Divulgaciones Matemáticas Vol. 10 No. 2(2002), pp. 101–129

A One Dimensional Deterministic Free Boundary Problem

Un Problema Determinístico Unidimensional de Frontera Libre

Guillermo Ferreyra (ferreyra@math.lsu.edu)
Louisiana State University
Baton Rouge, Louisiana
Jesus A. Pascal (pascal@luz.ve)
Facultad Experimental de Ciencias
Universidad del Zulia
Maracaibo, Venezuela

Abstract

A general one dimensional deterministic infinite horizon singular optimal control problem with unbounded control set is considered in this paper. Using the dynamic programming approach we prove that the value function is convex, and C^1 along the free boundary. Also, we find the free boundary in terms of the parameters of the problem.
Key words and phrases: deterministic optimal control, viscosity solutions, dynamic programming.

Resumen

En este artículo se considera un problema general de control óptimo singular con horizonte infinito y conjunto de control no acotado, unidimensional y determinístico. Usando el enfoque de la programación dinámica probamos que la función valor es convexa y C^1 a lo largo de la frontera libre. También encontramos la frontera libre en términos de los parámetros del problema.
Palabras y frases clave: control óptimo determinista, soluciones de viscosidad, programación dinámica.

Recibido 2001/12/19. Revisado 2002/09/10. Aceptado 2002/10/01.
MSC (2000): Primary: 35F30, 49L25; Secondary: 49L20, 93E20, 34B15.

1 Introduction

This paper refers to a class of infinite horizon singular optimal control problems which are optimal control problems with a set of control values which is unbounded and where the control appear linearly in the dynamics and in the running cost. We consider the scalar control system

$$\dot{x} = f(x) + u, \qquad x(0) = x \in \mathbb{R}, \tag{1}$$

where f is a differentiable function with bounded derivatives and the control $u(\cdot)$ is a measurable function of time in the family

$$\mathcal{U} = L^{\infty}([0,\infty), \mathbb{R}).$$

The optimal control problem consists of minimizing over all controls $u(\cdot) \in \mathcal{U}$ the infinite horizon discounted cost functional

$$v^u(x) = \int_0^{\infty} e^{-t}[L(x(t)) + |u(t)|]dt, \tag{2}$$

with a positive function L specified as in section 2. The value function for this optimal control problem is a function of the initial state x defined as the infimum of the costs, that is ,

$$v(x) = \inf\{v^u(x) \colon u(\cdot) \in \mathcal{U}\}, \tag{3}$$

and the optimal control $u^*(\cdot)$, if it exists, is the argument that minimizes the cost functional.

Note that the more general problem with

$$\dot{x} = f(x) + \alpha u, \qquad x(0) = x \in \mathbb{R},$$

and cost functional

$$v^u(x) = \int_0^{\infty} e^{-t}[L(x(t)) + \rho|u(t)|]dt,$$

with $\rho > 0$, $\alpha \in \mathbb{R}$, can be reduced to (1), (2), by rescaling f and L.
The dynamic programing equation, also called the Hamilton-Jacobi-Bellman (HJB) equation, for a deterministic optimal control problem is in general a first order nonlinear partial differential equation (PDE) that provides an approach to solving optimal control problems. It is well known, see [7], that if the value function is smooth enough, then it is a classical solution of the HJB

equation. But also using a weaker notion of solution, called viscosity solution, introduced by Crandall and Lions [3], the dynamic programming method can be pursued when the value function is not smooth enough. In fact, the HJB equation is a necessary condition that the value function must satisfy. The dynamic programming equation for the above deterministic optimal control problem is of the form

$$\max \left[F^1(x, v(x), v'(x)), F^2(x, v(x), v'(x))\right] = 0, \quad -\infty < x < \infty,$$

for suitable continuous functions F^1, F^2. The subset B of $\mathbb{R}$ where both

$$F^1(x, v(x), v'(x)) = F^2(x, v(x), v'(x)) = 0,$$

is called the free boundary. Our control problem is homogeneous of degree 1 in the control, thus we expect the optimal control to be extreme or to be singular. Moreover, since our running cost is nonnegative we expect optimal controls to equal zero, plus or minus infinity, or to be singular. By the control being plus or minus infinity we mean that it is an impulse. The free boundary (where the optimal control is in some cases singular)separates the null region (where the optimal control is zero) and the jump region (where the optimal control is impulsive). Nonsmoothness of the value function often occurs only along the free boundary B. The property of smooth fit is said to hold for a particular optimal control problem if the value function is smooth enough, C^1 in our case, along the free boundary B so that it solves the HJB equation in the classical sense. The dynamic programming equation gives rise to a free boundary problem since the crucial step in solving it is to locate the subset B where there is a switch between the conditions

$$F^1(x, v(x), v'(x)) \leq 0, \quad F^2(x, v(x), v'(x)) = 0,$$

and

$$F^1(x, v(x), v'(x)) = 0, \quad F^2(x, v(x), v'(x)) \leq 0.$$

Ferreyra and Hijab [5] studied the optimal control problem (1), (2), (3), assuming linearity of the function f and convexity of the function L, with controls taking values in $[0, \infty)$. This enables them to present a complete analysis of the solution of the control problem. They used the dynamic programming method and proved that the free boundary is just a single point giving its location in terms of the parameters of the problem. Also, they found that smoothness of v depends on the parameters of the problem. We consider the optimal control problem (1), (2), (3), with the same assumptions on f and L as in [5], but allowing the controls to take values in the whole real line. We use the dynamic programming method to prove that the free boundary is a pair of points in $\mathbb{R}$, locating them in terms of the parameters of the problem. We determine the optimal control on each one of the regions separated by the free boundary. We also see that C^2-fit is a property that depends on the parameters of the problem.

2 The Main Results

Let's consider the optimal control problem (1), (2), (3) with the following assumptions,

(i) L is C^2 and $L(x) \geq 0$,

(ii) $|L'(x)| \leq C_1(1 + L(x))$,

(iii) $0 < \mu \leq L''(x) \leq C_2(1 + L(x))$,

(iv) $f(x)$ is linear and $f'(x) < 0$,

(v) the control $u(\cdot)$ is a measurable function, $u(\cdot) \in L^\infty([0, \infty), \mathbb{R})$.

For clarity we set $f(x) = \beta x$, with $\beta < 0$.

Theorem 1. *The value function v for the control problem is a classical C^1-solution of the Hamilton-Jacobi-Bellman equation*

$$\max\,[v(x) - \beta x v'(x) - L(x), |v'(x)| - 1] = 0, \quad -\infty < x < \infty. \tag{4}$$

Moreover, there exist $\alpha^-, \alpha^+ \in \mathbb{R}$ such that

$$\begin{aligned} -v'(x) - 1 &= 0, & \forall x \in J^- = (-\infty, \alpha^-], \\ v(x) - \beta x v'(x) - L(x) &= 0, & \forall x \in N = [\alpha^-, \alpha^+], \\ v'(x) - 1 &= 0, & \forall x \in J^+ = [\alpha^+, +\infty). \end{aligned}$$

The value function v is never C^2 on $\mathbb{R}$ but

$$v \in C^2(\mathbb{R} \setminus \{\alpha^-, \alpha^+\}), \quad \textit{and}$$

$$v \in C^2 \textit{ at } \alpha^- \iff 0 < \alpha^-, \quad \textit{and}$$

$$v \in C^2 \textit{ at } \alpha^+ \iff \alpha^+ < 0.$$

The quantities α^- and α^+ can be computed in terms of the parameters of the problem.

Theorem 2. **(i)** *$\forall x \in \mathbb{R} \setminus [\alpha^-, \alpha^+]$, the optimal control is impulsive.*

(ii) *If $\alpha^- \le 0 \le \alpha^+$, then $\forall x \in [\alpha^-, \alpha^+]$ the zero control is optimal.*

(iii) ***Case*** $0 < \alpha^- < \alpha^+$ **.**
At $x = \alpha^-$, the optimal control is singular, with value

$$u^*(t) \equiv -\beta\alpha^-, \quad \forall t \ge 0$$

For each $x \in [\alpha^-, \alpha^+]$, the optimal control is

$$u^*(t) = \begin{cases} 0, & 0 \le t < T, \\ -\beta\alpha^-, & t \ge T, \end{cases}$$

where $T > 0$ is such that the corresponding solution $x^*(t), 0 \le t \le T$, satisfies,

$$x^*(t) = xe^{\beta T} = \alpha^-.$$

(iv) ***Case*** $\alpha^- < \alpha^+ < 0$ **.**
This case is similar to the previous one where $0 < \alpha^- < \alpha^+$.
At $x = \alpha^+$, the optimal control is singular, with value

$$u^*(t) \equiv -\beta\alpha^+, \quad \forall t \ge 0$$

For each $x \in [\alpha^-, \alpha^+]$, the optimal control is

$$u^*(t) = \begin{cases} 0, & 0 \le t < T, \\ -\beta\alpha^+, & t \ge T, \end{cases}$$

where $T > 0$ is such that the corresponding solution $x^*(t), 0 \le t \le T$, satisfies,

$$x^*(t) = xe^{\beta T} = \alpha^+.$$

3 Convexity and Differentiability of the Value Function

Lemma 3. *The value function v is convex, C^1, and a classical solution of the Hamilton-Jacobi-Bellman (HJB) equation* (4)*. Moreover, v'' exists almost everywhere and*

(i) $0 \le v(x) \le L^*(x)$,

(ii) $|v'(x)| \leq C_1(1 + L^*(x))$,

(iii) $0 \leq v''(x) \leq C_2(1 + L^*(x))$ *for almost every* x,

where $L^*(x)$ *denotes the maximum value of the function* L *over the line segment joinning* x *and the origin.*

Proof.
Note that since L is convex we have

$$L^*(x) =: \max\{L(y)\colon 0 \leq y \leq x\} = \max(L(x), L(0)).$$

It is clear that $v(x) \geq 0, \quad \forall x \in \mathbb{R}$. Let's show that v is convex. Let $x_0^0, x_0^1 \in \mathbb{R}$, and $s \in [0, 1]$. Given $\varepsilon > 0$, there exist $u_0, u_1 \in \mathcal{U}$ such that

$$v^{u_0}(x_0^0) \leq v(x_0^0) + \varepsilon \qquad \text{and} \qquad v^{u_1}(x_0^1) \leq v(x_0^1) + \varepsilon.$$

Let $u = (1 - s)u_0 + su_1$. It is clear that u is a measurable function, hence $u \in \mathcal{U}$.
Let $x_0 = (1-s)x_0^0 + sx_0^1$. Let $x_i(t)$ be the solution of $\dot{x} = f(x) + u$, with initial value $x(0) = x_0^i$, $i = 1, 2$. Then, $x(t) = (1 - s)x_0(t) + sx_1(t)$ is the solution of $\dot{x} = \beta x + u$, with initial value $x(0) = (1 - s)x_0^0 + sx_0^1 = x_0$. In fact, since f is a linear function

$$\frac{d}{dt}[x(t)] = \beta(x(t)) + u.$$

By definition of v, convexity of L and using the triangle inequality, we have

$$v[(1 - s)x_0^0 + sx_0^1] \leq (1 - s)v(x_0^0) + sv(x_0^1) + \varepsilon.$$

Since ε was arbitrary, this implies v is convex.
To conclude the proof of (i) note that when $u(\cdot) \equiv 0$, $x(t)$ lies on the line segment joining x to 0 because $\beta < 0$. This implies

$$v(x) \leq v^0(x) \leq \int_0^\infty e^{-t} L^*(x)dt = L^*(x).$$

Then we need only to consider controls $u(\cdot)$ in (3) satisfying $v^u(x) \leq L^*(x)$.
Now, using ∇ to mean first derivative with respect to x,

$$|\triangledown v^u(x)| \leq \int_0^\infty e^{-t}|\triangledown L(x(t))|dt \leq \int_0^\infty e^{-t}[C_1(1+L(x(t)))]dt \leq C_1[1+L^*(x)].$$

Similarly,

$$|\triangledown^2 v^u(x)| \leq \int_0^\infty e^{-t}|\triangledown^2 f(x(t))|dt \leq C_2[1 + f^*(x)].$$

Since the right hand side of this last inequality is bounded on every compact interval, we conclude that for each $a, b \in \mathbb{R}, a < b$ there exists a $k(a, b) > 0$, independent of u, such that $k(a, b)x^2 - v^u(x)$ is convex on [a,b]. Taking the supremum over all u it follows that $k(a, b)x^2 - v(x)$ is convex on $[a, b]$. Thus, v is semiconcave. Since v is also convex, then v is C^1 and v'' exists almost everywhere.

Finally, the estimates on v' and v'' follow from the above estimates for $\bigtriangledown v^u, \bigtriangledown^2 v^u$. Then, reasoning as in Fleming-Soner [8, VIII], [5], [6], and [4], the value function v is a viscosity solution of the HJB equation, hence v is classical solution of the dynamic programming equation

$$\max\,[v(x) - \beta x v'(x) - L(x) \quad , \quad H(v'(x))] = 0, \quad -\infty < x < \infty,$$

where

$$H(p) = \sup_{|u|=1} (-pu - |u|) = \sup_{|u|=1} (-pu - 1) = |p| - 1.$$

Therefore,

$$\max\,[v(x) - \beta x v'(x) - L(x) \quad , \quad |v'| - 1] = 0, \quad -\infty < x < \infty. \quad \square$$

4 The Cost of Using the Control Zero

In the next lemma we consider the cost of the control $u(\cdot) \equiv 0$ which we define as $\omega(x) = v^0(x)$.

Lemma 4. *The function ω is in $C^2(\mathbb{R})$, it is strictly convex and satisfies*

(i) $0 \leq \omega(x) \leq L^*(x)$,

(ii) $|\omega'(x)| \leq C_1(1 + L^*(x))$,

(iii) $0 < \mu \leq \omega''(x) \leq C_2(1 + L^*(x))$,

(iv) $\omega(x) - \beta x \omega'(x) - L(x) = 0, \qquad -\infty < x < \infty.$

Proof of (i).
By definition $\omega(x) = v^0(x) = \int_0^\infty e^{-t} L(x(t))dt$, with $x(t) = xe^{\beta t}$. Then by differentiating under the integral sign it follows that ω is in $C^2(\mathbb{R})$, and $0 \leq \omega(x) \leq L^*(x)$.
Proof of (ii).

Let $z \in \mathbb{R}$ and let $x(t)$ be the solution of (1) for the control $u(\cdot) \equiv 0$, with initial data $x(0) = z$. Then

$$|\omega'(z)| \leq \int_0^\infty e^{-t} \left| L'(x(t)) \frac{dx(t)}{dz} \right| dt,$$

where $x(t) = ze^{\beta t}$, hence $\frac{dx(t)}{dz} = e^{\beta t}$.
Thus, using the bounds on L', we get

$$|\omega'(z)| \leq \int_0^\infty e^{(\beta-1)t} C_1 [1 + L^*(z)] dt = \widetilde{C_1}[1 + L^*(z)].$$

Proof of (iii)
Similarly,

$$\omega''(z) = \int_0^\infty e^{-t} L''(ze^{\beta t}) e^{\beta t} e^{\beta t} dt = \int_0^\infty e^{(2\beta-1)t} L''(ze^{\beta t}) dt.$$

Using the bounds on L'', $\quad 0 < \mu \leq \omega''(z) \leq C_2(1 + L^*(z))$.
Proof of (iv).
Let $x \in \mathbb{R}$. Then, integrating by parts

$$\begin{aligned} \omega(x) - \beta x \omega'(x) - L(x) &= \int_0^\infty e^{-t} L(xe^{\beta t}) dt - \beta x \int_0^\infty e^{-t} L'(xe^{\beta t}) e^{\beta t} dt - L(x) \\ &= 0. \quad \Box \end{aligned}$$

5 The Free Boundary $\mathbf{B} = \{\boldsymbol{\alpha}^-, \boldsymbol{\alpha}^+\}$

In this section we find the free boundary of our control problem (1), (2), (3), (4) which is a pair of points $\alpha^-, \alpha^+ \in \mathbb{R}$. We will prove that α^-, α^+ are finite in Lemmas 8, 9.

Lemma 5. *There exist* α^-, α^+ *with* $-\infty \leq \alpha^- < \alpha^+ \leq \infty$, *such that*

$$\begin{aligned} -v'(x) - 1 &= 0, \qquad \forall x \in J^- = (-\infty, \alpha^-], \\ v(x) - \beta x v'(x) - L(x) &= 0, \qquad \forall x \in N = [\alpha^-, \alpha^+], \\ v'(x) - 1 &= 0, \qquad \forall x \in J^+ = [\alpha^+, +\infty). \end{aligned}$$

Proof.
By the Lemma 4 **(iii)** and by hypothesis the functions $\omega', L' : \mathbb{R} \longrightarrow \mathbb{R}$ are

respectively increasing and onto $\mathbb{R}$. Thus, we can define a^- , a^+, b^- and b^+ by

$$\omega'(a^-) = -1 \qquad \text{and} \qquad \omega'(a^+) = 1. \tag{5}$$

$$L'(b^-) = \beta - 1 \qquad \text{and} \qquad L'(b^+) = 1 - \beta. \tag{6}$$

We set

$$A^+ = \{x : v'(x) - 1 < 0\} \quad \text{and} \quad A^- = \{x : -v'(x) - 1 < 0\}.$$

A^+ and A^- are not empty because v is bounded below and because v satisfies the HJB equation (4). Then we define

$$\alpha^+ = \sup A^+ > -\infty \quad \text{and} \quad \alpha^- = \inf A^- < +\infty.$$

Since the function v' is increasing, by the HJB equation (4)

$$v'(x) = -1, \quad \forall x \le \alpha^-, \qquad \text{and} \qquad v'(x) = 1, \quad \forall x \ge \alpha^+.$$

Since v' is increasing and continuous, then $\alpha^- < \alpha^+$ and

$$-1 < v'(x) < 1, \qquad \forall x \in (\alpha^-, \alpha^+).$$

Thus, by the HJB equation (4), and since $|v'(x)| - 1 < 0, \quad \forall x \in (\alpha^-, \alpha^+)$

$$v(x) - \beta x v'(x) - L(x) = 0; \qquad \forall x \in (\alpha^-, \alpha^+). \tag{7}$$

Notice that if α^-, α^+ are finite then

$$\begin{aligned} -v'(x) - 1 &= 0, \qquad \forall x \in J^- = (-\infty, \alpha^-], \\ v(x) - \beta x v'(x) - L(x) &= 0, \qquad \forall x \in N = [\alpha^-, \alpha^+], \\ v'(x) - 1 &= 0, \qquad \forall x \in J^+ = [\alpha^+, +\infty). \end{aligned}$$

In particular,

$$v(\alpha^-) = L(\alpha^-) - \beta(\alpha^-), \quad \text{and } v'(\alpha^-) = -1, \tag{8}$$

and

$$v(\alpha^+) = L(\alpha^+) + \beta(\alpha^+), \quad \text{and } v'(\alpha^+) = 1. \tag{9}$$

Moreover, the value function verifies,

$$\forall x \in J^- = (-\infty, \alpha^-], \qquad v(x) = -x + (1-\beta)\alpha^- + L(\alpha^-), \tag{10}$$

$$\forall x \in J^+ = [\alpha^+, +\infty), \qquad v(x) = x + (\beta - 1)\alpha^+ + L(\alpha^+). \tag{11}$$

6 The Control Zero on (α^-, α^+)

Proposition 6. *We consider the optimal control problem* (1), (2), (3). *Let* $x \in (\alpha^-, \alpha^+)$. *Let* $x(t)$ *be the solution of* $\dot{x} = \beta x, \quad x(0) = x$, *for the control* $u(\cdot) \equiv 0$. *Let's suppose that there exists* $T > 0$ *such that* $x(t) \in (\alpha^-, \alpha^+), \quad \forall t \in [0, T)$. *Then*

$$v(x) = e^{-T} v(x(T)) + \int_0^T e^{-t} L(x(t)) dt. \tag{12}$$

Proof.
Let $x \in (\alpha^-, \alpha^+)$, let $x(t)$ be the solution of $\dot{x} = \beta x, \quad x(0) = x$, for the control $u(\cdot) \equiv 0$, and let $T > 0$ be such that $x(t) \in (\alpha^-, \alpha^+), \quad \forall t \in [0, T)$. Therefore, differentiating the function $t \longrightarrow e^{-t} v(x(t))$, and using equation (7)

$$\frac{d}{dt}[e^{-t} v(x(t))] = -e^{-t}[v(x(t)) - \beta x(t) v'(x(t))] = -e^{-t} L(x(t)), \quad \forall t \geq 0.$$

Now, integrating, over the interval [0, T], we get equation (12) . □

Proposition 7. *We consider the optimal control problem* (1), (2), (3).

(i) *Suppose* $\alpha^- \leq 0 \leq \alpha^+$, *then on* (α^-, α^+) *the control* $u(\cdot) \equiv 0$ *is optimal. Hence* $v = \omega$ *on* (α^-, α^+), *where* ω *is the cost of the control* $u(\cdot) \equiv 0$ *studied in Lemma 4.*

(ii) *Suppose* $0 < \alpha^- < \alpha^+$, *then the control* $u^*(t) \equiv -\beta\alpha^-, \quad \forall t \geq 0$, *is optimal at* α^-

(iii)*Suppose* $\alpha^- < \alpha^+ < 0$, *then the control* $u^*(t) \equiv -\beta\alpha^+, \quad \forall t \geq 0$, *is optimal at* α^+

Proof of (i).
Let $x \in (\alpha^-, \alpha^+)$ and let $x(t)$ be the solution of $\dot{x} = \beta x, \quad x(0) = x$, for the control $u(\cdot) \equiv 0$. Since $0 \in (\alpha^-, \alpha^+)$ and $\beta < 0$, then $x(t) \in (\alpha^-, \alpha^+), \quad \forall t \geq 0$. Hence, by Proposition 6 the equation (12) holds for all $T > 0$. That is,

$$v(x) = e^{-T} v(x(T)) + \int_0^T e^{-t} L(x(t)) dt, \quad \forall T > 0.$$

Letting $T \longrightarrow \infty$, yields

$$v(x) = \int_0^\infty e^{-t} L(x(t))\, dt = v^0(x) = \omega(x). \quad \square$$

Proof of (ii).
According to (7) and inserting $x = \alpha^-$ yields $v(\alpha^-) = L(\alpha^-) - \beta\alpha^-$. On the other hand, note that $x(t) = \alpha^-$ is the solution of $\dot{x} = \beta(x-\alpha^-), \quad x(0) = \alpha^-$. Therefore,

$$v^{u^*}(\alpha^-) = \int_0^\infty e^{-t}[L(\alpha^-) + (-\beta\alpha^-)]\,dt = L(\alpha^-) - b\alpha^-.$$

Thus, $u^*(t) \equiv -b\alpha^-, \quad \forall t \geq 0$ is optimal at α^-. $\square$
Proof of (iii).
According to (7) and inserting $x = \alpha^+$ yields $v(\alpha^+) = L(\alpha^+) + \beta\alpha^+$. On the other hand, note that $x(t) \equiv \alpha^+$ is the solution of

$$\dot{x} = \beta(x - \alpha^+), \quad x(0) = \alpha^+.$$

Therefore,

$$v^{u^*}(\alpha^+) = \int_0^\infty e^{-t}[L(\alpha^+) + \beta\alpha^+]\,dt = L(\alpha^-) + \beta\alpha^+.$$

Thus, $u^*(t) \equiv -\beta\alpha^+, \quad \forall t \geq 0$ is optimal at α^+. $\square$

7 α^-, α^+ Are Finite

Lemma 8. *α^- is finite.*

Proof.
We know that $-\infty \leq \alpha^- < \alpha^+ \leq +\infty$, let's suppose that $\alpha^- = -\infty$.
Case (i) $\alpha^+ \geq 0$.
Then $\alpha^- \leq 0 \leq \alpha^+$. Therefore, by Proposition 7 the control $u(\cdot) \equiv 0$ is optimal in (α^-, α^+) and $v(x) = v^0(x) = \omega(x), \quad \forall x \in (\alpha^-, \alpha^+)$. Then,

$$v'(x) = \omega'(x); \quad \forall x \in (\alpha^-, \alpha^+).$$

In particular, by continuity of v' and ω', and by (5), $v'(a^-) = \omega'(a^-) = -1$. This means that $a^- \leq \alpha^- = -\infty$. This is a contradiction, since $a^- \in \mathbb{R}$.
Case (ii) $\alpha^+ < 0$.
Let $x \in (\alpha^-, \alpha^+)$. Let $x(t)$ be the solution of $\dot{x} = \beta x, \quad x(0) = x$, for the control $u(\cdot) \equiv 0$. Since $\dot{x}(t) > \beta\alpha^+$, there exists $T > 0$ such that

$$x(T) = \alpha^+, \qquad \textit{and} \qquad x(t) \in (\alpha^-, \alpha^+), \quad \forall t \in [0, T).$$

Therefore, by Proposition 6 the equation (12) holds. So,

$$v(x) = e^{-T}v(\alpha^+) + \int_0^T e^{-t}L(x(t))dt.$$

To compute $v'(x)$ and $v''(x)$ we need to express T as a function of x. But $xe^{\beta T} = \alpha^+$. Solving for T and replacing above we get

$$v(x) = (\frac{\alpha^+}{x})^{-\frac{1}{\beta}}v(\alpha^+) + \int_0^{\varphi(x)} e^{-t}L(xe^{\beta t})dt, \quad \text{with} \quad \varphi(x) = \frac{1}{\beta}\log(\frac{\alpha^+}{x}).$$

Therefore,

$$\begin{aligned} v'(x) &= v(\alpha^+)(-\frac{1}{\beta})(\frac{\alpha^+}{x})^{-\frac{1}{\beta}-1}(-\frac{\alpha^+}{x^2}) + \int_0^{\varphi(x)} e^{-t}L'(xe^{\beta t})e^{\beta t}dt \\ &+ e^{-\varphi(x)}L(xe^{\beta\varphi(x)})\varphi'(x) \\ &= (\frac{\alpha^+}{x})^{\frac{\beta-1}{\beta}} + \int_0^{\varphi(x)} e^{(\beta-1)t}L'(xe^{\beta t})dt. \end{aligned}$$

Now, let's compute the second derivative at x

$$\begin{aligned} v''(x) &= \frac{\beta-1}{\beta}(\frac{\alpha^+}{x})^{-\frac{1}{\beta}}(-\frac{\alpha^+}{x^2}) + \int_0^{\varphi(x)} e^{-t}L''(xe^{\beta t})e^{2\beta t}dt \\ &+ e^{-\varphi(x)}L'(xe^{\beta\varphi(x)})e^{\beta\varphi(x)}\varphi'(x) \\ &= \left[-\frac{1}{\beta x}(\frac{\alpha^+}{x})^{\frac{\beta-1}{\beta}}(\beta - 1 + L'(\alpha^+))\right] + \int_0^{\varphi(x)} e^{-t}L''(xe^{\beta t})e^{2\beta t}dt. \end{aligned}$$

Let

$$\psi(x) = -\frac{1}{\beta x}(\frac{\alpha^+}{x})^{\frac{\beta-1}{\beta}}(\beta - 1 + L'(\alpha^+)).$$

It is clear that $\psi(x) \longrightarrow 0$ and $(\frac{\alpha^+}{x})^{\frac{2\beta-1}{\beta}} \longrightarrow 0$ as $x \longrightarrow 0$. Then, given $\varepsilon > 0$ there exists $K < 0$ such that for $x < K$ we have $0 < (\frac{\alpha^+}{x})^{\frac{2\beta-1}{\beta}} < 1$ and

$$\begin{aligned} v''(x) &> \int_0^{\varphi(x)} e^{-t}\mu e^{2\beta t}dt - \varepsilon = \mu[\frac{1}{2\beta-1}(e^{(2\beta-1)T} - 1)] - \varepsilon \\ &= \mu[\frac{1}{2\beta-1}((\frac{\alpha^+}{x})^{\frac{2\beta-1}{\beta}} - 1)] - \varepsilon > \mu[\frac{1}{2\beta-1}(-1)] - \varepsilon. \end{aligned}$$

Thus, taking $\varepsilon > 0$ small, and the corresponding $K < 0$

$$v''(x) \geq \gamma > 0; \forall x \in (-\infty, K).$$

Now, integrating over the interval $[x, K]$, for $-\infty < x < K$ yields

$$v'(K) - v'(x) \geq \gamma(K - x). \quad \text{Thus,} \quad v'(x) \leq \gamma(x - K) + v'(K).$$

Therefore,

$$v'(x) \longrightarrow -\infty, \qquad \text{as} \qquad x \longrightarrow -\infty.$$

This is a contradiction since the function v' can never be less than -1. Case (i) and (ii) imply $\alpha^- \neq -\infty$. Thus $-\infty < \alpha^- < +\infty$. $\square$

Lemma 9. α^+ *is finite.*

Proof.
We know that $-\infty \leq \alpha^- < \alpha^+ \leq \infty$. Let's suppose that $\alpha^+ = +\infty$.
Case (i) $\alpha^- \leq 0$.
Then $\alpha^- \leq 0 \leq \alpha^+$. Therefore, by Proposition 7 the control $u(\cdot) \equiv 0$ is optimal and $v(x) = v^0(x) = \omega(x)$, for $x \in (\alpha^-, \alpha^+)$. Then, $v'(x) = \omega'(x)$, $\forall x \in (\alpha^-, \alpha^+)$. In particular, by continuity of v' , ω' and by (5) $v'(\alpha^+) = 1 = \omega'(\alpha^+)$. This means that $a^+ \geq \alpha^+ = +\infty$. This is a contradiction, since $a^+ \in \mathbb{R}$.
Case (ii) $\alpha^- > 0$.
Let $x \in (\alpha^-, \alpha^+)$ and let $x(t)$ be the solution of $\dot{x} = \beta x$, $x(0) = x$, for the control $u(\cdot) \equiv 0$. Then, there exists $T > 0$ such that $x(T) = \alpha^-$, *and* $x(t) \in (\alpha^-, \alpha^+)$, $\forall t \in [0, T)$. Therefore, by Proposition 6 the equation (12) holds for $T > 0$. To compute $v'(x)$ and $v''(x)$ we need to express T as a function of x. But $xe^{\beta T} = \alpha^+$. Solving for T and replacing in equation (12) we get

$$v(x) = (\frac{\alpha^-}{x})^{-\frac{1}{\beta}} v(\alpha^-) + \int_0^{\varphi(x)} e^{-t} L(xe^{\beta t}) dt.$$

Therefore,

$$\begin{aligned} v'(x) &= v(\alpha^-)(-\frac{1}{\beta})(\frac{\alpha^-}{x})^{-\frac{1}{\beta}-1}(-\frac{\alpha^-}{x^2}) + \int_0^{\varphi(x)} e^{-t} L'(xe^{\beta t}) e^{\beta t} dt \\ &+ e^{-\varphi(x)} L(xe^{\beta\varphi(x)})\varphi'(x) \\ &= -\beta\alpha^-(\frac{\alpha^-}{x})^{-\frac{1}{\beta}} \frac{1}{\beta x} + \int_0^{\varphi(x)} e^{(\beta-1)t} L'(xe^{\beta t}) dt. \end{aligned}$$

So,

$$v'(x) = -(\frac{\alpha^-}{x})^{\frac{\beta-1}{\beta}} + \int_0^{\varphi(x)} e^{(\beta-1)t} L'(xe^{\beta t}) dt. \tag{13}$$

Now, let's compute the second derivative at x

$$\begin{aligned} v''(x) &= -\frac{\beta-1}{\beta}(\frac{\alpha^-}{x})^{-\frac{1}{\beta}}(-\frac{\alpha^-}{x^2}) + \int_0^{\varphi(x)} e^{(\beta-1)t}L''(xe^{\beta t})e^{\beta t}dt \\ &+ e^{-\varphi(x)}L'(xe^{\beta\varphi(x)})e^{\beta\varphi(x)}\varphi'(x) \\ &= \frac{\beta-1}{\beta x}(\frac{\alpha^-}{x})^{\frac{\beta-1}{\beta}} + \int_0^{\varphi(x)} e^{(2\beta-1)t}L''(xe^{\beta t})dt \\ &+ (\frac{\alpha^-}{x})^{-\frac{1}{\beta}}L'(\alpha^-)(\frac{\alpha^-}{x})(-\frac{1}{\beta x}). \end{aligned}$$

Then,

$$v''(x) = [\frac{1}{\beta x}(\frac{\alpha^-}{x})^{\frac{\beta-1}{\beta}}(\beta-1-L'(\alpha^-))] + \int_0^{\varphi(x)} e^{(2\beta-1)t}L''(xe^{\beta t})dt. \qquad (14)$$

Let

$$\psi(x) = \frac{1}{\beta x}(\frac{\alpha^-}{x})^{\frac{\beta-1}{\beta}}(\beta-1-L'(\alpha^-)).$$

It is clear that $\psi(x) \longrightarrow 0$ and $(\frac{\alpha^-}{x})^{\frac{2\beta-1}{\beta}} \longrightarrow 0$ as $x \longrightarrow +\infty$. Then, given $\varepsilon > 0$ there exists $K < 0$ such that for $x > K$ we have $0 < (\frac{\alpha^-}{x})^{\frac{2\beta-1}{\beta}} < 1$ and

$$\begin{aligned} v''(x) &> \int_0^{\varphi(x)} e^{(2\beta-1)t}\mu dt - \varepsilon = \mu[\frac{1}{2\beta-1}(e^{(2\beta-1)\varphi(x)} - 1)] - \varepsilon \\ &= \mu[\frac{1}{2\beta-1}((\frac{\alpha^-}{x})^{\frac{2\beta-1}{\beta}} - 1)] - \varepsilon > \mu[\frac{1}{2\beta-1}(-1)] - \varepsilon. \end{aligned}$$

Thus, taking $\varepsilon > 0$ small, and the corresponding $K > 0$, $v''(x) \geq \gamma > 0; \forall x \in [K, +\infty)$. Now, integrating over the interval $[K, x]$, for $K < x < +\infty$ we have

$$-v'(K) + v'(x) \geq \gamma(x-K). \text{Thus, } v'(x) \geq \gamma(x-K) + v'(K).$$

Therefore,

$$v'(x) \longrightarrow +\infty, \qquad \text{as} \qquad x \longrightarrow +\infty.$$

This is a contradiction since the function v' can never be greater than 1.
Case (i) and (ii) imply $\alpha^+ \neq +\infty$. Thus $-\infty < \alpha^+ < +\infty$. $\square$

8 The optimal control outside the interval $[\alpha^-, \alpha^+]$

First, we need to prove a verification theorem.
Let $U \subset \mathbb{R}^k$ the control set. Let $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ be a continuous function

such that satisfies the global Lipchitz continuity in the state variable and uniformly in the control variable.
We consider the control system

$$\dot{x} = f(x(t), u(t)), \; x(0) = x \in \mathbb{R}^n. \tag{15}$$

The controls $u(\cdot)$ are functions of time in the family,

$$\mathcal{U} = \mathcal{L}^{\infty}([\prime, \infty), \mathcal{U})$$

We set, for each $x \in \mathbb{R}^n$ and any control $u(\cdot) \in \mathcal{U}$ the Cost Functional

$$J(x, u(\cdot) = \int_0^{\infty} e^{-t} L(x(t), u(t))\, dt, \tag{16}$$

where $x(t)$ is the solution of (15), for the initial value $x(0) = x$, and for the control $u(\cdot)$.
We define the Value Function as,

$$v(x) = \inf_{u(\cdot) \in \mathcal{U}} J[x(t), u(t)] \tag{17}$$

The value function v solves the Hamilton-Jacobi-Bellman equation

$$v(x) + H(x, Dv(x)) = 0, \tag{18}$$

where

$$H(x, p) = \sup_{u \in U} \{-f(x, u)p - L(x, u)\}$$

Theorem 10. (A Verification Theorem) *We consider the optimal control problem (15), (16), (17).*
Let $W \in \mathbb{C}^1(\mathbb{R}^n)$ such that satisfies

$$W(x) + H(x, W'(x)) = 0, \forall x \in \mathbb{R}^n$$

and for all solution $x(t)$ of (15) to any initial value x given,

$$\lim_{t \to \infty} e^{-t} W(x(t)) = 0$$

Then,

i) $W(x) \leq V(x)$, $\forall x \in \mathbb{R}^n$

ii) *Given $x \in \mathbb{R}^n$, if there exists $u^*(\cdot) \in \mathcal{U}$ such that*

$$H[x^*(s), W'(x^*(s))] = -f(x^*(s), u^*(s))W'(x^*(s)) - L(x^*(s), u^*(s))$$

where $x^*(s)$ is the solution of (15) for the given control $u^*(s)$ and the initial value $x^*(s) = x$,
Then $u^*(s)$ is optimal control for the initial data x and

$$V(x) = W(x)$$

iii) *Given $x \in \mathbb{R}^n$, if there exists a sequence of controls $\Big\{u_n(\cdot)\Big\}_{n=1}^{\infty} \subset \mathcal{U}$ such that*

$$\lim_{n\to\infty} J(x, u_n(\cdot)) = W(x),$$

Then,

$$V(x) = W(x).$$

Proof. *i)* Let $x \in \mathbb{R}^n$, and let $u(\cdot) \in \mathcal{U}$ be any control. Let $x(t)$ be the solution of (15), for the control $u(\cdot)$ given and the initial value $x(0) = x$.

$$\frac{d}{dt}e^{-t}[W(x(t))] = -e^{-t}[W(x(t) - f(x(t), u(t)W'(x(t))]$$

Integrating over the interval $[0, T]$, for $T > 0$.

$$e^{-T}W(x(T)) + W(x) = \int_0^T e^{-t}[W(x(t) - f(x(t), u(t))W'(x(t))]\,dt \quad (19)$$

On the other hand, notice that

$$W(x(t)) - f(x(t), u(t))W'(x(t)) - L(x(t), u(t)) \leq$$
$$W(x(t)) + \sup_{u\in U}\Big\{ - f(x(t), u)W'(x(t)) - L(x(t), u)\Big\} = 0,$$

since W is a solution of (18). Thus,

$$W(x(t)) - f(x(t), u(t))W'(x(t)) \leq L(x(t), u(t)) \quad (20)$$

Now, combining 19) y (20) we have,

$$-e^{-T}W(x(t)) + W(x) \leq \int_0^T e^{-t}L(x(t), u(t))\,dt$$

Letting $T \uparrow \infty$,

$$W(x) \leq \int_0^\infty e^{-t} L(x(t), u(t))dt$$

since $e^{-t}W(x(t)) \to 0$, as $T \uparrow \infty$ by hypothesis. The control $u(\cdot)$ is arbitrary, then, we take the infimum over all control $u(\cdot)$

$$W(x) \leq \inf_{u \in U} \int_0^\infty e^{-t} L(x(t), u(t))dt = V(x) \qquad \square$$

ii) Given $x \in \mathbb{R}$; let's suppose that there exists $u^*(\cdot) \in \mathcal{U}$ such that

$$-L(x^*(s), u^*(s)) - f(x^*(s), u^*(s))W'(x^*(s)) = +H(x^*(s), W'(x^*(s)),$$

for almost every $s \in [0, +\infty]$. Since W is solution of (15), we can write

$$\begin{aligned} 0 =& W(x^*(s)) + H(x^*(s), W'(x^*(s))) \\ =& W(x^*(s)) - f(x^*(s), u^*(s))W'(x^*(s)) - L(x^*(s), u^*(s)), \end{aligned}$$

So

$$W(x^*(s)) - f(x^*(s), u^*(s))W'(x^*(s)) = L(x^*(s), u^*(s)) \qquad (21)$$

thus according to (19)we can write for the control $u^*(\cdot)$,

$$\begin{aligned} -e^T W(x^*(T)) + W(x) &= \int_0^T e^{-t}[W(x^*(t)) - f(x^*(t), u^*(t))W'(x^*(t))]dt \\ &= \int_0^T e^{-t} L(x^*(t), u^*(t))dt \end{aligned}$$

using (21). Letting $T \uparrow \infty$, since $e^{-T}W(x^*(T)) \to 0$, as $\uparrow \infty$, by hypothesis, we get

$$W(x) = \int_0^\infty e^{-t} L(x*(t), u^*(t))dt \geq V(x)$$

by definition of V. Therefore, since (15), we get,

$$W(x) = V(x)$$

iii) Let $x \in \mathbb{R}^n$, and let's suppose that there exists a a sequence of controls $\{U_n\}_{n=1}^\infty \subset U$ such that

$$\lim_{n \to \infty} J(x, u_n(\cdot)) = W(x)$$

By definition, $V(x) \leq J(x, u(\cdot))$; for any $u(\cdot) \in U$, In particular, for the given sequence of controls this inequality holds,

$$V(x) \leq J(x, u_n(\cdot)),$$

for all natural number n.
Letting $n \uparrow \infty$, we have

$$V(x) \leq \lim_{n\to\infty} J(x, u_n(\cdot)).$$

So,

$$V(x) \leq W(x),$$

by hypotesis.

□

Let's go back to our original optimal control problem (1), (2), (3).

Proposition 11. *For all $x \in \mathbb{R}$ such that $x \notin [\alpha^-, \alpha^+]$, there exists a sequence of controls $(u_n(\cdot)) \subset \mathcal{U}$ with $lim_{n\to\infty} u_n(\cdot) = \delta\gamma$, where δ is the Delta function and γ is the distance from x to the interval $[\alpha^-, \alpha^+]$, such that,*

$$\lim_{n\to\infty} v^{u_n(\cdot)}(x) = v(x) \tag{22}$$

Therefore, since the verification theorem, 10, outside the interval $[\alpha^-, \alpha^+]$, the optimal control is impulsive.

Proof.Case $x \in \mathbb{R}, x \notin [\alpha^-, \alpha^+], x < \alpha^-$ **.**
Let's consider the sequence of controls $(u_n(\cdot)) \subset \mathcal{U}$ defined by, for each $n \in N$

$$u_n(t) = \begin{cases} n(\alpha^- - x), & 0 \leq t < \frac{1}{n}, \\ 0, & t \geq \frac{1}{n}. \end{cases}$$

For each $n \in N$, we have the scalar control system,

$$\dot{x} = \beta x + u_n, \quad x(0) = x,$$

whose solution is,

$$x(t) = \begin{cases} x_n(t), & 0 \leq t < \frac{1}{n}, \\ x_n(\frac{1}{n}) e^{\beta(t - \frac{1}{n})}, & t \geq \frac{1}{n}, \end{cases}$$

where,

$$x_n(t) = (x + \frac{n(\alpha^- - x)}{\beta})e^{\beta t} - \frac{n(\alpha^- - x)}{\beta}, \qquad 0 \leq t < \frac{1}{n}.$$

For each $n \in N$ the cost functional is

$$\begin{aligned} v^{u_n}(x) &= \int_0^\infty e^{-t}[L(x(t)) + |u_n(t)|]dt, \\ &= \int_0^{\frac{1}{n}} e^{-t}L(x_n(t))dt, \\ &+ \int_0^{\frac{1}{n}} e^{-t}n(\alpha^- - x)dt, \\ &+ \int_{\frac{1}{n}}^\infty e^{-t}L[x_n(\frac{1}{n})e^{\beta(t-\frac{1}{n})}]dt, \end{aligned}$$

Observe that for n large enough, $n(\alpha^- - x) > \beta x$, so $x'_n(t) > 0$, hence $x_n(t)$ is increasing, then

$$x_n(t) < x_n(\frac{1}{n}), \quad \forall t,\ 0 \leq t < \frac{1}{n}.$$

Also,

$$\lim_{n\to\infty} x_n(\frac{1}{n}) = \alpha^-.$$

On the other hand, since L is convex, $L \geq 0$ and $x \leq x_n(t) \leq \alpha^-, \quad 0 \leq t < \frac{1}{n}$, there exists $K > 0$ such that

$$L[x_n(t)] \leq \max[L(x), L(\alpha^-)] \leq K, \quad \forall t, 0 \leq t < \frac{1}{n}, \forall n, \text{large enough},$$

Then,

$$0 \leq \lim_{n\to\infty} \int_0^{\frac{1}{n}} e^{-t}L(x_n(t))\,dt \leq \lim_{n\to\infty} \int_0^{\frac{1}{n}} e^{-t}K\,dt = 0.$$

This means,

$$\lim_{n\to\infty} \int_0^{\frac{1}{n}} e^{-t}L(x_n(t))\,dt = 0 \tag{23}$$

Also,

$$\lim_{n\to\infty} \int_0^{\frac{1}{n}} e^{-t}n(\alpha^- - x)\,dt = \alpha^- - x. \tag{24}$$

We may also apply the Dominated Convergence theorem to get

$$\lim_{n\to\infty}\int_{\frac{1}{n}}^{\infty} e^{-t}L[x_n(\frac{1}{n})e^{\beta(t-\frac{1}{n})}]dt = \int_0^{\infty} e^{-t}L[\alpha^- e^{\beta t}]\,dt = v(\alpha^-). \qquad (25)$$

Therefore, combining 23, 24, 25, 8 and 10, we have

$$\begin{aligned}\lim_{n\to\infty} v^{u_n}(x) &= \alpha^- - x + v(\alpha^-),\\ &= \alpha^- - x + L(\alpha^-) - \beta\alpha^-,\\ &= -x + (1-\beta)\alpha^- + L(\alpha^-),\\ &= v(x). \qquad \square\end{aligned}$$

Proof.Case $x \in \mathbb{R}, x \notin [\alpha^-, \alpha^+], x > \alpha^+$**.**
Let's consider the sequence of controls $(u_n(\cdot)) \subset \mathcal{U}$ defined by, for each $n \in N$

$$u_n(t) = \begin{cases} n(\alpha^+ - x), & 0 \le t < \frac{1}{n},\\ 0, & t \ge \frac{1}{n}.\end{cases}$$

For each $n \in N$, we have the scalar control system,

$$\dot{x} = \beta x + u_n, \quad x(0) = x,$$

whose solution is,

$$x(t) = \begin{cases} x_n(t), & 0 \le t < \frac{1}{n},\\ x_n(\frac{1}{n})e^{\beta(t-\frac{1}{n})}, & t \ge \frac{1}{n},\end{cases}$$

where,

$$x_n(t) = [x + \frac{n(\alpha^+ - x)}{\beta}]e^{\beta t} - \frac{n(\alpha^+ - x)}{\beta}, \qquad 0 \le t < \frac{1}{n}.$$

For each $n \in N$ the cost functional is

$$\begin{aligned} v^{u_n}(x) &= \int_0^{\infty} e^{-t}[L(x(t)) + |u_n(t)|]dt,\\ &= \int_0^{\frac{1}{n}} e^{-t}L(x_n(t))\,dt,\\ &+ \int_0^{\frac{1}{n}} e^{-t}n(x-\alpha^+)\,dt,\\ &+ \int_{\frac{1}{n}}^{\infty} e^{-t}L[x_n(\frac{1}{n})e^{\beta(t-\frac{1}{n})}]dt,\end{aligned}$$

Observe that for n large enough, $\beta x + n(\alpha^+ - x) < 0$, so $x'_n(t) < 0$, hence $x_n(t)$ is decreasing over $[0, \frac{1}{n}]$, then

$$x_n(t) > x_n(\frac{1}{n}), \quad \forall t,\, 0 \le t < \frac{1}{n}.$$

Also,

$$\lim_{n\to\infty} x_n(\frac{1}{n}) = \alpha^+.$$

On the other hand, since L is convex, $L \ge 0$ and $x \ge x_n(t) \ge \alpha^+, \quad 0 \le t < \frac{1}{n}$, there exists $K > 0$ such that

$$L[x_n(t)] \le \max[L(x), L(\alpha^+)] \le K, \quad \forall t, 0 \le t < \frac{1}{n}, \forall n, \text{large enough},$$

Then,

$$0 \le \lim_{n\to\infty} \int_0^{\frac{1}{n}} e^{-t} L(x_n(t))\, dt \le \lim_{n\to\infty} \int_0^{\frac{1}{n}} e^{-t} K\, dt = 0.$$

This means,

$$\lim_{n\to\infty} \int_0^{\frac{1}{n}} e^{-t} L(x_n(t))\, dt = 0 \tag{26}$$

Also,

$$\lim_{n\to\infty} \int_0^{\frac{1}{n}} e^{-t} n(x - \alpha^+)\, dt = x - \alpha^+. \tag{27}$$

We may also apply the Dominated Convergence theorem to get

$$\lim_{n\to\infty} \int_{\frac{1}{n}}^{\infty} e^{-t} L[x_n(\frac{1}{n}) e^{\beta(t-\frac{1}{n})}] dt = \int_0^{\infty} e^{-t} L[\alpha^+ e^{\beta t}]\, dt = v(\alpha^+). \tag{28}$$

Therefore, combining (26), (27), (28), (9) and (11), we have

$$\begin{aligned} \lim_{n\to\infty} v^{u_n}(x) &= x - \alpha^+ + v(\alpha^+), \\ &= x - \alpha^+ + L(\alpha^+) + \beta\alpha^+, \\ &= x + (\beta - 1)\alpha^+ + L(\alpha^+), \\ &= v(x). \qquad \square \end{aligned}$$

Reasoning as in [8, Lemma 7.1, p. 27, Chapter I] and using the verification theorem, the optimal control outside the interval $[\alpha^-, \alpha^+]$ is impulsive.

9 The Second Order Derivative of the Value Function

Proposition 12. *The value function v is in $C^2(\mathbb{R} \setminus \{\alpha^-, \alpha^+\})$. Also, the C^2 condition at $\{\alpha^-, \alpha^+\}$ for the value function v is as follows:*

$$\textit{(i)} \qquad v \text{ is } C^2 \text{ at } \alpha^- \iff 0 < \alpha^- \iff 0 < a^-.$$

$$\textit{(ii)} \qquad v \text{ is } C^2 \text{ at } \alpha^+ \iff \alpha^+ < 0 \iff a^+ < 0.$$

As a consequence v is never C^2 on $\mathbb{R}$. Moreover, in any case the free boundary set $\{\alpha^-, \alpha^+\}$ is determined in terms of the parameters of the control problem as

$$\alpha^- = \min(a^-, b^-) \quad \textit{and} \quad \alpha^+ = \max(a^+, b^+).$$

Proof. Case $\alpha^- \leq 0 \leq \alpha^+$ **.**
By Proposition 7 the control $u(\cdot) \equiv 0$ is optimal on (α^-, α^+). Hence $v = \omega$ in (α^-, α^+), where ω is the cost of the control $u(\cdot) \equiv 0$ studied in Lemma 4 Thus, $v'(x) = \omega'(x), \quad \forall x \in (\alpha^-, \alpha^+)$. In particular, by continuity of v' and ω', $v'(\alpha^-) = -1 = \omega'(\alpha^-)$ and $v'(\alpha^+) = 1 = \omega'(\alpha^+)$. Therefore, since $\omega'(a^-) = -1$, and $\omega'(a^+) = 1$, and since ω' is strictly increasing $\alpha^- = a^-$ and $\alpha^+ = a^+$. Also since the function ω is strictly convex, and since the value function v is an affine function to the left of α^- and to the right of α^+, we have $0 = v''_-(\alpha^-) = v''_+(\alpha^+)$. But $0 < \omega''_+(\alpha^-) = v''_+(\alpha^-)$ and $0 < \omega''_-(\alpha^+) = v''_-(\alpha^+)$. Therefore, $v \in C^2(\mathbb{R} \setminus \{\alpha^-, \alpha^+\})$ and v is C^2 neither at α^- nor at α^+.
Now, let's show that $a^- < b^-$. By Lemma 4 **(iv)** $\omega(x) - \beta x \omega'(x) - L(x) = 0, \quad \forall x \in (\alpha^-, \alpha^+) = (a^-, a^+)$. Thus, differentiating,

$$\omega'(x) - \beta x \omega''(x) - \beta \omega'(x) - L'(x) = 0, \qquad \forall x \in (\alpha^-, \alpha^+) = (a^-, a^+), \quad (29)$$

and inserting $x = a^- = \alpha^-$, yields

$$L'(a^-) - \beta = -1 - \beta a^- \omega''(a^-) < -1 = L'(b^-) - \beta.$$

Thus, $L'(a^-) < L'(b^-)$, hence $a^- < b^-$. So,

$$\alpha^- = a^- = \min(a^-, b^-).$$

Now, let's show that $b^+ \leq a^+$. By Lemma 4 **(iv)** inserting $x = b^+$ and using (6) yields

$$(\omega'(b^+) - 1)(1 - \beta) = \beta b^+ \omega''(b^+).$$

Let's suppose that $a^+ < b^+$. Thus, $b^+ > 0$, since $a^+ \geq 0$. So, $\beta b^+ \omega''(b^+) < 0$, which implies $\omega'(b^+) - 1 < 0$. Therefore, $\omega'(a^+) < \omega'(b^+) < 1$, since the function ω' is increasing. This is a contradiction since we know that $\omega'(a^+) = 1$. Therefore, $b^+ \leq a^+$. Hence

$$\alpha^+ = \max(a^+, b^+).$$

Case $0 < \alpha^- < \alpha^+$ **.**
Observe that by Proposition 7, the control $u^*(t) \equiv -\beta\alpha^-, \quad \forall t \geq 0$, is optimal at α^-.
Let's try to pin down the parameter α^-. For any bounded control u consider the family of bounded controls defined by

$$u_\varepsilon(t) = -\beta\alpha^- + \varepsilon u(t), \qquad \forall t \geq 0, \quad \varepsilon > 0.$$

Let $v^{u_\varepsilon}(\alpha^-)$ be the corresponding cost starting at α^-. Then

$$v^{u^*}(\alpha^-) \leq v^{u_\varepsilon}(\alpha^-), \quad \text{for all}\, \varepsilon > 0,$$

since u^* is the optimal control at α^-. Then

$$\frac{d}{d\varepsilon}(v^{u_\varepsilon}(\alpha^-))_{|\varepsilon=0} = 0.$$

Given $\varepsilon > 0$, and the control u_ε, let $x_\varepsilon(t)$ be the solution of

$$\dot{x} = \beta x + [-\beta\alpha^- + \varepsilon u], \quad x_\varepsilon(0) = \alpha^-.$$

Interchanging $\frac{d}{dt}$ and $\frac{d}{d\varepsilon}$ we see that $\frac{d}{d\varepsilon}[x_\varepsilon(t)]$ is the solution of

$$\dot{z} = F(z, u), \qquad z(0) = 0, \qquad \text{where } F(z, u) = \beta z + u.$$

Then, the variation of the constants formula gives us that

$$\frac{d}{d\varepsilon}[x_\varepsilon(t)] = e^{\beta t}[\int_0^t e^{-\beta s} u(s) ds] = \int_0^t e^{\beta(t-s)} u(s) ds.$$

Thus,

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}(v^{u_\varepsilon}(\alpha^-))_{|\varepsilon=0} = \int_0^\infty e^{-t}\{L'(x_\varepsilon(t))\frac{d}{d\varepsilon}[x_\varepsilon(t)] + u(t)\}_{|\varepsilon=0} dt \\ &= L'(\alpha^-)\int_0^\infty e^{-t}[\int_0^t e^{\beta(t-s)} u(s) ds] dt + \int_0^\infty e^{-t} u(t) dt. \end{aligned}$$

Now, integrating by parts

$$\begin{aligned} 0 &= L'(\alpha^-)\left\{(\int_0^t e^{-\beta s}u(s)ds)(\frac{1}{\beta-1}e^{(\beta-1)t})\right\}_0^\infty \\ &- L'(\alpha^-)\int_0^\infty \frac{1}{\beta-1}e^{(\beta-1)t}(e^{-\beta t}u(t))dt + \int_0^\infty e^{-t}u(t)dt. \end{aligned}$$

Therefore,

$$0 = [\frac{L'(\alpha^-)}{1-\beta}+1]\int_0^\infty e^{-t}u(t)dt.$$

Since u was arbitrary

$$\frac{L'(\alpha^-)}{1-\beta}+1=0, \qquad \text{hence } L'(\alpha^-)=\beta-1.$$

By definition $L'(b^-)=\beta-1$. Therefore, $\alpha^-=b^-$.
Now, let's show that $b^-<a^-$. Using Lemma 4 **(iv)** for $x=\alpha^-=b^-$ yields

$$\omega'(b^-)-\beta\omega'(b^-)-\beta b^-\omega''(b^-)-L'(b^-)=0.$$

Since $\beta b^-<0$ and ω is strictly convex

$$(\omega'(b^-)+1)(1-\beta)=\beta b^-\omega''(b^-)<0.$$

Then,

$$\omega'(b^-)+1<0, \qquad \text{hence } \omega'(b^-)<-1.$$

Therefore, $b^-<a^-$, since the function ω' is increasing and since $\omega'(a^-)=-1$. So,

$$\alpha^-=b^-=\min(a^-,b^-).$$

Now, let's show that $b^+<a^+$. Using Lemma 4 **(iv)** and inserting $x=b^+$ yields

$$\omega'(b^+)-\beta\omega'(b^+)-\beta b^+\omega''(b^+)-L'(b^+)=0.$$

Thus,

$$(\omega'(b^+)-1)(1-\beta)=\beta b^+\omega''(b^+)<0.$$

Then,

$$\omega'(b^+)-1<0, \qquad \text{hence } \omega'(b^+)<1.$$

Therefore,

$$b^+<a^+,$$

since the function ω' is increasing and since $\omega'(a^+) = 1$.
Now, let's show that

$$\alpha^+ = a^+ = \max{(a^+, b^+)}.$$

In fact,

$$1 = v'(\alpha^+) = \omega'(\alpha^+) = \omega'(\alpha^+).$$

Thus,

$$\alpha^+ = a^+,$$

since the function ω' is strictly increasing .
Now, let's prove that the value function v is C^2 at α^- but not at α^+. By (7)

$$v(x) - \beta x v'(x) - L(x) = 0, \qquad \forall x \in (\alpha^-, \alpha^+).$$

Thus, differentiating on the right hand side of α^-, and since

$$v'_+(\alpha^-) = -1, \quad \text{ and } L'_+(\alpha^-) = L'(b^-) = \beta - 1,$$

we have

$$-\beta\alpha^- v''_+(\alpha^-) = 0. \qquad \text{Thus, } v''_+(\alpha^-) = 0.$$

On the other hand, by the Lemma 5 $v'(x) = -1, \quad \forall x \in (\infty, \alpha^-]$.
So, $v''_-(\alpha^-) = 0$. Therefore, $v''(\alpha^-) = 0$. Hence, the value function v is C^2 at α^-.
Now, let's prove that the value function v is not C^2 at α^+. By the Lemma 5 $v'(x) = 1, \quad \forall x \in [\alpha^+, \infty)$. So, $v''_+(\alpha^+) = 0$. It suffices to show that $v''_-(\alpha^+) \neq 0$. Given $x \in (\alpha^-, \alpha^+)$, note that $0 < \alpha^-$, then there exists $T > 0$ such that

$$x(T) = \alpha^+, \text{ and } x(t) \in (\alpha^-, \alpha^+), \forall t \in [0, T).$$

Therefore, by Proposition 6 equation (12) holds. So,

$$v(x) = e^{-T} v(\alpha^-) + \int_0^T e^{-t} L(x(t)) dt.$$

Then

$$v'(x) = -(\frac{\alpha^-}{x})^{\frac{\beta-1}{\beta}} + \int_0^{\varphi(x)} e^{(\beta-1)t} L'(xe^{\beta t}) dt.$$

So

$$v''(x) = [\frac{1}{\beta x}(\frac{\alpha^-}{x})^{\frac{\beta-1}{\beta}}(\beta - 1 - L'(\alpha^-))] + \int_0^{\varphi(x)} e^{(2\beta-1)t} L''(xe^{\beta t}) dt.$$

Note that $\alpha^- = b^-$. Then $\beta - 1 - L'(\alpha^-)) = 0$. So, if $T > 0$ is such that $\varphi(x) = T$, then

$$v''(x) = \int_0^{\varphi(x)} e^{(2\beta-1)t} L''(xe^{\beta t})dt > \int_0^T e^{(2\beta-1)t}\mu dt > Te^{(2\beta-1)T}\mu > 0.$$

Letting $x \uparrow \alpha^+$, we get

$$v''_-(\alpha^+) \geq Te^{(2\beta-1)T}\mu > 0. \text{Therefore, } v''_-(\alpha^+) > v''_+(\alpha^+) = 0.$$

Hence the value function v is not C^2 at α^+.
Case $\alpha^- < \alpha^+ < 0$ **.**
Observe that by Proposition 7, the control $u^*(t) \equiv -\beta\alpha^+, \quad \forall t \geq 0$, is optimal at α^+.
Let's try to pin down the parameter α^+. For any bounded control u consider the family of bounded controls defined by

$$u_\varepsilon(t) = -b\alpha^+ - \varepsilon u(t), \qquad \forall t \geq 0.$$

where $\varepsilon > 0$ is small enough so that $u_\varepsilon(t) < 0$. Let $v^{u_\varepsilon}(\alpha^+)$ be the corresponding cost starting from α^+. Then

$$v^{u^*}(\alpha^+) \leq v^{u_\varepsilon}(\alpha^+), \quad \text{for all}\, \varepsilon > 0,$$

since u^* is the optimal control at α^+. Then

$$\frac{d}{d\varepsilon}(v^{u_\varepsilon}(\alpha^+))_{|\varepsilon=0} = 0.$$

Given $\varepsilon > 0$, and the control u_ε, let $x_\varepsilon(t)$ be the solution of

$$\dot{x} = \beta x + [-\beta\alpha^+ - \varepsilon u], \quad x_\varepsilon(0) = \alpha^+.$$

Interchanging $\frac{d}{dt}$ and $\frac{d}{d\varepsilon}$ we see that $\frac{d}{d\varepsilon}[x_\varepsilon(t)]$ is the solution of

$$\dot{z} = F(z, u), \quad z(0) = 0, \quad \text{where } F(z, u) = \beta z - u.$$

Then, the variation of the constants formula gives us that

$$\frac{d}{d\varepsilon}[x_\varepsilon(t)] \quad = \quad e^{\beta t}[\int_0^t e^{-\beta s}(-u(s))ds] = -\int_0^t e^{\beta(t-s)}u(s)ds.$$

Thus,

$$0 = \frac{d}{d\varepsilon}(v^{u_\varepsilon}(\alpha^+))_{|\varepsilon=0} = -L'(\alpha^+)\int_0^\infty e^{-t}[\int_0^t e^{\beta(t-s)}u(s)ds]dt + \int_0^\infty e^{-t}u(t)dt.$$

Now, integrating by parts

$$\begin{aligned} 0 &= -L'(\alpha^+)\{(\int_0^t e^{-\beta s}u(s)ds)(\frac{1}{\beta-1}e^{(\beta-1)t})\}_0^\infty \\ &+ L'(\alpha^+)\int_0^\infty \frac{1}{\beta-1}e^{(\beta-1)t}(e^{-\beta t}u(t))dt + \int_0^\infty e^{-t}u(t)dt. \end{aligned}$$

But

$$\{(\int_0^t e^{-\beta s}u(s)ds)(\frac{1}{\beta-1}e^{(\beta-1)t})\}_0^\infty = 0.$$

Therefore,

$$0 = [-\frac{L'(\alpha^+)}{1-\beta} + 1]\int_0^\infty e^{-t}u(t)dt.$$

Since u was arbitrary

$$-\frac{L'(\alpha^+)}{1-\beta} + 1 = 0.$$

Hence, by definition $L'(\alpha^+) = 1-\beta = L'(b^+)$. Thus,$\alpha^+ = b^+$. since the function L' is strictly increasing.
Let's prove that

$$b^+ > a^+. \quad \text{Hence} \quad \alpha^+ = \max(b^+, a^+).$$

Using Lemma 4 **(iv)** and inserting $x = b^+ = \alpha^+$, yields

$$\omega'(b^+) - \beta\omega'(b^+) - \beta b^+\omega''(b^+) - L'(b^+) = 0.$$

But $L'(b^+) = L'(\alpha^+) = 1-\beta$, ω is strictly convex and since $b^+ = \alpha^+ < 0$, we get

$$(\omega'(b^+) - 1)(1-\beta) = \beta b^+\omega''(b^+) > 0,$$

so, $\omega'(b^+) - 1 > 0$, hence $\omega'(b^+) > 1$. Therefore,$b^+ > a^+$, since the function ω' is increasing and since $\omega'(a^+) = 1$.
Let's prove that

$$b^- > a^-. \quad \text{Hence} \quad \alpha^- = \min(b^-, a^-).$$

By Lemma 4 **(iv)**, differentiating and inserting $x = b^-$,yields

$$\omega'(b^-) - \beta\omega'(b^-) - \beta b^-\omega''(b^-) - L'(b^-) = 0.$$

But $L'(b^-) = \beta - 1$, ω is strictly convex and since $b^- < b^+ = \alpha^+ < 0$, we have

$$(\omega'(b^-) + 1)(1 - \beta) = \beta b^- \omega''(b^-) > 0.$$

Then, $\omega'(b^-) + 1 > 0$, hence $\omega'(b^-) > -1$. Therefore, $b^- > a^-$, since ω' is increasing and since $\omega'(a^-) = -1$. On the other hand, note that $-1 = v'(\alpha^-) = \omega'(\alpha^-) = \omega'(a^-)$. Then $\alpha^- = a^-$, since ω' is strictly increasing. Therefore, $\alpha^- = \min(a^-, b^-) = a^-$.
Now, let's prove that the value function v is C^2 at α^+. But v is not C^2 at α^+.
We recall (7) $v(x) - bxv'(x) - f(x) = 0$, $\forall x \in (\alpha^-, \alpha^+)$. Thus, differentiating on the left hand side of α^+, and since $v'_-(\alpha^+) = 1$, and $L'_-(\alpha^+) = L'(b^+) = 1 - \beta$, yields

$$-\beta\alpha^+ v''_-(\alpha^+) = 0. \quad \text{Thus } v''_-(\alpha^+) = 0.$$

On the other hand, by Lemma 5 $v'(x) = 1$, $\forall x \in (\alpha^+, \infty)$. So, $v''_+(\alpha^+) = 0$. Therefore, $v''(\alpha^+) = 0$. Hence, the value function v is C^2 at α^+.
Now, let's prove that the value function v is not C^2 at α^-. By Lemma 5 $v'(x) = -1$, $\forall x \in (-\infty, \alpha^-)$. So, $v''_-(\alpha^-) = 0$. It suffices to show that $v''_+(\alpha^-) \neq 0$. Given $x \in (\alpha^-, \alpha^+)$, note that $\alpha^+ < 0$, then there exists $T > 0$ such that $x(T) = \alpha^+$, and $x(t) \in (\alpha^-, \alpha^+), \forall t \in [0, T)$. Therefore, by Proposition 6 equation (12) holds. So,

$$v(x) = e^{-T} v(\alpha^+) + \int_0^T e^{-t} L(x(t)) dt.$$

So,

$$v'(x) = \left(\frac{\alpha^+}{x}\right)^{\frac{\beta-1}{\beta}} + \int_0^{\varphi(x)} e^{(\beta-1)t} L'(xe^{\beta t}) dt.$$

Then

$$v''(x) = \left[-\frac{1}{\beta x}\left(\frac{\alpha^+}{x}\right)^{\frac{\beta-1}{\beta}} (\beta - 1 + L'(\alpha^+))\right] + \int_0^{\varphi(x)} e^{(2\beta-1)t} L''(xe^{\beta t}) dt.$$

Note that $\alpha^+ = b^+$. Then $\beta - 1 + L'(\alpha^+)) = 0$. So, if $T > 0$ is such that $\varphi(x) = T$, then

$$v''(x) = \int_0^{\varphi(x)} e^{(2\beta-1)t} L''(xe^{\beta t}) dt > \int_0^T e^{(2\beta-1)t} \mu dt > T e^{(2\beta-1)T} \mu > 0.$$

Letting $x \downarrow \alpha^-$, yields $v''_+(\alpha^-) \geq T e^{(2\beta-1)T} \mu > 0$. Therefore, $v''_+(\alpha^-) > v''_-(\alpha^-) = 0$. Hence the value function v is not C^2 at α^-.

References

[1] I. Capuzzo Dolcetta, P.L. Lions, Viscosity Solutions and Applications, *Lecture Notes in Mathematics, Springer Verlag*, **1660**, Montecatini Terme, (1995), 134–185.

[2] M.G.Crandall, H. Ishii, & P.L. Lions, User's Guide to Viscosity Solutions of Second Order Partial Differential Equations, *Bull. Amer. Math. Soc.* **32** (1994), 1322–1331.

[3] M.G. Crandall, & P.L. Lions, Viscosity Solutions of Hamilton-Jacobi Equations, *Trans. Amer.Math.Soc.*, **277**(1984) 1–42.

[4] J.R. Dorroh and G. Ferreyra, A Free Boundary Problem in $\mathbb{R}^d$ with Both Smooth and Nonsmooth Fit *SIAM J. Control Optim.*, **2** Vol 36, March 1998, 579–589.

[5] Ferreyra, G. and Hijab, O., Smooth Fit For Some Bellman Equation, *Pitman Research Notes in Math.* , **324**(1995) 116–122.

[6] Ferreyra, G. and Hijab, O., A Simple Free Boundary Problem in $\mathbb{R}^d$ *SIAM J. Control and Optimization* **2** , Vol 32 (1994), 501–515.

[7] W.H. Fleming & R.W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York, 1975.

[8] W.H. Fleming & H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Appl. Math., Vol 25, Springer-Verlag, New York, 1993.

[9] S.E. Shreve, H.M. Soner, Optimal Investment and Consumption with Transaction Costs, *The Annals of Applied Probability* , **3**, Vol. 4, 1994, 609–692.