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Improved bounds for $(kn)!$ and the factorial polynomial

Cotas mejoradas para $(kn)!$ y el polinomio factorial

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Abstract

In this article, improved bounds for $(kn)!$ and the factorial polynomial are obtained using Kober's inequality and Lagrange's identity.
Key words and phrases: factorial, factorial polynomial, Kober's inequality, Lagrange's inequality.

Resumen

En este artículo se obtienen cotas mejoradas para $(kn)!$ y el polinomio factorial usando la desigualdad de Kober y la identidad de Lagrange.
Palabras y frases clave: factorial, polinomio factorial, desigualdad de Kober, identidad de Lagrange.

1 Introduction

Factorials are ubiquitous in mathematics and the natural sciences. Thus, the evaluation of factorials is necessary in many areas. Unfortunately, there is no formula for the factorials of n. This fact was already recognised by Euler who, refering to factorials, stated that its general term cannot be exhibited algebraically [3]. However, approximations and bounds for $n!$ are known. On one hand, we have the famous Stirling's approximation given by $n! \approx \sqrt{2\pi n}(n/e)^n$ for large n. On the other hand, recently, bounds for $n!$, $(2n)!, \ldots, (kn)!$ and the factorial polynomial have been found [2,6] using the arithmetic-geometric

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inequality (AGI). Thus, the problem of finding better bounds and approximations for the factorial is an interesting line of inquiry.

In this note, we show that improved bounds for the factorial polynomial can be found using the Kober inequality [5] together with the Lagrange identity [1].

The Lagrange identity can be stated as follows [1]: Let $\{a_1,\, a_2,\, ...,\, a_N\}$ and $\{b_1,\, b_2,\, ...,\, b_N\}$ be any two sets of real numbers; then
$\left(\sum_{k=1}^{N} a_k b_k\right)^2 = \left(\sum_{k=1}^{N} a_k^2\right)\left(\sum_{k=1}^{N} b_k^2\right) - \sum_{1\le k<j\le N}(a_k b_j - a_j b_k)^2.$

The Kober inequality can be stated as follows [5]: Let $\{a_1, a_2, ..., a_N\}$ be any set of positive numbers with arithmetic and geometric means M_a and M_g, respectively; then
$N(M_a - M_g) \le \sum_{j<k}^{N}\left(\sqrt{a_j}-\sqrt{a_k}\right)^2 \le N(N-1)(M_a - M_g).$

This last inequality is less known than the AGI, however it is known that it gives sharper bounds than the AGI [1]. A new improved bound for $(kn)!$ is found as a particular case of the bound for the factorial polynomial.

The factorial polynomial (also known as falling factorial, binomial polynomial, or lower factorial) is defined by $x^{(0)} = 1$, $x^{(n)} = x(x-1)(x-2)\cdots(x-n+1)$ [4].

2 Results

Theorem 1.
$\left(x^2+(1-n)x-\frac{n^3-4n^2+5n-2}{12}\right)^{n/2} \le x^{(n)} \le \left(x^2+(1-n)x+\frac{3n^2-7n+2}{12}\right)^{n/2}.$

Proof. The Kober inequality, applied to the set $\{x, (x-1), \ldots, (x-n+1)\}$, with $x \ge n$ and $a_i = (x-i+1)^2$ gives

$$n(M_a - M_g) \le \sum_{j<k}(k-j)^2 \le n(n-1)(M_a - M_g), \tag{1}$$

where

$$M_a = \frac{1}{n}\sum_{i=1}^{n}(x-i+1)^2 = x^2+(1-n)x+\frac{n^2}{3}-\frac{n}{2}+\frac{1}{6}$$

and

$$M_g = \left(\prod_{i=1}^{n}(x-i+1)^2\right)^{1/n} = \left(\prod_{i=1}^{n}(x-i+1)\right)^{2/n} = \left(x^{(n)}\right)^{2/n}.$$

Now, from the Lagrange identity with $a_i = i$ and $b_i = 1$ we find

$$\sum_{j<k}(k-j)^2 = \frac{n^2(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{4} = \frac{(n+1)n^2(n-1)}{12}.$$

Finally, substitution of the last three expressions into (1) gives the result. $\square$

Corollary 1. $\left(\frac{(n+1)}{12}(5n+2-n^2)\right)^{n/2} \leq n! \leq \left(\frac{(n+1)}{12}(3n+2)\right)^{n/2}.$

Proof. Set $x = n$ and use $n^{(n)} = n!$. $\square$

Corollary 2.

$$\left((kn)^2 + (1-n)kn - \frac{n^3-4n^2+5n-2}{12}\right)^{n/2} [((k-1)\,n)!] \leq (kn)!$$
$$\leq \left((kn)^2 + (1-n)kn + \frac{3n^2-7n+2}{12}\right)^{n/2} [((k-1)\,n)!]\,.$$

Proof. Set $x = kn$ and use the property $(kn)^{(n)} = \frac{(kn)!}{((k-1)n)!}$. $\square$

Since the lower bounds are only valid for small values of n ($n \leq 5$) we will concentrate in what follows in the upper bounds.

Theorem 2.

$$\begin{aligned}(kn)! \quad &\leq \quad \left(\prod_{i=0}^{k-1}\left[\left((k-i)n + \frac{1-n}{2}\right)^2 - \frac{n+1}{12}\right]\right)^{n/2} \\ &\leq \quad \big(\tfrac{1}{720}[20 + 20(6k-1)n + 5(-15-24k+52k^2)n^2 \\ &\qquad +10(1-18k-16k^2+24k^3)n^3 \\ &\qquad +(21+16k-104k^2-64k^3+80k^4)n^4]\big)^{nk/4}. \end{aligned} \tag{2}$$

Proof. From *Corollary 2*, $(kn)! \leq \left(\left(kn + \frac{1-n}{2}\right)^2 - \frac{n+1}{12}\right)^{n/2} [((k-1)\,n)!]$

$$\leq \left(\left(kn + \tfrac{1-n}{2}\right)^2 - \tfrac{n+1}{12}\right)^{n/2} \left(\left((k-1)n + \tfrac{1-n}{2}\right)^2 - \tfrac{n+1}{12}\right)^{n/2} [((k-2)\,n)!]$$
$$\leq \left(\left(kn + \tfrac{1-n}{2}\right)^2 - \tfrac{n+1}{12}\right)^{n/2} \left(\left((k-1)n + \tfrac{1-n}{2}\right)^2 - \tfrac{n+1}{12}\right)^{n/2}$$
$$\left(\left((k-2)n + \tfrac{1-n}{2}\right)^2 - \tfrac{n+1}{12}\right)^{n/2} [((k-3)\,n)!]$$
$$\leq \cdots$$

$$\leq \left(\left(kn+\tfrac{1-n}{2}\right)^2-\tfrac{n+1}{12}\right)^{n/2}\left(\left((k-1)n+\tfrac{1-n}{2}\right)^2-\tfrac{n+1}{12}\right)^{n/2}$$
$$\left(\left((k-2)n+\tfrac{1-n}{2}\right)^2-\tfrac{n+1}{12}\right)^{n/2}\cdots$$
$$\left(\left((k+1-k)n+\tfrac{1-n}{2}\right)^2-\tfrac{n+1}{12}\right)^{n/2}[((k-k)\,n)!]$$
$$\leq \left(\left(kn+\tfrac{1-n}{2}\right)^2-\tfrac{n+1}{12}\right)^{n/2}\left(\left((k-1)n+\tfrac{1-n}{2}\right)^2-\tfrac{n+1}{12}\right)^{n/2}$$
$$\left(\left((k-2)n+\tfrac{1-n}{2}\right)^2-\tfrac{n+1}{12}\right)^{n/2}\cdots\left(\tfrac{1}{12}(n+1)(3n+2)\right)^{n/2}$$
$$\leq \left[\left(\left(kn+\tfrac{1-n}{2}\right)^2-\tfrac{n+1}{12}\right)\left(\left((k-1)n+\tfrac{1-n}{2}\right)^2-\tfrac{n+1}{12}\right)\right.$$
$$\left.\left(\left((k-2)n+\tfrac{1-n}{2}\right)^2-\tfrac{n+1}{12}\right)\cdots\left(\tfrac{1}{12}(n+1)(3n+2)\right)\right]^{n/2}.$$

The set $\Big\{\left(kn+\frac{1-n}{2}\right)^2-\frac{n+1}{12},\left((k-1)n+\frac{1-n}{2}\right)^2-\frac{n+1}{12},\cdots,$ $\frac{1}{12}(n+1)(3n+2)\Big\}$ contains k terms. Applying Kober's theorem again with $a_i=\left[\left((k-i)n+\frac{1-n}{2}\right)^2-\frac{n+1}{12}\right]^2$ gives

$$\begin{aligned} k(M_a-M_g) &\leq \sum_{i<j}^{k}\left[\left((k-i)n+\frac{1-n}{2}\right)^2-\left((k-j)n+\frac{1-n}{2}\right)^2\right]^2 \\ &\leq k(k-1)(M_a-M_g), \end{aligned} \tag{3}$$

where

$$\begin{aligned} M_a &= \frac{1}{k}\sum_{i=0}^{k-1}\left[\left((k-i)n+\frac{1-n}{2}\right)^2-\frac{n+1}{12}\right]^2 \\ &= \frac{1}{720}\left[20+20(-1+6k)n+5(-15-12k+64k^2)n^2+\right. \\ &\quad \left.10(1-18k-4k^2+36k^3)n^3+3(7-40k^2+48k^4)n^4\right] \end{aligned}$$

and

$$M_g=\left(\prod_{i=0}^{k-1}\left[\left((k-i)n+\frac{1-n}{2}\right)^2-\frac{n+1}{12}\right]\right)^{2/k}.$$

Now, using the Lagrange identity with $a_i=\left((k-i)n+\frac{1-n}{2}\right)^2$ and $b_i=1$ we obtain

$$\sum_{i<j}^{k}\left[\left((k-i)n+\frac{1-n}{2}\right)^2-\left((k-j)n+\frac{1-n}{2}\right)^2\right]^2=$$
$$k\sum_{i=0}^{k-1}\left((k-i)n+\frac{1-n}{2}\right)^4-\left[\sum_{i=0}^{k-1}\left((k-i)n+\frac{1-n}{2}\right)^2\right]^2$$
$$=\frac{1}{180}k^2(k^2-1)n^2\left[15+30kn-4n^2+16k^2n^2\right].$$

Substitution of the last expression into (3) yields (2). □

Corollary 3. $(2n)!\le\left[\left(2n+\frac{1-n}{2}\right)^2-\frac{n+1}{12}\right]^{n/2}\left[\left(n+\frac{1-n}{2}\right)^2-\frac{n+1}{12}\right]^{n/2}$.

Proof. Set $k=2$. □

Corollary 4. $(3n)!\le\left[\left(3n+\frac{1-n}{2}\right)^2-\frac{n+1}{12}\right]^{n/2}\left[\left(2n+\frac{1-n}{2}\right)^2-\frac{n+1}{12}\right]^{n/2}$
$\left[\left(n+\frac{1-n}{2}\right)^2-\frac{n+1}{12}\right]^{n/2}$.

Proof. Set $k=3$. □

What remains to be proven is that our result is indeed sharper than the previous one. The result of [6] can be rewritten concisely as

$$(kn)!\le\left[\prod_{i=0}^{k-1}\left(\frac{(2k-(2i+1))n+1}{2}\right)\right]^n. \tag{4}$$

Then, we have

Theorem 3: $\left(\prod_{i=0}^{k-1}\left[\left((k-i)n+\frac{1-n}{2}\right)^2-\frac{n+1}{12}\right]\right)^{n/2}\le\left[\prod_{i=0}^{k-1}\left(\frac{(2k-(2i+1))n+1}{2}\right)\right]^n$.

Proof: It is only necessary to show that the *ith* term of the right hand side of (4) is indeed greater than the *ith* term of the right hand side of the first inequality of (2). We will proceed by *reductio ad absurdum* and suppose the contrary, that is,
$\frac{(2k-(2i+1))n+1}{2}\le\left((k-i)n+\frac{1-n}{2}\right)^2-\frac{n+1}{12}$. After some simplifications on the last expression we arrive at $n+1\le 0$, which is a contradiction since $n\ge 0$.

3 Examples

In order to compare with previous estimates, we evaluate bounds for 9!, 10! and 12!. By *Corollary 3* we obtain

$$10! \leq \left(\frac{127}{2}\right)^{5/2}\left(\frac{17}{2}\right)^{5/2} \approx 6.76833 \times 10^6$$

which is a better bound than

$$24^5 \approx 7.96262 \times 10^6$$

obtained from Corollary 3 of [6]. By *Corollary 3* we get

$$100! \leq (5696)^{50/2}(646)^{50/2} \approx 1.39657 \times 10^{164}$$

which a better bound than

$$\left(\frac{51 \times 151}{4}\right)^{50} \approx 1.67632 \times 10^{164}$$

obtained from *Corollary 3* of [6]. By *Corollary 4* we have

$$9! \leq \left(\frac{191}{3}\right)^{3/2}\left(\frac{74}{3}\right)^{3/2}\left(\frac{11}{3}\right)^{3/2} \approx 4.36959 \times 10^5$$

which a better bound than

$$80^3 \approx 5.12000 \times 10^5$$

found by *Corollary 4* of [6]. Finally, by *Corollary 4* we obtain

$$12! \leq \left(\frac{659}{6}\right)^{2}\left(\frac{251}{6}\right)^{2}\left(\frac{35}{6}\right)^{2} \approx 7.18368 \times 10^8$$

which is a better bound than

$$\frac{1365^4}{2^{12}} \approx 8.47560 \times 10^8$$

found by *Corollary 4* of [6].

We believe that other types of bounds, along the lines of [2] but using Kober's inequality instead of the AGI could be found for the factorial of n.

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