ such that $(Q,K)\in R_c$ and $D\subseteq K$. So, $\Gamma\subseteq K$. Then, by the inductive hypothesis, $R_{\varphi(\psi)}(Q,P,n)$ and $(Q,K)\in R_c$.

The proof in the other direction is easy. The proof of the other cases are similar and left to the reader. □

Corollary 11. *Let $\mathcal{I}$ be an intuitionistic modal logic such that it contains the logic $\mathbf{IK}+\{ILS(\alpha_n^m)\}$. Then the canonical frame $\mathcal{F}_c$ of $\mathcal{I}$ satisfies the first-order condition $R(\alpha,\vec{n},\vec{m})$ Therefore, the logic $\mathbf{IK}+\{ILS(\alpha_n^m)\}$ is canonical.*

Proof. Let $Q,P_1,P_2,\ldots,P_k\in X_c$. Suppose that $(Q,P_1)\in R^{m_1}$,...,$(Q,P_k)\in R^{m_k}$. By Proposition 10 we have to prove that

$$\{\alpha(\psi_1,\psi_2,\ldots,\psi_k):\Box^{n_i}\psi_i\in P_i,i\leq k\}\subseteq Q.$$

Let $\Box^{n_i}\psi_i\in P_i,i\leq k$. Then $\bigwedge_{i=1}^{k}\Diamond^{m_i}\Box^{n_i}\psi_i\in Q$. As $ILS(\alpha_n^m)\in Q$, $\alpha(\psi_1,\psi_2,\ldots,\psi_k)\in Q$ Thus, $\mathcal{F}_c$ satisfies the condition $R(\alpha,\vec{n},\vec{m})$, and consequently $\mathbf{IK}+\{ILS(\alpha_n^m)\}$ is canonical. □

5 Conclusions

In this paper we prove that the logic **IK** extended with the Lemmon-Scott axiom is canonical and frame complete. By these results we can deduce that, for instance, the logic $\mathbf{IK}+\left\{\Box\varphi\to\Box^2\varphi,\Diamond\Box\varphi\to\Box\Diamond\varphi\right\}$ is canonical and its class of frames are the frames $\mathcal{F}=\langle X,\leq,R\rangle$ where $R_\Box=R\circ\leq$ is transitive and $R_\Box\circ R_\Diamond\subseteq R_\Diamond\circ R_\Box$. These results generalize and extend the results given by G. Fischer-Servi [6] on extensions of the logic **IK**.

An important fact of the logic **IK** is that it embodies a fully acceptable interpretation of the modal operators $\Box$ and $\Diamond$ by means of the axioms $\Diamond(\varphi\to\psi)\to(\Box\varphi\to\Diamond\psi)$ and $(\Diamond\varphi\to\Box\psi)\to\Box(\varphi\to\psi)$. Moreover, in the classical modal logic **K,** the formula $\Box(\varphi\vee\psi)\to(\Box\varphi\vee\Diamond\psi)$ is equivalent to any of the two above formulas. This is not valid when we consider a modal logic based on intuitionistic logic. This motivated define other classes of intuitionistic modal logic with operators $\Box$ and $\Diamond$ using different combinations of these formulas. These problems will be investigated in a future paper.

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