Maps onto Certain Fano Threefolds

Ekaterina Amerik

Received: May 15, 1997
Revised: July 11, 1997

Communicated by Thomas Peternell

Abstract. We prove that if X is a smooth projective threefold with $b_2 = 1$ and Y is a Fano threefold with $b_2 = 1$, then for a non-constant map $f : X \to Y$, the degree of f is bounded in terms of the discrete invariants of X and Y. Also, we obtain some stronger restrictions on maps between certain Fano threefolds.

1991 Mathematics Subject Classification: 14E99, 14J45

1. Introduction

Let X, Y be smooth complex n-dimensional projective varieties with $Pic(X) \cong Pic(Y) \cong \mathbf{Z}$. Let $f : X \to Y$ be a non-constant morphism. It is a trivial consequence of Hurwitz's formula

$$K_X = f^*K_Y + R$$

that if Y is a variety of general type, then $deg(f)$ is bounded in terms of the numerical invariants of X and Y, and in particular all the morphisms from X to Y fit in a finite number of families.

If we drop the assumption that Y is of general type, then this assertion is no longer quite true. Indeed, if Y is a projective space $\mathbf{P}^n$, for any X we can construct infinitely many families of maps $X \to Y$: take an embedding of X in $\mathbf{P}^N$ by any very ample divisor on X and then project the image to $\mathbf{P}^n$. However, one might ask if $\mathbf{P}^n$ is the only variety with this property (the following conjectures are suggested by A. Van de Ven) :

Conjecture A: *Let X, Y be as above and $Y \not\cong \mathbf{P}^n$. Then there is only finitely many families of maps from X to Y. Moreover, the degree of a map $f : X \to Y$ can be bounded in terms of the discrete invariants of X and Y.*

A weaker version is the following

Conjecture B: *Let X, Y be smooth n-dimensional projective varieties with $b_2(X) = b_2(Y) = 1$. Suppose $Y \not\cong \mathbf{P}^n$ and, if $n = 1$, that Y is not an elliptic curve. Then the degree of a map $f : X \to Y$ can be bounded in terms of the discrete invariants of X and Y.*

Remark: If $n = 1$, the Conjecture A is empty and the Conjecture B is trivial. If $n = 2$, one must check the Conjecture A with Y a K3-surface, and at the moment I do not know how to do this. This problem, of course, does not arise for Conjecture B, which again becomes a triviality in dimension two (note that if for a smooth complex projective variety V we have $b_1(V) \neq 0$ and $b_2(V) = 1$, then V is a curve). The assumption in the Conjecture B that Y is not an elliptic curve is , of course, necessary: any torus has endomorphisms of arbitrarily high degree given by multiplication by an integer.

Evidence: It seems likely that "the more ample is the canonical sheaf on Y, the more difficult it becomes to produce maps from X to Y". Of course, the projective space has the "least ample" canonical sheaf: $K_{\mathbf{P}^n} = -(n+1)H$, where H is a hyperplane. The next case is that of a quadric: $K_{Q_n} = -nH$ with H a hyperplane section. For $n = 3$, it has been proved by C.Schuhmann ([S]) that the degree of a map from a smooth threefold X with Picard group $\mathbf{Z}$ to the three-dimensional quadric is bounded in terms of the invariants of X. In [A], I have suggested a simpler method to prove results of this kind, which also generalizes to higher dimensions.

The main purpose of this paper is to show by a rather simple method that for Fano threefolds Y, at least for those with very ample generator of the Picard group, the above Conjecture B is true (we also show that for many of such threefolds Conjecture A holds). The boundedness results are proved in the next section. In Section 3, we obtain in a similar way a strong restriction on maps between "almost all" Fano threefolds with Picard group $\mathbf{Z}$. This is related to the "index conjecture" of Peternell which states that if $f : X \to Y$ is a map between Fano varieties of the same dimension with cyclic Picard group, then the index of Y is not smaller than that of X. This conjecture is studied for Fano threefolds by C.Schuhmann in her thesis, and one of her main results is that there are no maps from such a Fano threefold of index two to a Fano threefold of index one with reduced Hilbert scheme of lines. An extension of this result appears also in Theorem 3.1 of this paper ; however, there is at least one Fano threefold of index one with non-reduced Hilbert scheme of lines, namely, Mukai and Umemura's V_{22}. The last section of this paper deals with this variety: it is proved that a Fano threefold of index two with Picard group $\mathbf{Z}$ does not admit a map onto it. One would think that the Mukai-Umemura V_{22} is the only Fano threefold of genus at least four with cyclic Picard group and non-reduced Hilbert scheme of lines. The proof of this would be a solution to the "index conjecture" in the three-dimensional case (recall that a Fano threefold of index one and genus at most three has the third Betti number which is bigger than the third Betti number of any Fano threefold of index two ([I1] ,table 3.5), so we do not have to consider the case of genus less than four to prove the index conjecture). In fact even a weaker statement would suffice (see Theorem 3.1).

This paper can be viewed as a very extensive appendix to [A], as a large part of the method is described there.

We will often use the following notations: Generally, for $X \subset \mathbf{P}^n$, H_X denotes the hyperplane section divisor on X. Also, for X with cyclic Picard group, we will call H_X the ample generator of $Pic(X)$ (in this paper it will mostly be assumed that H_X is very ample). By V_k, following Iskovskih, we will often denote a Fano threefold with cyclic Picard group, which has index one and for which $H_X^3 = k$ (k will be called the degree of this Fano threefold). For Grassmann varieties, we use projective notation: $G(k,n)$ denotes the variety of projective k-subspaces in the projective n-space.
Finally, throughout the paper we work over the field of complex numbers.

Acknowledgments: I would like to thank Professor A. Van de Ven for many helpful discussions. I am grateful to Frank-Olaf Schreyer for explaining me many facts on V_{22} and for letting me use his unfinished manuscript [Sch], and also to Aleksandr Kuznetsov for giving me his master's thesis [K]. The final version of this paper was written during my stay at the University of Bayreuth, to which I am grateful for its hospitality and support.

2. Boundedness

Let Y be a Fano threefold such that $Pic(Y) \cong \mathbf{Z}$, and suppose that the positive generator of the Picard group is very ample. When speaking of $deg(Y)$ and other notions related to the projective embedding (e.g. the sectional genus $g(Y)$ of Y) we will suppose that this embedding is given by global sections of the generator.
It is well-known ([I],I, section 5) that if Y is of index two, then lines on Y are parameterized by a smooth surface F_Y (the Fano surface) on Y. A general line on Y has trivial normal bundle, and there is a curve on F which parametrizes lines with the normal bundle $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$ (let us call them (-1,1)-lines). If Y is of index one, than Y contains a one-dimensional family of lines ([I], II, section 3); the normal bundle of a line is then either $\mathcal{O}_{\mathbf{P}^1}(-1) \oplus \mathcal{O}_{\mathbf{P}^1}$, or $\mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. In the last case such a line is of course a singular point of the Hilbert scheme. In the sequel we will use the simple fact that if the Hilbert scheme of lines on a Fano threefold of index one is non-reduced, i.e. every line of one of its irreducible components is (-2,1), then the surface covered by the lines of this component is either a cone, or a tangent surface to a curve.
If the generator H_Y of $Pic(Y)$ is not very ample, there still exist "lines" on Y: we call a curve C a line if $C \cdot H_Y = 1$. In this case, however, there exist other possibilities for the normal sheaf $N_{C,Y}$. If Y is a threefold of index 2 and $H_Y^3 = 1$, C can even be a singular curve and, moreover, if we want our "lines" to fit into a Hilbert scheme, we must also allow embedded points ([T]).
At this point, it is convenient to recall from [I] which Fano threefolds have very ample/not very ample generator of the Picard group. For index two, the threefolds with very ample generator are cubics, intersections of two quadrics and the linear section of $G(1,4)$; the other threefolds are double covers of $\mathbf{P}^3$ branched in a quartic (quartic double solids) and double covers of the Veronese cone branched in a cubic section of it (double Veronese cones). For index one, we have nine families of threefolds

with very ample generators, plus double covers of the quadric branched in a quartic section and double covers of $\mathbf{P}^3$ branched in a sextic.
Often we will assume here for simplicity that H_Y is very ample, and discuss the other case in remarks.
We start by proving the following

Proposition 2.1 *A) If Y is a Fano threefold (with $Pic(Y) \cong \mathbf{Z}$, H_Y very ample) of index 2 such that the surface $U_Y \subset Y$ which is the union of all (-1,1)-lines on Y is in the linear system $|iH_Y|$ with $i \geq 5$, then for any threefold X, $Pic(X) \cong \mathbf{Z}$, the degree of a map $f : X \to Y$ is bounded in terms of the discrete invariants of X.*
B) If Y is a Fano threefold of index 1 with $Pic(Y) \cong \mathbf{Z}$, H_Y very ample, such that the surface $S_Y \subset Y$ which is the union of all lines on Y is in the linear system iH_Y with $i \geq 3$, then for any threefold X, $Pic(X) \cong \mathbf{Z}$, the degree of a map $f : X \to Y$ is bounded in terms of the discrete invariants of X.

Proof: Let m be such that $f^*H_Y = mH_X$. Notice that by Hurwitz' formula, our conditions on U_Y resp. S_Y just mean that if $deg(f)$ is big enough, then not the whole inverse image of U_Y resp. S_Y is contained in the ramification. Indeed, if Y is, say, of index one, we have $K_Y = -H_Y$. The Hurwitz formula reads

$$K_X = -mH_X + R.$$

If the whole inverse image of S_Y is in the ramification, then R is at least $\frac{3}{2}mH_X$, so m cannot get very big. Therefore one gets that the inverse image D of a general (-1,1)-line on Y (in the index-two case) or a general line on Y (in the index-one case) has a reduced irreducible component C.
Let Y be a Fano threefold of index two satisfying $U_Y = iH_Y$ with $i \geq 5$. For C and D as above, there is a natural morphism

$$\phi : (\mathcal{I}_C/\mathcal{I}_C^2)^* \to (\mathcal{I}_D/\mathcal{I}_D^2)^*|_C = \mathcal{O}_C(m) \oplus \mathcal{O}_C(-m),$$

and this map must be an isomorphism at a smooth point of D, i.e. at a sufficiently general point of C, as C is reduced. Now, also due to the fact that C is reduced, the natural map

$$\psi : T_X|_C \to (\mathcal{I}_C/\mathcal{I}_C^2)^*$$

is a generic surjection. Therefore if we find an integer j such that $T_X(j)$ is globally generated, we must have $m \leq j$.
Such j depends only on the discrete invariants of X. Indeed, let A be a very ample multiple of H_X. A linear subsystem of the sections of A gives an embedding of a threefold X into $\mathbf{P}^7$. We have

$$T_X(K_X) = \Lambda^2\Omega_X.$$

$\Lambda^2\Omega_X$ is a quotient of $\Lambda^2\Omega_{\mathbf{P}^7}|_X$, and we deduce from this that $\Lambda^2\Omega_X(3A)$ is generated by the global sections. So $T_X(K_X + 3A)$ is generated by the global sections, and j can be taken such that $K_X + 3A = jH_X$. So one only needs to know which multiple of H_X is very ample, and this can be expressed in terms of the discrete invariants of X (see for example [D] for many results in this direction).

The case of index one is completely analogous: a normal bundle of any line on a Fano threefold of index one has a negative summand.

REMARK A: The assumption on the very ampleness of the generator of $Pic(Y)$ is not really necessary to prove Proposition 2.1. Otherwise, we call "lines" curves C satisfying $C \cdot H_Y$=1. These curves are rational. One has then to count with the possibility that e. g. some of the "lines" on such a Fano 3-fold of index two can have normal bundle $\mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1}(2)$, but this is not really essential for the argument: as soon as we can find sufficiently big 1-parameter family of smooth rational curves with a negative summand in the normal bundle, our method works.

EXAMPLES OF FANO THREEFOLDS Y SATISFYING OUR ASSUMPTIONS ON S_Y, U_Y:

1) Y a cubic in $\mathbf{P}^4$ and
2) Y an intersection of two quadrics in $\mathbf{P}^5$. To check this is more or less standard and almost all details can be found in [CG] for a cubic and in [GH] (Chapter 6) for an intersection of two quadrics. For convenience of the reader, we give here the argument for Y an intersection of two quadrics in $\mathbf{P}^5$:
Let $F \subset G(1,5)$ be a surface which parametrizes lines on Y (Fano surface) , and let $\mathcal{U} \to F$ be the family of these lines. The ramification locus of the natural finite map $\mathcal{U} \to Y$ consists exactly of (-1,1)-lines, that is, the surface M covered by (-1,1)-lines on Y is exactly the set of points of Y through which there pass less than four lines (of course there are four lines through a general point of Y). F is the zero-scheme of a section of the bundle $S^2U^* \oplus S^2U^*$ on $G(1,5)$. A standard computation with Chern classes yields then that $K_F = \mathcal{O}_F$ (in fact, F is an abelian variety ([GH])). For a general line $l \subset Y$ consider a curve $C_l \subset F$ which is the closure in F of lines intersecting l and different from l. C_l contains l iff l is (-1,1). C_l is smooth for any l ([GH]). By adjunction, C_l has genus 2. So the ramification R of the natural 3:1 morphism $h_l : C_l \to l$ sending l' to $l \cap l'$ (with l general, i.e. not a (-1,1)-line) has degree 8. The branch locus of h consists of intersection points of l and the surface M of (-1,1)-lines, and so we have that this surface is in $|iH_Y|$ with $i \geq 4$ and $i = 4$ only if there are only 2 lines through a general point of M. This is again impossible: otherwise, for l a (-1,1)-line, C_l would be birational to l. In fact, one gets that $i = 8$.
3) Y a quartic double solid. The computations are rather similar, and the best reference is [W]. Bitangent lines to the quartic surface give pairs of "lines" on Y as their inverse images under the covering map. Welters proves the following results: the Fano surface F_Y has only isolated singularities (and is smooth for a general Y); the curve C_l for a general l is smooth except for one double point; there are 12 "lines" through a general point of Y; $p_a(C_l) = 71$. We use these results to conclude that Y satisfies our assumptions.
4) Y is a "sufficiently general" Fano threefold of index one (of course we assume that $Pic(Y) \cong \mathbf{Z}$ and that the positive generator of $Pic(Y)$ is very ample), $deg(Y) \neq 22$: see [I], II, proof of th. 6.1. It is computed there that a Fano threefold Y of index one (with very ample H_Y) with reduced scheme of lines satisfies our assumption on S_Y iff $deg(Y) \neq 22$. By the classification of Mukai ([M]), any Fano threefold of index one as above except V_{22}'s is a hyperplane section of a smooth (Fano) fourfold. Clearly, a general line on a Fano fourfold of index two has trivial normal bundle. So a general

hyperplane section of such a fourfold has reduced Hilbert scheme of lines.
5) Y any Fano threefold of index one and genus 10: Prokhorov shows in [P] that the Hilbert scheme of lines on *any* such threefold is reduced.
6) Y any Fano threefold V_{14} of index one and genus 8: such a threefold is a linear section of $G(1,5)$ in the Plücker embedding. Iskovskih shows in [I], II, proof of th. 6.1 (vi), that on such a threefold with reduced scheme of lines, lines will cover a surface which is linearly equivalent to $5H$. So one sees that if the lines cover only H or $2H$, the scheme of lines is non-reduced and the surface covered by lines consists of one or two components which are hyperplane sections of Y. Moreover, as a V_{14} does not contain cones, all the lines in one of the components must be tangent to some curve A. One checks easily that this curve is a rational normal octic. A is then the Gauss image of a rational normal quintic B in $\mathbf{P}^5$ ([A], proof of Proposition 3.1(ii)). This makes it possible to check that there is no smooth three-dimensional linear section of $G(1,5)$ containing the tangent surface to A. Indeed, one can assume that B is given as

$$(x_0^5 : x_0^4 x_1 : ... : x_1^5), (x_0 : x_1) \in \mathbf{P}^1;$$

one computes then that the Gauss image of B in $G(1,5) \subset \mathbf{P}^{14}$ (where $G(1,5)$ is embedded to $\mathbf{P}^{14}$ by Plücker coordinates (z_i), the order of which we take as follows: for a line l through $p = (p_0 : ... : p_5)$ and $q = (q_0 : ... : q_5)$ we take $z_0 = p_0q_1 - p_1q_0; z_1 = p_0q_2 - p_2q_0; ...; z_5 = p_1q_2 - p_2q_1; ...; z_{14} = p_4q_5 - p_5q_4$) generates the linear subspace L given by the following equations:

$$z_2 = 3z_5, z_3 = 2z_6, z_4 = 5z_9,$$

$$z_7 = 3z_9, z_8 = 2z_{10}, z_{11} = 3z_{12}.$$

So we must consider all the projective 9-subspaces through L and prove that the intersection of every such space with $G(1,5)$ is singular. This can be done for example as follows: let $\mathcal{L} \cong \mathbf{P}^5$ be a parametrizing variety for these 9-subspaces. Notice that the points $x = (1 : 0 : ... : 0)$ and $y = (0 : ... : 0 : 1)$ belong to our curve A. Notice that if t is a point of A, then the set $\mathcal{L}_t = \{M \in \mathcal{L} : M \cap G(1,5) \text{ is singular at } t\}$ is a hyperplane in $\mathcal{L}$. If we see that these sets are different at different points t, we are done. It is not difficult to check explicitly (writing down the matrix of partial derivatives) that for $x = (1 : 0 : ... : 0) \in A$ and $y = (0 : ... : 0 : 1) \in A$, $\mathcal{L}_x \neq \mathcal{L}_y$: if a 9-space M through L is given by the equations

$$a_{1i}(z_2 - 3z_5) + a_{2i}(z_3 - 2z_6) + a_{3i}(z_7 - 3z_9) +$$

$$+ a_{4i}(z_8 - 2z_{10}) + a_{5i}(z_{11} - 3z_{12}) + a_{6i}(z_4 - 5z_9) = 0$$

for $i = 1, ..., 5$, then $M \in \mathcal{L}_x$ if and only if

$$det(a_{ki})_{k=1,2,3,4,6}^{i=1,2,3,4,5} = 0$$

and $M \in \mathcal{L}_y$ if and only if

$$det(a_{ki})_{k=1,2,3,4,5}^{i=1,2,3,4,5} = 0.$$

These conditions are clearly different.

EXAMPLES OF FANO THREEFOLDS NOT SATISFYING ASSUMPTIONS OF PROPOSITION 2.1:

1) Y is a linear section of $G(1,4)$ in the Plücker embedding: the surface U_Y has degree 10.
2) Y is a Fano variety of index one and genus 12 (V_{22}). The surface of lines belongs to $|-2K_Y|$ for all V_{22}'s but one ([P]), for which the scheme of lines is non-reduced and the surface covered by lines belongs to $|-K_Y|$. This threefold with non-reduced Hilbert scheme of lines (the Mukai-Umemura variety) will be denoted V_{22}^s.

QUESTION: *Are these the only examples?*

REMARK B: Though any V_{22} violates the assumption of the Proposition 2.1, for a V_{22} with the reduced Hilbert scheme of lines (therefore for all V_{22}'s but one) the boundedness of the degree of a map $f : X \to V_{22}$ can be proved. The point is that a general line on such a V_{22} has the normal bundle $\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-1)$, so if U is the universal family of lines on V_{22} and $\pi : U \to V_{22}$ is the natural map, then π is an immersion along a general line. Now if the preimage of a general line l is not contained in the ramification R, one can proceed as before. If it is, then let C be the reduction of an irreducible component of $f^{-1}(l)$, and let k be such that at a general point of the component of R containing C, the ramification index is $k-1$ (i.e. "k points come together".) It turns out that using our observation about π, we can then estimate the arithmetic genus of C (see [A], section 5). Namely, let $f^*H_{V_{22}} = mH_X$ and let $K_X = rH_X$. We get then

$$2p_a(C) - 2 \leq (r - \frac{m}{k})CH_X.$$

Suppose now that $k-1$ is a smallest ramification index for R. Hurwitz' formula implies that if $r < \frac{m}{3}$, then $k = 2$. So if m gets big, $p_a(C)$ becomes negative, and this is impossible.

Concerning the remaining Fano threefolds (in particular, V_{22}^s and $G(1,4) \bigcap \mathbf{P}^6$), we can prove a weaker result (as in Conjecture B):

PROPOSITION 2.2 *Let Y be a Fano threefold with $Pic(Y) = \mathbf{Z}$ and with H_Y very ample, let X be a smooth threefold with $b_2(X) = 1$ and let $f : X \to Y$ be a morphism. If either Y is of index two, or Y is of index one with non-reduced Hilbert scheme of lines, then the degree of f is bounded in terms of the discrete invariants of X.*

Proof: Consider for example the index one case. We have that Y has a one-dimensional family of $(-2,1)$-lines. If we take a smooth hyperplane section H through a line l of this family, the sequence of the normal bundles

$$0 \to N_{l,H} \to N_{l,Y} \to N_{H,Y}|_l \to 0$$

splits.
Therefore, if M is the inverse image of H and C is the inverse image of l (scheme-theoretically), the sequence

$$0 \to N_{C,M} \to N_{C,X} \to N_{M,X}|_C \to 0$$

also splits.
It is not difficult to see that for a general choice of l and H, the surface M has only isolated singularities. As M is a Cartier divisor on a smooth variety X (say $M \in |\mathcal{O}_X(m)|$), M is normal.
Now we are in the situation which is very similar to that of the following

Theorem (R. Braun, [B]): *Let W be a Cartier divisor on a variety V of dimension n, $2 \leq n < N$, in $\mathbf{P}^N$ such that W has an open neighborhood in V which is locally a complete intersection in $\mathbf{P}^N$. If the sequence of the normal bundles*

$$0 \to N_{W,V} \to N_{W,\mathbf{P}^N} \to N_{V,\mathbf{P}^N}|_W \to 0 \ (*)$$

splits, then W is numerically equivalent to a multiple of a hyperplane section of V.

It turns out that if we replace here W, V, $\mathbf{P}^N$ by C, M, X as in our situation, the similar statement is true. The only additional assumption we must make is that M is sufficiently ample, i.e. m is sufficiently big:
Claim: Let X be a smooth projective 3-fold with $b_2(X) = 1$, and let M be a sufficiently ample normal Cartier divisor on X. If C is a Cartier divisor on M and the sequence

$$0 \to N_{C,M} \to N_{C,X} \to N_{M,X}|_C \to 0$$

splits, then C is numerically equivalent to a multiple of $H_X|_M$.
The proof of this claim is almost identical to that of Braun's theorem (which is itself a refinement of the argument of [EGPS] where the theorem is proved for V a smooth surface). Recall that the main steps of this proof are:

1) The sequence $(*)$ splits iff W is a restriction of a Cartier divisor from the second infinitesimal neighborhood V_2 of V in $\mathbf{P}^N$;
2)The image of the natural map $Pic(V_2) \to Num(V)$ is one-dimensional.

In the situation of the lemma, 1) goes through without changes with W, V, $\mathbf{P}^N$ replaced by C, M, X (M_2 will of course denote the second infinitesimal neighborhood of M in X). The second step is an obvious modification of that in [B], [EGPS]: as in these works, it is enough to prove that the image of the natural map

$$Pic(M_2) \to H^1(M, \Omega^1_M)$$

is contained in a one-dimensional complex subspace, and this follows from the commutative diagram

$$\begin{array}{ccccccc}
Pic(M_2) & \xrightarrow{restr.} & Pic(M) & \longrightarrow & Num(M) & \longrightarrow & H^1(M,\Omega^1_M) \\
\downarrow{\scriptstyle dlog} & & & & & \nearrow & \\
H^1(M_2,\Omega^1_{M_2}) & \longrightarrow & H^1(M,\Omega^1_{M_2}|_M) & \xrightarrow{\alpha} & H^1(M,\Omega^1_X|_M) & &
\end{array}$$

(where α exists because the sheaves $\Omega^1_{M_2}|_M$ and $\Omega^1_X|_M$ are isomorphic) and the fact that for sufficiently ample M,

$$H^1(M, \Omega^1_X|_M) \cong H^1(X, \Omega^1_X) \cong \mathbf{C}$$

as follows from the restriction exact sequence

$$0 \to \Omega^1_X(-M) \to \Omega^1_X \to \Omega^1_X|_M \to 0.$$

Note that we can give an effective estimate for "sufficient ampleness" of M in terms of numerical invariants of X using the Griffiths vanishing theorem ([G]).
Applying this to our situation of a map onto a Fano threefold Y of index one with non-reduced Hilbert scheme of lines, we get that $C = f^{-1}(l)$ must be numerically equivalent to a multiple of the hyperplane section divisor on $M = f^{-1}(H)$ if the number m (defined by $f^*H_Y = mH_X$) is large enough. As it is easy to show that C and the hyperplane section of M are independent in $Num(M)$, it follows that m and therefore $deg(f)$ must be bounded. The case of index two is exactly the same (use the existence of a divisor covered by (-1,1)-lines). So the Proposition is proved.

We summarize our results in the following

Theorem 2.3 *Let X be a smooth projective threefold with $b_2(X) = 1$, let Y be a Fano threefold with $b_2(Y) = 1$ and very ample H_Y and let $f : X \to Y$ be a morphism. If $Y \not\cong \mathbf{P}^3$, then the degree of f is bounded in terms of the discrete invariants of X, Y.*

Proof: Indeed, there are only four possibilities:
a) Y is a quadric: this is proved in [S], [A].
b) Proposition 2.1 applies;
c) Y is V_{22} with reduced scheme of lines: the boundedness for $deg(f)$ is obtained in Remark B;
d) Y is either $G(1,4) \cap \mathbf{P}^6$, or a Fano threefold with non-reduced Hilbert scheme of lines: then Proposition 2.2 applies.
Notice that in the first three cases it is sufficient that $Pic(X) \cong \mathbf{Z}$.

3. Maps between Fano threefolds

It turns out that we obtain especially strong bound if X is also a Fano variety. In many cases,this even implies non-existence of maps:

Theorem 3.1 *Let X, Y be Fano threefolds, $Pic(X) \cong Pic(Y) \cong \mathbf{Z}$. Suppose that H_X, H_Y are very ample. If either*
i) Y is of index one and S_Y is at least $2H_Y$,
or
ii) Y is of index two and U_Y is at least $4H_Y$(where S_Y, U_Y are as in Proposition 2.1), then for a non-constant map $f : X \to Y$ we must have

$$f^*(H_Y) = H_X,$$

i.e.

$$deg(f) = \frac{H_X^3}{H_Y^3}.$$

Before starting the proof, we formulate the following result from [S]:
Let $f : X \to Y$ be a non-trivial map between Fano threefolds with Picard group $\mathbf{Z}$. *Then:*
A) If X, Y are of index two, then the inverse image of any line is a union of lines;
B) If X, Y are of index one, then the inverse image of any conic is a union of conics;
C) If X is of index one and Y is of index two, then the inverse image of any line is a union of conics;
D) If X is of index two and Y is of index one, then the inverse image of any conic is a union of lines.
(here a conic is allowed to be reducible or non-reduced. Unions of lines and conics are understood in set-theoretical sense, i.e. a line or a conic from this union can, of course, have a multiple structure.)

We will also need some facts on conics on a Fano threefold V of index one, with very ample $-K_V$ and cyclic Picard group. Iskovskih proves ([I],II, Lemma 4.2) that if C is a smooth conic on such a threefold, then $N_{C,V} = \mathcal{O}_{\mathbf{P}^1}(-a) \oplus \mathcal{O}_{\mathbf{P}^1}(a)$ with a equal to 0,1,2 or 4. The following lemma is an almost obvious refinement of this:

Lemma 3.2 *a) Let $C \subset V$ be a smooth conic. Then $N_{C,V} = \mathcal{O}_{\mathbf{P}^1}(-4) \oplus \mathcal{O}_{\mathbf{P}^1}(4)$ if and only if there is a plane tangent to V along C. In particular, such conics exist only if V is a quartic.*
b) Let $C \subset V$ be a reducible conic: $C = l_1 \bigcup l_2$, $l_1 \neq l_2$. Let N be the (locally free with trivial determinant) normal sheaf of C in V. Then $N|_{l_i} = \mathcal{O}_{\mathbf{P}^1}(-a_i) \oplus \mathcal{O}_{\mathbf{P}^1}(a_i)$ with $0 \leq a_i \leq 2$, and if $a_i = 2$ for both i, then there is a plane tangent to V along C (and V is a quartic).

Proof: a) This is a simple consequence of the fact that for $C \subset V \subset \mathbf{P}^n$, $N_{C,V} \subset N_{C,\mathbf{P}^n}$, and the only subbundle of degree 4 in $N_{C,\mathbf{P}^n}$ is $N_{C,P}$ with P the plane containing C. One concludes that V is a quartic as all the other Fano threefolds V considered here are intersections of quadrics and cubics which contain this V ([I], II, sections 1,2) and therefore must contain this P, which is impossible.
b) We have embeddings

$$0 \to N_{l_i,V} \to N|_{l_i},$$

this implies the first statement: $0 \leq a_i \leq 2$. If $a_i = 2$, then l_i should be a (-2,1)-line; therefore there are planes P_i tangent to V along l_i, giving the degree 1 subbundle of $N_{l_i,V}$ and the exceptional section in $\mathbf{P}(N_{l_i,V}) \cong F_3$. In fact $P_1 = P_2$. This is easy to see using so-called " elementary modifications" of Maruyama (of which I learned from [AW] ,p.11): if we blow $\mathbf{P}(N_{l_1,V})$ up in the point p corresponding to the direction of l_2 and then contract the proper preimage of the fiber, we will get $\mathbf{P}(N|_{l_1})$. Under our circumstances, p must lie on the exceptional section of $\mathbf{P}(N_{l_1,V})$, so $l_2 \subset P_1$. In the same way, $l_1 \subset P_2$, q.e.d..

Proof of the Theorem:
Let $f : X \to Y$ be a finite map between Fano threefolds as above.
Again, the condition on S_Y, T_Y means that not the whole inverse image of S_Y, T_Y can be contained in the ramification. If Y is of index one resp. index two, we will denote by C be a reduced irreducible component of the inverse image of a general line

resp. (-1,1)-line l on Y (so C is not contained in the ramification), and by D the full scheme-theoretic inverse image of such a line.
Let $f^*\mathcal{O}_Y(1) = \mathcal{O}_X(m)$. If X is of index two, then $T_X(1)$ is globally generated. As in the Proposition 2.1, we conclude that $m = 1$.
If X is of index one and Y is of index two, then, by the result quoted in the beginning of this section, C is a line or a conic.
If C is a smooth conic, we look at the generic isomorphism

$$\phi : (\mathcal{I}_C/\mathcal{I}_C^2)^* \to (\mathcal{I}_D/\mathcal{I}_D^2)^*|_C = \mathcal{O}_C(m) \oplus \mathcal{O}_C(-m).$$

Immediately we get that m is equal to one or two. Suppose $m = 2$. Then, by the Lemma, X is a quartic and there is a plane P tangent to X along C. Choose the coordinates so that P is given by $x_3 = x_4 = 0$. Then the equation of X can be written as

$$(q(x_0, x_1, x_2))^2 + x_3F + x_4G = 0,$$

where q defines C and F, G are cubic polynomials. Denote as A and B the curves cut out on P by these cubics. The necessary condition for smoothness of X is

$$A \cap B \cap X = \emptyset.$$

Now recall that C resp. P varies in a one-dimensional (complete) family C_t resp. P_t. A and B also vary, and for every t we must have

$$A_t \cap B_t \cap X = \emptyset.$$

This means that all the planes P_t pass through the same point, not lying on X. Projecting from this point, we see that the surface W formed by our conics C_t is in the ramification locus of this projection. The Hurwitz formula then gives $W \in |\mathcal{O}_X(i)|$ with $i \leq 3$. Now Y is, by assumption, a cubic or an intersection of two quadrics. But then, as we saw, the surface U_Y of (-1,1)-lines is at least $5H_Y$, and an elementary calculation shows that it is impossible that the inverse image of the surface of (-1,1)-lines U_Y consists only from W and the ramification.
If C is a line, then the argument is similar. One only needs to prove the following
Claim:In this situation, if $m = 2$, the scheme D has another reduced irreducible component C_1, which intersects C.
Then of course either C_1, or $C \bigcup C_1$ is a conic, and one can proceed as above. The proof of this claim is elementary algebra. We will sketch it after finishing the following last step of the Theorem:
If X and Y are both of index one, we have that the inverse image of a line l on Y should consist of lines and conics; for C as above, we have a map

$$\phi : (\mathcal{I}_C/\mathcal{I}_C^2)^* \to \mathcal{O}_C \oplus \mathcal{O}_C(-m),$$

if l is (0,-1), or

$$\phi' : (\mathcal{I}_C/\mathcal{I}_C^2)^* \to \mathcal{O}_C(m) \oplus \mathcal{O}_C(-2m),$$

if l is (1,-2). As these maps must be generic isomorphisms, we get that in both cases $m = 1$, whether C is a conic or a line.

Proof of the claim: Notice that C must be (1,-2)-line. The cokernel of the natural map

$$\beta : \mathcal{I}_D/\mathcal{I}_D^2|_C \to \mathcal{I}_C/\mathcal{I}_C^2$$

is the sheaf $\mathcal{I}_{C,D}/\mathcal{I}_{C,D}^2$, supported on intersection points of C and other components of D. We see from our assumptions that it must have length one (so be supported at one point x). Suppose that C intersects non-reduced components of D at x. Let A be a local ring of D at x and $\mathfrak{p} \subset A$ a fiber of $\mathcal{I}_{C,D}$. Of course $\mathfrak{p}/\mathfrak{p}^2 \neq 0$ by Nakayama. To see that $dim\mathfrak{p}/\mathfrak{p}^2 \geq 2$, we find an ideal $\mathfrak{a} \subset \mathfrak{p}$, not contained in $\mathfrak{p}^2$. For example, we can take an ideal defining the union of C and the reduction of an irreducible but non-reduced component of D intersecting C. We have a surjection

$$\mathfrak{p}/\mathfrak{p}^2 \to (\mathfrak{p}/\mathfrak{a})/(\mathfrak{p}^2/(\mathfrak{p}^2 \cap \mathfrak{a})) \to 0,$$

which has non-trivial (again by Nakayama) image and non-trivial kernel, q. e. d..

Corollary 3.3 *Let X, Y be Fano threefolds of index one as in Theorem 3.1 i). Then any map between X and Y is an isomorphism.*

Proof: Iskovskih computed the third Betti numbers of all Fano threefolds (see also [M]), and in fact as soon as $deg(X) > deg(Y)$, then $b_3(X) < b_3(Y)$, so a morphism $f : X \to Y$ cannot exist.

Remark C: Some part of the argument of Theorem 3.1 goes through without assumptions on the very ampleness of the generator H of the Picard group. For example, when X is a quartic double solid, which is a Fano threefold of index two, all the "lines" C on X except possibly a finite number, have either trivial normal bundle, or the normal bundle $\mathcal{O}_C(H) \oplus \mathcal{O}_C(-H)$ (in other words, the surface which parametrizes lines on X, has only isolated singularities). One can then replace the words "$T_X(H)$ is globally generated", which are not true in general, by some "normal bundle arguments" as in the above proof. The same should hold for the Veronese double cone. See [W], [T] for details. As for maps *to* the quartic double solid, the argument goes through without changes.

Examples: Any cubic in $\mathbf{P}^4$ satisfies the assumption we made on Y. By our Theorem 3.1 , we get that if a Fano threefold X of index one with cyclic Picard group is mapped onto a cubic, then the degree of this map can only be $\frac{degX}{3}$. So if X admits such a map, then $deg(X)$ is divisible by 3. Of course there are Fano threefolds of index one which admit a map onto a cubic: we can take an intersection of a cubic cone and a quadric in $\mathbf{P}^5$. Theorem 3.1 shows that if a smooth complete intersection of type (2,3) in $\mathbf{P}^5$ maps to a cubic, then it is contained in a cubic cone and the map is the projection from the vertex of this cone.
The same applies of course to maps from a complete intersection of three quadrics in $\mathbf{P}^6$ to a complete intersection of two quadrics in $\mathbf{P}^5$. Notice that any smooth complete intersection of two quadrics in $\mathbf{P}^5$ admits a map g onto a quadric in $\mathbf{P}^4$ such that the inverse image of the hyperplane section is the hyperplane section (any pencil of quadrics with non-singular base locus contains a quadratic cone). Therefore if a smooth intersection of three quadrics in $\mathbf{P}^6$ can be mapped onto a smooth complete

intersection of two quadrics in $\mathbf{P}^5$, it must lie in a quadric of corank 2 in $\mathbf{P}^6$. Of course a general intersection of three quadrics in $\mathbf{P}^6$ does not have this property, as the space of quadrics of corank 2 is of codimension four in the space of all quadrics.

Additional examples of varieties satisfying the assumption of Theorem 3.1:

1) any complete intersection of a cubic and a quadric in $\mathbf{P}^5$ or
2) any complete intersection of three quadrics in $\mathbf{P}^6$. Indeed, if lines on these varieties cover only a hyperplane section divisor, then the scheme of lines must be non-reduced, i.e. each line must have normal bundle $\mathcal{O}_{\mathbf{P}^1}(-2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)$. So the surface of lines is either a cone or the tangent surface to a curve. But one can check that these varieties do not contain cones; neither do they contain a tangent surface to a curve as a hyperplane section, because by a version of Zak's theorem on tangencies (see for example [FL]), a hyperplane section of a complete intersection has only isolated singularities.
3) Any V_{22} with reduced Hilbert scheme of lines. By ([P]), there is exactly one V_{22} such that its Hilbert scheme of lines is non-reduced.
4) any Fano threefold V_{16} of index one and genus 9. This can be shown by the method of Prokhorov ([P]) :
First, notice that if the lines on V_{16} cover only a hyperplane section, the scheme of lines is non-reduced. So all the lines are tangent to a curve. This is actually a rational normal curve, so the lines never intersect.
For convenience of the reader, we recall from [I2] the notion of double projection from a line and its application to V_{16} :
Let X be a Fano threefold of index one, $g(X) \geq 7$, and let l be a line on X. On $\tilde{X}$, the blow-up of X, we consider the linear system $|\sigma^* H - 2E|$, where σ is the blow-up, $H = K_Y$ and E is the exceptional divisor. This is not base-point-free, namely, its base locus consists of proper preimages of lines intersecting l, and, if l is (-2,1), from the exceptional section of the ruled surface $E \cong F_3$. However, after a flop (i.e. a birational transformation which is an isomorphism outside this locus) we can make it into a base-point-free system $|(\sigma^* H)^+ - 2E^+|$ on the variety $\tilde{X}^+$.
If $g(X) = 9$, i.e.X is a V_{16}, the variety $\tilde{X}^+$ is birationally mapped by this linear system onto $\mathbf{P}^3$. This map, say g, is a blow-down of the surface of conics intersecting l to a curve $Y \subset \mathbf{P}^3$, which is smooth of degree 7 and genus three (smoothness of Y is obtained from Mori's extremal contraction theory). Y lies on a cubic surface which is the image of E^+. Moreover, the inverse rational map from $\mathbf{P}^3$ to X is given by the linear system $|7H - 2Y|$.
One has therefore that the lines from X, different from l, must be mapped by g to trisecants of Y. Note that if lines on X form only a hyperplane section, the desingularization of the surface of lines on X is rational ruled, and it remains so after the blow-up and the flop. So, as in [P], we must have a morphism $F_e \to \mathbf{P}^3$, which is given by some linear system $|C + kF|$ with C the canonical section and f a fiber, such that the inverse image of Y belongs to the system $|3C + lF|$. $deg(Y) = 7$ implies

$$(3C + lF)(C + kF) = -3e + 3k + l = 7,$$

and as $degK_Y = 4$,

$$(C + (l - 2 - e)F)(3C + lF) = -6e + 4l - 6 = 4,$$

Combining these two equations, we get

$$2k - e = 3,$$

However, we must have $e \geq 0$ and $k \geq e$, as otherwise the linear system $|C + kF|$ does not define a morphism. This leaves only two possibilities for k and e: either $e = k = 3$, or $e = 1, k = 2$. The first case actually cannot occur: this would imply that Y is singular. So the image of $F_e = F_1$ in $\mathbf{P}^3$ is a cubic which is a projection of F_1 from $\mathbf{P}^4$. By assumption, Y is also contained in another irreducible cubic (the image of E^+). But one check that this cannot happen, using e.g. a theorem by d'Almeida ([Al]), which asserts that if a smooth non-degenerate curve Y of degree $d \geq 6$ and genus g in $\mathbf{P}^3$ satisfies $H^1(\mathcal{I}_Y(d-4)) \neq 0$, then Y has a (d-2)-secant provided that $(d, g) \neq (7, 0), (7, 1), (8, 0)$.

4. V_{22}

Let us now take $Y = V_{22}^s$, i.e. the only variety of type V_{22} which has non-reduced Hilbert scheme of lines. This V_{22} violates the assumptions of Theorem 3.1. However, using Mukai's and Schreyer's descriptions of conics on varieties of type V_{22}, it is still possible to say something on maps from Fano threefolds onto Y. We will show the following:

Proposition 4.1 *A Fano threefold X of index two with cyclic Picard group and irreducible Hilbert scheme of lines does not admit a map onto V_{22}^s.*

As for the last assumption on X, one believes that this is always satisfied. In fact this is easy to check (and well-known) for a cubic or a complete intersection of two quadrics (the Hilbert scheme is smooth in this case, so it is enough to show that it is connected). The irreducibility is also known for V_5, in fact, the Hilbert scheme is isomorphic to $\mathbf{P}^2$ ([I], I, Corollary 6.6). For a quartic double solid, see [W]. As for a double Veronese cone, in [T] it is proven that a general double Veronese cone has irreducible Hilbert scheme of lines. So the only possible exception could be a special double Veronese cone.

In fact our argument will work for a sufficiently general V_{22}, but for all of them except V_{22}^s this assertion is already proved in the last paragraph.

Proof: Let S be the Fano surface (= reduced Hilbert scheme) of lines on X and T the Fano surface of conics on the V_{22}. If $f : X \to V_{22}$ is a finite map, then, as Schuhmann proves in [S], the inverse image of any conic is a union of lines, and, moreover, in this way f induces a finite surjective morphism $g : S \to T$ (thanks to irreducibility of S, any line on X is in the inverse image of a conic on V_{22}).
F.-O. Schreyer ([Sch]) gives the following description of a general conic on V_{22}: Consider V_{22} as the variety of polar hexagons of a plane quartic curve $C \subset \mathbf{P}^2$ (a polar hexagon of C is the union of six lines $l_1, ...l_6$ given by equations $L_1 = 0, ..., L_6 = 0$,

such that $L_1^4 + ... + L_6^4 = F$ where $F = 0$ defines C; "the variety of polar hexagons" means here the closure of the set of 6-tuples $l_1, ...l_6$ with $L_1^4 + ... + L_6^4 = F$ in the Hilbert scheme $Hilb_6(\mathbf{P}^{2^*})$; a general V_{22} is isomorphic to such a variety for a certain curve C; V_{22}^s is the variety of polar hexagons of a double conic). Then there is a birational isomorphism between $(\mathbf{P}^2)^*$ and T given as follows:
for a general $l \subset \mathbf{P}^2$ the curve of polar hexagons to C containing l is a conic on V_{22}.
This description and the fact that through any point on a V_{22} there is only a finite number of conics ([I], II, Theorem 4.4) gives that
there are six conics through a general point of V_{22}.
In [M], Mukai claims that the Fano surface of conics on a V_{22} is even isomorphic to $\mathbf{P}^2$. Unfortunately, this paper does not contain a proof of this fact. The proof appears in the paper of A. Kuznetsov ([K]): he uses another description of a general V_{22} as a subvariety of $G(2,6)$. Namely, if V and N are 7- and 3-dimensional vector spaces respectfully and $f : N \to \Lambda^2 V^*$ is a general net of skew-symmetric forms on V, then a general V_{22} (including V_{22}^s, [Sch]) appears as a set of all 3-subspaces of V which are isotropic with respect to this net (i.e. to all forms of the net simultaneously). Let U (resp. Q) denote restriction on a V_{22} of the universal (resp. universal quotient) bundle on $G(2,6)$. Kuznetsov proves that every (possibly singular) conic on a V_{22} is a degeneracy locus of a homomorphism $U \to Q^*$; the Fano surface of conics is thus $\mathbf{P}(Hom(U, Q^*)) = \mathbf{P}^2$.
Now if there is a finite map $f : X \to V_{22}$ as above, then X must be a cubic: indeed, a Fano threefold with cyclic Picard group and with 6 lines through a general point is a cubic. Let $f^* H_{V_{22}} = m H_X$, then one easily computes that the inverse image of a general conic consists of $deg(g) = s = \frac{3}{11} m^2$ lines.
For simplicity, we will use the same notation for points of T (resp. S) and corresponding conics on V_{22} (resp. lines on X). We have $T \cong \mathbf{P}^2$. Let a be such that conics on V_{22} intersecting a given (general) conic A, form a divisor D_A from $|\mathcal{O}_{\mathbf{P}^2}(a)|$
On S, denote as E_L the divisor of lines intersecting a given line L. It is well-known and easy to compute that $E_L \cdot E_M = 5$ for any L, M.
If $g^{-1}(A) = \{L_1, ..., L_s\}$, then

$$g^*(\mathcal{O}_{\mathbf{P}^2}(a)) = \mathcal{O}_S(E_{L_1} + ... + E_{L_s}).$$

We therefore have another formula for $deg(g)$:

$$deg(g) = \frac{5s^2}{a^2}.$$

From the equality $s = \frac{5s^2}{a^2}$ we get that $(\frac{m}{a})^2 = \frac{11}{15}$, however, this is impossible as $\frac{11}{15}$ is not a square of a rational number.

References

[Al] J. d'Almeida: Courbes de l'espace projectif: Series lineaires incompletes et multisecantes, J. Reine Angew. Math., 370 (1986), 30-51.

[A] E. Amerik: On a problem of Noether-Lefschetz type, to appear in Compositio Mathematica.

[AW] M. Andreatta, J. Wisniewski: On contractions of smooth varieties, MSRI e-print archive, preprint alg-geom/9605013

[B] R. Braun: On the normal bundles of Cartier divisors on projective varieties, Arch. Math. 59 (1992), 403-411.

[CG] H. Clemens, Ph. Griffiths: The intermediate Jacobian of the cubic threefold, Ann. Math. 95 (1972), 281-356.

[D] J.-P. Demailly: L^2-vanishing theorems for positive line bundles and adjunction theory, Cetraro Lectures, 1994, MSRI e-prints archive, preprint alg-geom/9410022.

[EGPS] G. Ellingsrud, L. Gruson, C. Peskine and S. Stromme: On the normal bundle of curves on smooth projective surfaces, Invent. Math. 80 (1985), 181-184.

[FL] W.Fulton, R. Lazarsfeld: Connectivity and its applications in algebraic geometry, in *Algebraic geometry, Proceedings, University of Illinois at Chicago Circle, 1980*, Lecture Notes in Math. 862.

[G] Ph. Griffiths: Hermitian differential geometry, Chern classes and vector bundles, in Global Analysis, papers in honor of K. Kodaira, Princeton University press, 1969, 185-251.

[GH] Ph. Griffiths, J. Harris: *Principles of algebraic geometry*, Wiley, 1978.

[I] V.A. Iskovskih: Fano threefolds I, II, Math. USSR Izv. 11 (1977), 485-527, and 12 (1978), 469-506.

[I1] V. A. Iskovskih: Anticanonical models of 3-dimensional algebraic varieties, in: R.V. Gamkrelidze (ed.), Itogi nauki i techniki, Sovremennie problemi matematiki, t. 12 (in Russian).

[I2] V.A. Iskovskih: Double projection from a line on Fano threefolds of the first kind, Math. USSR Sbornik, 66 (1990), no. 1, 265-284.

[K] A. Kuznetsov: "The derived category of coherent sheaves on V_{22}, master's thesis, Moscow, 1995.

[M] S. Mukai: Fano 3-folds, in Complex Projective Geometry, ed. by G. Ellingsrud et al., London Math. Soc. Lecture Notes 179, 255-263.

[P] Yu. Prokhorov: On exotic Fano varieties, Moscow Univ. Math. Bull. 45 (1990), no.3, 36-38.

[Sch] F.-O. Schreyer: Geometry and algebra of prime Fano threefolds of genus 12, preprint (preliminary partial version).

[S] C. Schuhmann: Morphisms between Fano threefolds, thesis, Leiden, 1997.

[T] A. Tikhomirov: The Fano surface of lines on the Veronese double cone, Math. USSR Izv. 19 (1982), 377-443.

[W] G. Welters: Abel-Jacobi isogenies for certain types of Fano threefolds, thesis, Utrecht, 1981.

Ekaterina Amerik
Institut Fourier
100 Rue des Maths
BP 74 38402 St.-Martin d'Heres
France
amerik@main.mccme.rssi.ru