Effective Freeness of Adjoint Line Bundles

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Abstract. This note shows how two existing approaches to providing effective (quadratic) bounds for the freeness of adjoint line bundles can be linked to establish a new effective bound which approximately differs from the linear bound conjectured by Fujita only by a factor of the cube root of the dimension of the underlying manifold. As an application, a new effective statement for pluricanonical embeddings is derived.

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1 Introduction and statement of the Main Theorem

Let L be an ample line bundle over a compact complex projective manifold X of complex dimension n. Let K_X be the canonical line bundle of X. The following conjecture is due to Fujita [Fuj87].

Conjecture 1.1 (Fujita). The adjoint line bundle $K_X + mL$ is base point free (i.e. spanned by global holomorphic sections) for $m \geq n+1$. It is very ample for $m \geq n+2$.

The standard example of the hyperplane line bundle on $X = \mathbb{P}^n$ shows that the conjectured numerical bounds are optimal. In the case of X being a compact Riemann surface, the conjecture is easily verified by means of the Riemann-Roch theorem. Moreover, Reider [Rei88] was able to validate the conjecture also in the case $n = 2$. In higher dimensions, the very ampleness part of the conjecture has proved to be quite intractable so far. In fact, no further results seem to be known here. On the other hand, several further results have been established towards the freeness conjecture. The case $n = 3$ was solved by

Ein and Lazarsfeld [EL93] (see also [Fuj93]), and $n = 4$ is due to Kawamata [Kaw97]. In arbitrary dimension n, the state-of-the-art is that $K_X + mL$ is base point free for any integer m that is no less than a number roughly of order n^2 (see below for exact statements).

To the author's knowledge, [Dem00] constitutes the most recent survey on the subject under discussion. It contains an extensive list of references (see also the references at the end of this article) and, furthermore, introduces the reader to various other effective results in algebraic geometry.

The above-mentioned bound in the case of arbitrary dimension n can be derived from each of the following two theorems, due to Angehrn and Siu [AS95] (see also [Siu96]) and Helmke [Hel97], [Hel99], respectively. Although the proofs of these two theorems adhere to the same inductive approach, the key ideas at their cores are of a different nature. In Proposition 3.5, we will show how to use the techniques in question seamlessly in a back-to-back manner. This insight, together with the numerical considerations in Section 2, will lead to the improved bound asserted in the Main Theorem and proved in Section 3.

First, let us state the bound given by [AS95].

Theorem 1.2 ([AS95]). *The line bundle $K_X + mL$ is base point free for $m \geq \frac{1}{2}n(n+1)+1$.*

Secondly, we state Helmke's result. Due to the nature of his technique, the assumptions of his theorem are formulated in a slightly different way. We quote the result in the way it is presented in [Hel97], because the slight improvement achieved in [Hel99] is not relevant for our purposes.

Theorem 1.3 ([Hel97]). *Assume that L has the additional properties that*

$$L^n > n^n$$

and for all $x \in X$:

$$L^d.Z \geq m_x(Z) \cdot n^d$$

for all subvarieties $Z \subset X$ with $x \in Z$, $d = \dim Z \leq n-1$ and multiplicity $m_x(Z) \leq \binom{n-1}{d-1}$ at x. Then $K_X + L$ is base point free.

If $n \geq 3$, it is clear that we need to set

$$m_0 := \max\{n \cdot \sqrt[d]{\binom{n-1}{d-1}} : d \in \mathbb{N} \text{ and } 1 \leq d \leq n\}$$

to determine the minimal bound m_0 deducible from Theorem 1.3 such that $K_X + mL$ is base point free for any integer $m \geq m_0$. Since

$$\sqrt[d]{\binom{n-1}{d-1}} \geq \frac{1}{\sqrt[d]{3}} \frac{n^{1-\frac{1}{d}}}{d}$$

according to our Lemma 2.3, we find that the m_0 which one can derive from Theorem 1.3 is essentially also of the order n^2.
We conclude this section with the statement of our Main Theorem, which asserts that the bound m_0 can be chosen to be a number of the order $n^{\frac{4}{3}}$.

Theorem 1.4 (Main Theorem). *The line bundle $K_X + mL$ is base point free for any integer m with*

$$m \geq (e + \frac{1}{2})n^{\frac{4}{3}} + \frac{1}{2}n^{\frac{2}{3}} + 1,$$

where $e \approx 2.718$ is Euler's number.

2 Estimates for binomial coefficients

In order to understand precisely the nature of the numerical conditions in the assumptions of Theorem 1.3, we prove some auxiliary estimates in this section. We begin with the following lemma.

Lemma 2.1. *For all $x \in]0,1[$: $1 < \left(\frac{1}{1-x}\right)^{\frac{1-x}{x}} < e$.*

Proof. It is obvious that 1 is a strict lower bound of the given expression, so it remains to show that

$$\left(\frac{1}{1-x}\right)^{\frac{1-x}{x}} < e.$$

Taking log on both sides of the inequality, we see that we are done if we can show that

$$g(x) := \frac{x-1}{x}\log(1-x) < 1$$

on the open unit interval. However, for this it suffices to prove that $\lim_{x\to 0^+} g(x) = 1$ and $g'(x) < 0$. The former is easily verified using L'Hôpital's rule, while the latter follows readily from a simple computation. □

In the proof of the subsequent Lemma 2.3, we will employ Lemma 2.1 in the form of the following corollary.

Corollary 2.2. *Let n be an integer ≥ 2. Let d be an integer with $1 \leq d \leq n-1$. Then*

$$1 < \left(\frac{n}{n-d}\right)^{\frac{n-d}{d}} < e.$$

Proof. We have

$$\left(\frac{n}{n-d}\right)^{\frac{n-d}{d}} = \left(\frac{1}{1-\frac{d}{n}}\right)^{\frac{1-\frac{d}{n}}{\frac{d}{n}}}.$$

Thus the corollary follows immediately from Lemma 2.1. □

The preceding considerations allow us to estimate the binomial coefficients from Theorem 1.3 in the form of the following lemma.

LEMMA 2.3. *Let $1 \leq d \leq n-1$. Then*

$$\frac{1}{\sqrt[d]{3}} \frac{n^{1-\frac{1}{d}}}{d} \leq \sqrt[d]{\binom{n-1}{d-1}} \leq e\frac{n}{d}.$$

Proof. In [Ahl78], page 206, Stirling's formula is stated as

$$\Gamma(x) = \sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x} e^{\frac{\theta(x)}{12x}}$$

for $x > 0$ with $0 < \theta(x) < 1$. In particular,

$$\sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x} \leq \Gamma(x) \leq \sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x} e^{\frac{1}{12}}$$

for any $x \geq 1$. Thus, for the proof of the desired estimate from above, Stirling's formula enables us to proceed as follows.

$$\begin{aligned}
\binom{n-1}{d-1} &= \frac{(n-1)!}{(d-1)!(n-d)!} = \frac{1}{n-d} \frac{\Gamma(n)}{\Gamma(d)\Gamma(n-d)} \\
&\leq \frac{1}{n-d} \frac{\sqrt{2\pi}\, n^{n-\frac{1}{2}} e^{-n} e^{\frac{1}{12}}}{\sqrt{2\pi}\, d^{d-\frac{1}{2}} e^{-d} \sqrt{2\pi}\,(n-d)^{n-d-\frac{1}{2}} e^{-(n-d)}} \\
&= \frac{e^{\frac{1}{12}}}{\sqrt{2\pi}} \sqrt{\frac{d}{(n-d)n}} \left(\frac{n}{d}\right)^d \left(\frac{n}{n-d}\right)^{n-d} \\
&\leq \left(\frac{n}{d}\right)^d \left(\frac{n}{n-d}\right)^{n-d}.
\end{aligned}$$

Resorting to Corollary 2.2, we eventually conclude:

$$\binom{n-1}{d-1}^{\frac{1}{d}} \leq \frac{n}{d} \left(\frac{n}{n-d}\right)^{\frac{n-d}{d}} \leq e\frac{n}{d}.$$

The desired estimate from below is proved analogously:

$$\begin{aligned}
\binom{n-1}{d-1} &= \frac{(n-1)!}{(d-1)!(n-d)!} = \frac{1}{n-d} \frac{\Gamma(n)}{\Gamma(d)\Gamma(n-d)} \\
&\geq \frac{1}{n-d} \frac{\sqrt{2\pi}\, n^{n-\frac{1}{2}} e^{-n}}{\sqrt{2\pi}\, d^{d-\frac{1}{2}} e^{-d} e^{\frac{1}{12}} \sqrt{2\pi}\,(n-d)^{n-d-\frac{1}{2}} e^{-(n-d)} e^{\frac{1}{12}}} \\
&= \frac{1}{e^{\frac{1}{6}} \sqrt{2\pi}} \sqrt{\frac{d}{(n-d)n}} \left(\frac{n}{d}\right)^d \left(\frac{n}{n-d}\right)^{n-d} \\
&\geq \frac{1}{3} \frac{1}{n} \left(\frac{n}{d}\right)^d \left(\frac{n}{n-d}\right)^{n-d}.
\end{aligned}$$

Using Corollary 2.2 again, we obtain:

$$\binom{n-1}{d-1}^{\frac{1}{d}} \geq \sqrt[d]{\frac{1}{3n}}\frac{n}{d} = \frac{1}{\sqrt[d]{3}}\frac{n^{1-\frac{1}{d}}}{d}.$$

□

3 Proof of the Main Theorem

The following theorem states the improved effective freeness bound which we shall prove at the end of this section.

Theorem 3.1 (Main Theorem). *The line bundle K_X+mL is base point free for any integer m with*

$$m \geq (e+\frac{1}{2})n^{\frac{4}{3}} + \frac{1}{2}n^{\frac{2}{3}} + 1,$$

where $e \approx 2.718$ is Euler's number.

First of all, let us recall how a result of this type can be proved by means of multiplier ideal sheaves.
Let $x \in X$ be an arbitrary but fixed point. The key idea of both [AS95] and [Hel97] is to find an integer m_0 (as small as possible) and a singular metric h of the line bundle m_0L with the following two properties:

1. Let h be given locally by $e^{-\varphi}$. Then the curvature current $i\partial\bar{\partial}\varphi$ dominates a positive definite smooth $(1,1)$-form on X in the sense of currents.
2. Let the *multiplier ideal sheaf* of h be defined stalk-wise by ($\chi \in X$):

   $$(\mathcal{I}_h)_\chi := \{f \in \mathcal{O}_{X,\chi} : |f|^2 e^{-\varphi} \text{ is locally integrable at } \chi\}.$$

   Then, in a neighborhood of x, the zero set of $\mathcal{I}_h$, which we denote by $V(\mathcal{I}_h)$, is just the point x. (This is the key property we are looking for. Note that the support of $V(\mathcal{I}_h)$ is just the set of points where h is not locally integrable.)

The first property implies that

$$H^q(X, \mathcal{I}_h(K_X+m_0L)) = (0) \quad (q \geq 1),$$

due to the vanishing theorem of Nadel [Nad89], [Nad90]. (In the special case when the singular metric is algebraic geometrically defined, Nadel's vanishing theorem is the same as the theorem of Kawamata and Viehweg [Kaw82], [Vie82].) With this information and the second property, it is easy to obtain an element of $\Gamma(X, K_X+m_0L)$ which does not vanish at x. Namely, consider the short exact sequence

$$0 \to \mathcal{I}_h(K_X+m_0L) \to K_X+m_0L \to (\mathcal{O}_X/\mathcal{I}_h)(K_X+m_0L) \to 0.$$

The relevant part of the pertaining long exact sequence reads:

$$\Gamma(X, K_X + m_0L) \to \Gamma(V(\mathcal{I}_h), (\mathcal{O}_X/\mathcal{I}_h)(K_X + m_0L)) \to 0,$$

which implies by virtue of the second property that

$$\Gamma(X, K_X + m_0L) \xrightarrow{\text{restr.}} \Gamma(\{x\}, \mathcal{O}_{\{x\}}(K_X + m_0L)) \to 0,$$

meaning that the restriction map to $\{x\}$ is surjective, which is what we intended to prove. Note that, since L is ample, there trivially exists a metric with the two aforementioned properties for every line bundle mL with $m \geq m_0$ (just multiply the metric for m_0L by the $(m - m_0)$-th power of a smooth positive metric of L).

In both [AS95] and [Hel97], the sought-after metric h is produced by an inductive method. First, here is the key statement proved in sections 7–9 of [AS95]. The cornerstone of its proof is a clever application of the theorem of Ohsawa and Takegoshi on the extension of L^2 holomorphic functions [OT87]. Note that, in contrast to [AS95], we are only concerned with freeness, and not point separation, so we can do without the complicated formulations found there.

Proposition 3.2 ([AS95]). *Let d be an integer with $1 \leq d \leq n-1$. Let k_d be a positive rational number, and let h_d be a singular metric of the line bundle k_dL. Assume that $x \in V(\mathcal{I}_{h_d})$ and $x \notin V(\mathcal{I}_{(h_d)^\gamma})$ for $\gamma < 1$. Moreover, assume that the dimensions of those components of $V(\mathcal{I}_{h_d})$ which contain x do not exceed d. Then there exist integers $d', k_{d'}$ with $0 \leq d' < d$ and $k_d < k_{d'} < k_d + d + \varepsilon$ (ε denotes a positive rational number which can be chosen to be arbitrarily small) and a singular metric $h_{d'}$ of $k_{d'}L$ such that $h_{d'}$ possesses the same properties as h_d, but with d and k_d replaced by d' and $k_{d'}$.*

Second, the key statement of [Hel97] is the following proposition. It is stated in such a way that it unites [Hel97], Proposition 3.2 (the inductive statement), and [Hel97], Corollary 4.6 (the multiplicity bound), into one ready-to-use statement. In its proof, the use of the aforementioned L^2 extension theorem is avoided by an explicit bound on the multiplicity of the minimal centers occurring in the inductive procedure.

Proposition 3.3 ([Hel97]). *Let d be an integer with $1 \leq d \leq n - 1$. Let*

$$L^n > n^n$$

and

$$L^{\tilde{d}}.Z \geq m_x(Z) \cdot n^{\tilde{d}}$$

for all subvarieties $Z \subset X$ such that $x \in Z$, $d \leq \tilde{d} = \dim Z \leq n - 1$ and multiplicity $m_x(Z) \leq \binom{n-1}{\tilde{d}-1}$ at x. Then there exists an integer $0 \leq d' < d$, a rational number $0 < c < 1$ and an effective $\mathbb{Q}$-divisor D such that D is $\mathbb{Q}$-linearly equivalent to cL, the pair (X, D) is log canonical at x and the minimal center of (X, D) at x is of dimension d'.

Let us briefly recall the definitions of some of the terms occurring in Proposition 3.3. First of all, for a pair (X, D) of a variety X and a $\mathbb{Q}$-divisor D, an *embedded resolution* is a proper birational morphism $\pi : Y \to X$ from a smooth variety Y such that the union of the support of the strict transform of D and the exceptional divisor of π is a normal crossing divisor. Next, we define the following notions, which are fundamental in the study of the birational geometry of pairs (X, D).

Definition 3.4. Let X be a normal variety and $D = \sum_i d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. If $\pi : Y \to X$ is a birational morphism (in particular, an embedded resolution of the pair (X, D)), we define the *discrepancy divisor* of (X, D) under π to be

$$\sum_j b_j F_j := K_Y - \pi^*(K_X + D).$$

The pair (X, D) is called *log canonical* (resp. *Kawamata log terminal*) at x, if there exists an embedded resolution π such that $b_j \geq -1$ (resp. $b_j > -1$) for all j with $x \in \pi(F_j)$. Moreover, a subvariety Z of X containing x is said to be a *center of a log canonical singularity* at x, if there exists a birational morphism $\pi : Y \to X$ and a component F_j with $\pi(F_j) = Z$ and $b_j \leq -1$.

It follows from Shokurov's connectedness lemma in [Sho86] that the intersection of two centers of a log canonical singularity is again a center of a log canonical singularity (for a proof, see [Kaw97]). Thus there exists a unique minimal center of a log canonical singularity at x with respect to the inclusion of subvarieties on X.

In order to connect Proposition 3.2 and Proposition 3.3 for our purposes, we derive from the conclusion of Proposition 3.3 a statement about the existence of a certain singular metric:

Proposition 3.5. *Let (X, D) be a pair of a smooth projective variety X and an effective $\mathbb{Q}$-divisor D. Let $x \in X$ be an arbitrary but fixed point. Assume that the pair (X, D) is a log canonical at x with its minimal center at x being non-empty. Let $0 < c < 1$ be a rational number such that D is $\mathbb{Q}$-linearly equivalent to cL. Then there exists a singular metric h_D and a rational number c' (which can be chosen to be arbitrarily close to c) such that h_D is a metric of $c'L$, $x \in V(\mathcal{I}_{h_D})$ and $V(\mathcal{I}_{h_D})$ is contained in the minimal center of (X, D) at x in a neighborhood of x. Moreover, $x \notin V(\mathcal{I}_{(h_D)^\gamma})$ for $\gamma < 1$.*

Proof. Let s be a multivalued holomorphic section of cL whose $\mathbb{Q}$-divisor is D. This means that for some positive integer p with cp being an integer, the p-th power of s is the canonical holomorphic section of pcL with divisor pD. Let Z denote the minimal center of (X, D) and $\pi : Y \to X$ a log resolution of (X, D) with discrepancy divisor $\sum_j b_j F_j$. We choose π such that there exists at least one index j_0 with $b_{j_0} = -1$ and $\pi(F_{j_0}) = Z$. Furthermore, we set $\sum_j \delta_j F_j := \pi^*(D)$.

Since L is ample, we can choose a finite number of multivalued holomorphic sections $s_1, \dots, s_q$ of L whose common zero set is exactly Z. Let $\delta_{i,j}$ denote the vanishing order of $\pi^* s_i$ along F_j at a generic point of F_j. If we set $\delta := \min\{\delta_{i,j_0} : i = 1, \dots, q\}$, then $\delta > 0$ holds because all s_i vanish on Z.
For small positive rational numbers $\varepsilon, \varepsilon'$, we define the following singular metric of, say, $\tilde{c}L$:

$$\tilde{h}_D := \frac{1}{|s|^{2(1-\varepsilon)}} \frac{1}{(\sum_{i=1}^q |s_i|^2)^{\varepsilon'}}.$$

Whatever the choice of $\varepsilon, \varepsilon'$ may be, $\tilde{h}_D$ is locally integrable outside of Z in a small neighborhood of x. Here is how to choose $\varepsilon, \varepsilon'$ in order to make $\tilde{h}_D$ not integrable at x. In a small neighborhood U of x, the integrability of $\tilde{h}_D$ is equivalent to the integrability of

$$\pi^* \tilde{h}_D |\mathrm{Jac}(\pi)|^2 = \frac{1}{|s \circ \pi|^{2(1-\varepsilon)}} \frac{1}{(\sum_{i=1}^q |s_i \circ \pi|^2)^{\varepsilon'}} |\mathrm{Jac}(\pi)|^2$$

over every small open subset W of $\pi^{-1}(U)$. Note that $|\mathrm{Jac}(\pi)|^2$ can only be defined locally, and over W we take it to be the quotient

$$\frac{\pi^*(\omega_U \wedge \bar{\omega}_U)}{\omega_W \wedge \bar{\omega}_W},$$

where ω_U, ω_W are arbitrary but fixed nowhere vanishing local holomorphic n-forms on U and W, respectively. As we continue, we observe that there exists a small open subset W of $\pi^{-1}(U)$ such that $W \cap F_{j_0} \neq \emptyset$ and $\pi^* \tilde{h}_D |\mathrm{Jac}(\pi)|^2$ has a pole along $W \cap F_{j_0}$, with its order at a generic point of $W \cap F_{j_0}$ being

$$b_{j_0} + \varepsilon \delta_{j_0} - \varepsilon' \delta.$$

This number equals -1 if we choose ε arbitrarily and set $\varepsilon' := \frac{1}{\delta} \varepsilon \delta_{j_0}$. We conclude that, with these choices for ε and ε', $\tilde{h}_D$ is not integrable at x.
Finally, we set $h_D := (\tilde{h}_D)^r$ with

$$r := \min\{\rho : 0 < \rho \leq 1,\ (\tilde{h}_D)^\rho \text{ is not integrable at } x\}$$

to obtain the desired singular metric for $c'L$. Notice that if we let $\varepsilon \to 0$, then $r \to 1$, $\tilde{c} \to c$ and $c' \to c$. □

Now we are in a position to prove our Main Theorem.

Proof. Fix $x \in X$. Our goal is to prove that, if m_0 is the smallest integer no less than

$$(e + \frac{1}{2}) n^{\frac{4}{3}} + \frac{1}{2} n^{\frac{2}{3}} + 1,$$

there exists a singular metric h of the line bundle $m_0 L$ such that the two properties listed at the beginning of this section are satisfied. As was explained before, this is all that is necessary to prove the Main Theorem.

Let a be the smallest integer which is no less than $e\,n^{\frac{4}{3}}$. Let d_0 be the integral part of $n^{\frac{2}{3}}$. According to Lemma 2.3, we have

$$\binom{n-1}{\tilde{d}-1}^{\frac{1}{\tilde{d}}} n \leq e\,\frac{n^2}{\tilde{d}}$$

for all integers $\tilde{d}$ with $1 \leq \tilde{d} \leq n-1$. Furthermore,

$$e\,\frac{n^2}{\tilde{d}} \leq e\,n^{\frac{4}{3}} \leq a$$

for $\tilde{d} \geq n^{\frac{2}{3}}$. Thus we can use Proposition 3.3 to produce an effective $\mathbb{Q}$-divisor D such that D is $\mathbb{Q}$-linearly equivalent to caL for some $0 < c < 1$, the pair (X, D) is log canonical at x and its minimal center at x is of dimension d' for some integer d' with $0 \leq d' \leq d_0$. By Proposition 3.5, this translates into the existence of a singular metric h_1 of $c'aL$ $(0 < c' < 1)$ such that $x \in V(\mathcal{I}_{h_1})$, $x \notin V(\mathcal{I}_{(h_1)^\gamma})$ for $\gamma < 1$ and the dimensions of those components of $V(\mathcal{I}_{h_1})$ that contain x do not exceed d'.
From this point onwards, we can use the method of [AS95] in the form of Proposition 3.2 to produce inductively a singular metric h_2 such that $V(\mathcal{I}_{h_2})$ is isolated at x. If the constructed metric h_2 is a metric for, say, kL, then

$$k \leq c'a + 1 + 2 + \ldots + d_0 + \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{d_0}.$$

Since the ε_i can be chosen to be arbitrarily small positive rational numbers and since $c' < 1$, we can assume that

$$c'a + 1 + 2 + \ldots + d_0 + \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_{d_0} < a + 1 + 2 + \ldots + d_0.$$

To obtain a metric of m_0L with the additional property that its curvature current dominates a positive definite smooth $(1,1)$-form on X in the sense of currents, we can simply multiply h_2 by the $(m_0 - k)$-th power of a smooth positive metric of L to obtain the desired metric h of m_0L. Note that $m_0 - k$ is a positive number because

$$\begin{aligned}
m_0 - k &> m_0 - (a + 1 + 2 + \ldots + d_0) \\
&\geq m_0 - (e\,n^{\frac{4}{3}} + 1 + 1 + 2 + \ldots + d_0) \\
&= m_0 - (e\,n^{\frac{4}{3}} + 1 + \frac{1}{2} d_0(d_0 + 1)) \\
&\geq m_0 - (e\,n^{\frac{4}{3}} + 1 + \frac{1}{2} n^{\frac{2}{3}}(n^{\frac{2}{3}} + 1)) \\
&= m_0 - ((e + \frac{1}{2}) n^{\frac{4}{3}} + \frac{1}{2} n^{\frac{2}{3}} + 1) \geq 0.
\end{aligned}$$

The proof of the Main Theorem is now complete. □

4 Applications

As was indicated before, not much is known about the very ampleness part of the Fujita conjecture. The theorems and techniques mentioned in the previous sections do not seem to be directly applicable to it. However, Angehrn and Siu [AS95] were able to prove the following weaker analog to the very ampleness part of Fujita's conjecture, in which they assume that L, in addition to being ample, is also base point free. Their result improves on previous results of Ein, Küchle and Lazarsfeld [EKL95] and Kollar [Kol93].

Theorem 4.1 ([AS95]). *Let L be an ample line bundle over a compact complex manifold X of complex dimension n such that L is free. Let A be an ample line bundle. Then $(n+1)L + A + K_X$ is very ample.*

In conjunction with our Main Theorem, Theorem 4.1 can readily be applied to the case of an ample canonical line bundle in order to give the following effective statement on pluricanonical embeddings. As far as the author knows, this is the best effective statement on pluricanonical embeddings currently on hand. Note that Fujita's conjecture indicates that the statement of Corollary 4.2 should hold true for any integer $m \geq n+3$.

Corollary 4.2. *If X is a compact complex manifold of complex dimension n whose canonical bundle K_X is ample, then mK_X is very ample for any integer $m \geq (e+\frac{1}{2})n^{\frac{7}{3}} + \frac{1}{2}n^{\frac{5}{3}} + (e+\frac{1}{2})n^{\frac{4}{3}} + 3n + \frac{1}{2}n^{\frac{2}{3}} + 5$.*

Proof. Let m_0 be the smallest integer no less than $(e+\frac{1}{2})n^{\frac{4}{3}} + \frac{1}{2}n^{\frac{2}{3}} + 1$. According to our Main Theorem, $m_0K_X + K_X = (m_0+1)K_X$ is base point free (and, of course, ample). Thus we can apply Theorem 4.1 with $L = (m_0+1)K_X$ and $A = K_X$ to obtain that mK_X is very ample for any integer $m \geq (n+1)(m_0+1)+2$. A simple estimate yields the following upper bound for $(n+1)(m_0+1)+2$:

$$\begin{aligned} & (n+1)(m_0+1)+2 \\ = \;& m_0(n+1)+n+3 \\ \leq \;& ((e+\frac{1}{2})n^{\frac{4}{3}} + \frac{1}{2}n^{\frac{2}{3}} + 1 + 1)(n+1) + n + 3 \\ = \;& (e+\frac{1}{2})n^{\frac{7}{3}} + \frac{1}{2}n^{\frac{5}{3}} + (e+\frac{1}{2})n^{\frac{4}{3}} + 3n + \frac{1}{2}n^{\frac{2}{3}} + 5. \end{aligned}$$

□

Finally, we remark that our effective statement on pluricanonical embeddings can be used to sharpen the best known bound for the number of dominant holomorphic maps from a fixed compact complex manifold with ample canonical bundle to any variable compact complex manifold with big and numerically effective canonical bundle.

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