Spectral Analysis of Relativistic Atoms – Dirac Operators with Singular Potentials

Matthias Huber

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Abstract. This is the first part of a series of two papers, which investigate spectral properties of Dirac operators with singular potentials. We examine various properties of complex dilated Dirac operators. These operators arise in the investigation of resonances using the method of complex dilations. We generalize the spectral analysis of Weder [50] and Šeba [46] to operators with Coulomb type potentials, which are not relatively compact perturbations. Moreover, we define positive and negative spectral projections as well as transformation functions between different spectral subspaces and investigate the non-relativistic limit of these operators. We will apply these results in [30] in the investigation of resonances in a relativistic Pauli-Fierz model, but they might also be of independent interest.

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1 Introduction and Definitions

A fascinating question in the mathematical analysis of operators describing atomic systems is the fate of eigenvalues embedded in the continuous spectrum if a perturbation is "turned on". Typically, these eigenvalues "vanish" and one has absolutely continuous spectrum. But the eigenvalues leave a trace: For example, the scattering cross section shows bumps near the eigenvalues, or certain states with energies close to the eigenvalues have an extended lifetime (described by the famous "Fermi Golden Rule" [13, Equation (VIII.2), p. 142] on a certain time scale). These energies are called resonances or resonance energies. Mathematically, resonances are described by poles of a holomorphic continuation of the resolvent (or matrix elements of it) or the scattering amplitude to a second sheet.

The generic systems in which resonances occur are many-particle systems. This can be many-electron systems, in which the electron-electron interaction is the perturbation. The corresponding physical effect is called "Auger effect": Excited states ("autoionizing states") relax by emission of electrons. Another typical system in a one- or many-electron atom interacting with the quantized electromagnetic field, in which case excited states can relax by emitting photons. Resonances can also occur in one-particle systems, although this is not typically the case. It is well known (see [8] for example) that for a Schrödinger operator with Coulomb potential the set of resonances is empty.

During the last decades numerous results were obtained in the mathematical investigation of resonances so that it seems hopeless to give a complete account of the available literature. Nevertheless we would like to give an overview and mention at least some of the relevant works.

The investigation of resonances as poles of holomorphic continuations of scattering amplitude and resolvent goes back to Weisskopf and Wigner [53] and Schwinger [45]. The mathematical theory of resonances was pushed further by Friedrichs [14], Livsic [36], and Howland [27, 28]. One of the mathematical methods in the spectral analysis is the method of complex dilation, which associates the "vanished" embedded eigenvalue with a non-real eigenvalue of a certain non-selfadjoint operator and was investigated by Aguilar and Combes [2] and Balslev and Combes [6] (see [43] for an overview). Resonances in the case of the Stark effect were investigated by Herbst [24] and by Herbst and Simon [25]. Simon [48] initiated the mathematical investigation of the time-dependent perturbation theory. This was carried on by Hunziker [32]. Herbst [23] proved exponential temporal decay for the Stark effect.

The spectral analysis of non-relativistic atoms in interaction with the radiation field was initiated by Bach, Fröhlich, and Sigal [4, 5]. It was carried on by Griesemer, Lieb und Loss [18], by Fröhlich, Griesemer und Schlein (see for example [15]) and many others (see for example Hiroshima [26], Arai and Hirokawa [3], Dereziński and Gérard [9], Hiroshima and Spohn [12]), Loss, Miyao and Spohn [37] or Hasler and Herbst [21, 20]). In particular, Bach, Fröhlich, and Sigal [5] proved a lower bound on the lifetime of excited states in non-relativistic QED. Later, an upper bound was proven by Hasler, Herbst, and Huber [22] (see also [29]) and by Abou Salem et al. [1]. Recently, Miyao and Spohn [38] showed the existence of a groundstate for a semi-relativistic electron coupled to the quantized radiation field.

Our overall aim is to show that the lifetime of excited states of a relativistic one-electron atom obeys Fermi's Golden Rule [30] and coincides with the non-relativistic result in leading order in the fine structure constant. We will investigate the necessary spectral properties of a Dirac operator with potential, projected to its positive spectral subspace, coupled to the quantized radiation field. Following Bach et al. [5] and Hasler et al. [22], our main technical tool is complex dilation in connection with the Feshbach projection method.

In this first part of the work, we investigate the necessary properties of one-particle Dirac operators with singular potentials. In particular, we will derive the necessary properties of complex dilated spectral projections and discuss the non-relativistic limit of complex dilated Dirac operators. This serves mainly as a technical input for the second part of our work [30]. However, we believe that some of the results presented in the first part are also of independent interest. Note that the method of complex dilation has already successfully been applied to Dirac operators (see Weder [50] and Šeba [46]). However, these authors assume the relative compactness of the electric potential so that their method does not apply to Coulomb type potentials. Note moreover that Weder [51] considers very general operators including relativistic spin-0-Hamiltonians with potentials with Coulomb singularity. The basic assumption of this work is, however, that the unperturbed operator is sectorial, which is not fulfilled for the Dirac operator. Our results cover a class of Dirac operators which includes Coulomb and Yukawa potentials (with exception of Lemma 11 and Lemma 12 which we prove for the Coulomb case only).
Our results about the spectral projections of the dilated Dirac operator can be used to generalize the Douglas-Kroll transformation (see Siedentop and Stockmeyer [47] and Huber and Stockmeyer [31]) to dilated operators.

2 Definitions and Overview

The free Dirac operator (with velocity of light $c > 0$)

$$D_{c,0} := -ic\boldsymbol{\alpha} \cdot \nabla + c^2\beta \tag{1}$$

is an operator on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4)$. It is self-adjoint on the domain $\mathrm{Dom}(D_{c,0}) := H^1(\mathbb{R}^3; \mathbb{C}^4)$ [49, Chapter 1.4]. Here $\boldsymbol{\alpha}$ is the vector of the usual Dirac α- matrices, and β is the Dirac β-matrix.
We define for $\epsilon > 0$ the strip $S_\epsilon := \{z \in \mathbb{C} | |\mathrm{Im}\, z| < \epsilon\}$. Let $\chi : \mathbb{R}^3 \to \mathbb{R}$ a bounded, measurable function. We will suppose that there is a $\Theta > 0$ such that $\theta \mapsto \chi(e^\theta x)$ admits a holomorphic continuation to $\theta \in S_\Theta$ for all $x \in \mathbb{R}^3$. We abbreviate $\chi_\theta := \chi(e^\theta \cdot)$. We will need the following two properties at different places:

$$\sup_{\theta \in S_\Theta,\, x \in \mathbb{R}^3} |\chi(e^\theta x)| \leq 1 \tag{H1}$$

$$\sup_{x \in \mathbb{R}^3} |\chi(e^\theta x) - \chi(x)| \leq \tilde{C}|\theta| \quad \text{for some } \tilde{C} > 0 \tag{H2}$$

It is easy to see that these properties are fulfilled for the Coulomb potential ($\chi(x) = 1$) or the Yukawa potential ($\chi(x) = e^{-ax}$ for some $a > 0$). The Dirac operator with potential $V := \chi / |\cdot|$

$$D_{c,\gamma} := -ic\boldsymbol{\alpha} \cdot \nabla + c^2\beta - \gamma V \tag{2}$$

is an operator on the Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^4)$ as well. It is self-adjoint on the domain $\mathrm{Dom}(D_{c,\gamma}) := \mathrm{Dom}(D_{c,0}) = H^1(\mathbb{R}^3; \mathbb{C}^4)$ for $\gamma \in \mathbb{R}$ with $|\gamma| <$

$c\sqrt{3}/2$ [49, Chapter 4.3.3]. γ is called coupling constant. The interacting Dirac operator describes a relativistic electron in the field of a nucleus, where the free operator yields the kinetic energy of the electron, whereas the electric potential gives its potential energy in the electric field of the nucleus.
The operator $D_{c,\gamma}$ has the set $(-\infty, -c^2] \cup [c^2, \infty)$ as essential spectrum. We assume that the operator has a nonempty set of positive eigenvalues, all of which have finite multiplicity. We number the eigenvalues by $\tilde{E}_{n,l}(c,\gamma)$ (not counting multiplicities). Here $n \in \mathbb{N}$ (or $n \in \{1, \ldots, N_{\max}\}$ for some $N_{\max} \in \mathbb{N}$ if there are only finitely many eigenvalues) denotes the principal quantum number and $l \in \{1, \ldots, N_n\}$ for some $N_n \in \mathbb{N}$ labels the fine structure components. We choose the numbering in such a way that for all $n' > n$, all $l \in \{1, \ldots, N_n\}$ and all $l' \in \{1, \ldots, N_{n'}\}$ the inequality $\tilde{E}_{n,l}(c,\gamma) < \tilde{E}_{n',l'}(c,\gamma)$ holds and such that $\tilde{E}_{n,l}(c,\gamma) < \tilde{E}_{n,l'}(c,\gamma)$ for $l < l'$. This numbering is natural for all values of c for the Coulomb potential, where the eigenvalues are explicitly known (see [35]). The spectrum of a Dirac operators can be shown to have this structure if c is large enough for general potentials (see [49]). We set $E_{n,l}(c,\gamma) := \tilde{E}_{n,l}(c,\gamma) - c^2$.
We define for $\theta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ the dilated operators

$$D_{c,0}(\theta) := -ice^{-\theta}\boldsymbol{\alpha} \cdot \nabla + c^2\beta \tag{3}$$

and

$$D_{c,\gamma}(\theta) := -ice^{-\theta}\boldsymbol{\alpha} \cdot \nabla + c^2\beta - \gamma V(\theta) \tag{4}$$

with $V(\theta) := e^{-\theta}\chi_\theta V_C$ on $\mathrm{Dom}(D_{c,0}(\theta)) = \mathrm{Dom}(D_{c,\gamma}(\theta)) = H^1(\mathbb{R}^3;\mathbb{C}^4)$, where $V_C = 1/|\cdot|$ is the Coulomb potential. It is clear that $D_{c,0}(\theta)$ is closed on this domain and that (because of Hardy's inequality) $D_{c,\gamma}(\theta)$ is at least well defined under assumption (H1). We shall prove further properties in Section 4. For technical reasons, we will assume $c \geq 1$ in the following. We will assume moreover that $\gamma \geq 0$. Further, we define for $\theta \in \mathbb{R}$ the unitary dilation $\mathcal{U}(\theta) : L^2(\mathbb{R}^3;\mathbb{C}^4) \to L^2(\mathbb{R}^3;\mathbb{C}^4)$, $(\mathcal{U}(\theta)f)(x) := e^{\frac{3}{2}\theta}f(e^\theta x)$. It fulfills the identity $\mathcal{U}(\theta)D_{c,\gamma}\mathcal{U}(\theta)^* = D_{c,\gamma}(\theta)$. The operators $D_{c,\gamma}(\theta)$ are extensions of the operators $\mathcal{U}(\theta)D_{c,\gamma}\mathcal{U}(\theta)^*$ for complex θ. Note that the mapping $\mathcal{U}(\theta)$ cannot be continued as a bounded operator to a complex domain, but the mapping $\theta \mapsto \mathcal{U}(\theta)\psi$ for an analytic vector ψ admits such an continuation, whose radius of convergence depends on the vector ψ (cf. [42, Chapter X.6]). However, we will prove in Section 8, that under certain conditions the restrictions of $\mathcal{U}(\theta)$ to certain spectral subspaces have bounded, bounded invertible extensions.
We add a short guide through the paper: We define a version of the Foldy-Wouthuysen transformation for non-self-adjoint Dirac operators in Section 3. Just as its analog for self-adjoint operators, it diagonalizes the free Dirac operator. It is however not a unitary operator any more so that one has to prove explicit estimates on its norm (see Theorem 1). The Foldy-Wouthuysen transformation serves as a technical input for the following sections.
We prove in Section 4 that the method of complex dilation can be successfully applied to Dirac operators with potentials with Coulomb singularities. In particular, we shall see that the dilated operators define a holomorphic family of

type (A) in the sense of Kato (see Theorem 2). Moreover, we provide a spectral analysis of such operators in Theorem 3. Just as in the case of Schrödinger operator, the real eigenvalues remain fixed under the complex dilation, whereas the essential spectrum swings into the complex plane and thus reveals possible non-real eigenvalues, which correspond to resonances of the original self-adjoint operator (see Figure ??). Note that there are no resonances for the Coulomb potential (see Remark 3).

In Section 5 we extend the notion of positive and negative spectral projections to the complex dilated Dirac operators. The definition of the spectral projections in Formula (32) is a straightforward extension of a well known formula from Kato's book (see [33, Lemma VI.5.6]). The rest of this section is devoted to the proof that the operators defined in (32) are actually well defined projections (see Theorem 4), that they commute with the dilated Dirac operator (see Theorem 5), and that their range is what one expects it to be (see Theorem 5 as well), which is not completely obvious in the non-self-adjoint case. Note that the projections themselves are not orthogonal projections.

These results enable us to define transformation functions between the positive spectral projections of the dilated and not dilated Dirac operators in Section 6, which is essential in order to show that also the projected Dirac operators are holomorphic families – even if they are coupled to the quantized radiation field. This will be accomplished in [30]. Moreover, these results can be used to generalize [47] to complex dilated operators. Transformation functions as defined in Formula (60) are similarity transformations between two (not necessarily orthogonal) projections (see Formula (57) in Theorem 6). Note that our definition requires that the norm difference between the projections be smaller than one, but there are more general approaches. For details on transformation functions we refer the reader to [33, Chapter II.4].

In Theorem 7 in Section 7 we prove a resolvent estimate for the dilated Dirac operator projected and restricted onto its positive spectral subspace. In particular, we prove that the norm of the resolvent converges (essentially) to zero as the inverse distance to the right complex half plane. Note that this really requires the restriction of the operator to its positive spectral subspace and that the norm of the resolvent of a non-self-adjoint operator is not bounded from above by the inverse distance of the spectral parameter to the spectrum.

In Section 8 we will investigate the non-relativistic limit of dilated Dirac operators and thereby generalize and extend the results in Thaller's book [49] in various directions. We prove in Theorem 8 and Corollary 2 that complex dilated Dirac operators converge to the corresponding (complex dilated) Schrödinger operators in the sense of norm resolvent convergence as the velocity of light goes to infinity. As in the undilated case, this convergence is needed to gain information about the spectral projections onto the eigenspaces belonging to the real eigenvalues and their behaviour in the nonrelativistic limit (see for example Lemma 7 or Lemma 8). In particular, the complex dilation, restricted to an eigenspace is a bounded operator (uniformly in the dilation parameter and the velocity of light – see Lemma 9) and the projections onto the fine structure components are uniformly bounded as well (see Corollary 5). These statements will be needed in [30]. Note that for Schrödinger operators and non-relativistic QED the above mentioned problems are absent, since there is neither a fine structure splitting nor the additional parameter of the velocity of light which has to be controlled.

Moreover, we show in Theorem 9 and Theorem 10 that the lower Pauli spinor of a normed eigenfunction of the Dirac operator converges to zero in the sense of the Sobolev space $H^1(\mathbb{R}^3;\mathbb{C}^2)$ and that the upper Pauli spinor is bounded in the sense of $H^1(\mathbb{R}^3;\mathbb{C}^2)$ as the velocity of light tends to infinity. This shows that the notion of "large" and "small" components of a Dirac spinor, which is frequently used by physicists, is also justified for dilated operators. Moreover, it follows that certain expectation values of the Dirac $\boldsymbol{\alpha}$-matrix vanish as the velocity of light tends to infinity. We will apply this fact in [30].

Note that in the discussion of the non-relativistic limit in Section 8 we need some estimates from Bach, Fröhlich, and Sigal [5] which we cite in Appendix A for the convenience of the reader.

3 Foldy-Wouthuysen-Transformation

In this section we investigate the complex continuation of the Foldy-Wouthuysen transformation and show some important properties in Theorem 1. We need this as a technical input for the spectral analysis in the following sections. Let us mention that a complex continuation of the Foldy-Wouthuysen transformation was implicitly used by Evans, Perry, and Siedentop [11] for the investigation of the spectrum of the Brown-Ravenhall operator. Also Balslev and Helffer [7] use holomorphic continuations of the Foldy-Wouthuysen transformation.

For $p \in \mathbb{R}^3$ we define the matrix $D_{c,0}(p;\theta) := ce^{-\theta}\boldsymbol{\alpha}\cdot p + c^2\beta$. We use the convention $\sqrt{\cdot} : \mathbb{C}\setminus\mathbb{R}_0^- \to \mathbb{C} : \sqrt{z} = re^{\mathrm{i}\,\phi/2}$ for the complex square root, where $z = re^{\mathrm{i}\,\phi}$ with $r \geq 0$ and $-\pi < \phi < \pi$. Note that for $w \in \mathbb{C}$ with $|\arg w| \leq \frac{\pi}{4}$ the estimate

$$\operatorname{Re}\sqrt{w} \geq \sqrt{\operatorname{Re} w} \geq 0 \tag{5}$$

holds, which follows immediately from the formula $\cos(2\phi) = (\cos\phi)^2 - (\sin\phi)^2 \leq (\cos\phi)^2$. Next, we define for $p \in \mathbb{R}^3$ and $\theta \in S_{\pi/2}$ the matrix

$$\hat{U}_{\mathrm{FW}}(c,p;\theta) := \frac{1}{N_c(p;\theta)}\begin{pmatrix}(c^2+E_c(p;\theta))\mathbf{1}_{2\times 2} & ce^{-\theta}\boldsymbol{\sigma}\cdot p\\ -ce^{-\theta}\boldsymbol{\sigma}\cdot p & (c^2+E_c(p;\theta))\mathbf{1}_{2\times 2}\end{pmatrix}, \tag{6}$$

where $E_c(p;\theta) := \sqrt{e^{-2\theta}c^2p^2+c^4}$ and $N_c(p;\theta) := \sqrt{2E_c(p;\theta)(c^2+E_c(p;\theta))}$. $\hat{U}_{\mathrm{FW}}(c;\theta)$ is the maximal multiplication operator on $L^2(\mathbb{R}^3;\mathbb{C}^4)$ which is generated by $U_{\mathrm{FW}}(p,c;\theta)$. Analogously, we define

$$\hat{V}_{\mathrm{FW}}(p,c;\theta) := \frac{c^2+E_c(p;\theta)-ce^{-\theta}\beta\boldsymbol{\alpha}\cdot p}{N_c(p;\theta)} \tag{7}$$

and $V_{\mathrm{FW}}(c;\theta)$. The corresponding Fourier transforms are $U_{\mathrm{FW}}(c;\theta) := \mathcal{F}^{-1}\hat{U}_{\mathrm{FW}}(c;\theta)\mathcal{F}$ and $V_{\mathrm{FW}}(c;\theta) := \mathcal{F}^{-1}\hat{V}_{\mathrm{FW}}(c;\theta)\mathcal{F}$. Note that these operators coincide with the usual Foldy-Wouthuysen transformation for $\theta = 0$ (see [49]), but are not unitary for $\theta \notin \mathbb{R}$. Nevertheless they define a similarity transformation, which diagonalizes the free Dirac operator. This will be important in the following sections, since the diagonalized operator $\sqrt{-c^2 e^{-2\theta}\Delta + c^4}\beta$ is normal, contrary to the operator $D_{c,0}(\theta)$.

Theorem 1. *Let $\theta \in S_{\pi/4}$. Then the following statements hold:*

a) *The operator $U_{\mathrm{FW}}(c;\theta)$ is a bounded operator on $L^2(\mathbb{R}^3;\mathbb{C}^4)$ with bounded inverse $V_{\mathrm{FW}}(c;\theta)$. There is a constant C_{FW} (independent of c and θ) such that*

$$\|U_{\mathrm{FW}}(c;\theta)\| \le \sqrt{1 + C_{\mathrm{FW}}|\sin\operatorname{Im}\theta|} \tag{8}$$

and

$$\|V_{\mathrm{FW}}(c;\theta)\| \le \sqrt{1 + C_{\mathrm{FW}}|\sin\operatorname{Im}\theta|}. \tag{9}$$

b) *The Foldy-Wouthuysen transformation diagonalizes the Dirac operator:*

$$U_{\mathrm{FW}}(c;\theta) D_{c,0}(\theta) V_{\mathrm{FW}}(c;\theta) = \sqrt{-c^2 e^{-2\theta}\Delta + c^4}\beta. \tag{10}$$

Proof.
a) A simple calculation shows

$$\hat{U}_{\mathrm{FW}}(p,c;\theta)\hat{V}_{\mathrm{FW}}(p,c;\theta) = \hat{V}_{\mathrm{FW}}(p,c;\theta)\hat{U}_{\mathrm{FW}}(p,c;\theta) = \mathbf{1}. \tag{11}$$

We have $\|U_{\mathrm{FW}}(c;\theta)\| \le \sup_{p\in\mathbb{R}^3} \|\hat{U}_{FW,c}(p;\theta)\|$. Thus, it suffices to consider the case $c = 1$ and $\operatorname{Re}\theta = 0$. In view of the identity $\|\hat{U}_{\mathrm{FW,c}}(p;\theta)\|^2 = \|\hat{U}_{\mathrm{FW,c}}(p;\theta)^*\hat{U}_{\mathrm{FW,c}}(p;\theta)\|$ we find with $\vartheta \in (-\pi/4, \pi/4)$

$$\begin{aligned}\hat{U}_{\mathrm{FW,c}}(p;\mathrm{i}\,\vartheta)^*\hat{U}_{\mathrm{FW,c}}(p;\mathrm{i}\,\vartheta) &= \frac{(1+E_1(p;\mathrm{i}\,\vartheta))(1+E_1(p;-\mathrm{i}\,\vartheta)) + p^2}{\tilde{N}} \\ &+ \frac{\beta\boldsymbol{\alpha}\cdot p(e^{-\mathrm{i}\,\vartheta}(1+E_1(p;-\mathrm{i}\,\vartheta)) - e^{\mathrm{i}\,\theta}(1+E_1(p;\mathrm{i}\,\vartheta)))}{\tilde{N}},\end{aligned} \tag{12}$$

where $\tilde{N} := \sqrt{4E_1(p;\mathrm{i}\,\vartheta)E_1(p;-\mathrm{i}\,\vartheta)(1+E_1(p;\mathrm{i}\,\vartheta))(1+E_1(p;-\mathrm{i}\,\vartheta))}$. Note that the expression under the square root is real, and that $|1 + E_1(p;\pm\mathrm{i}\,\vartheta)| \ge |E_1(p;\pm\mathrm{i}\,\vartheta)| = \sqrt[4]{1+2\cos(2\vartheta)p^2+p^4} \ge \sqrt[4]{1+p^4}$, where we used $|\vartheta| < \pi/4$. Thus the denominator in (12) can be estimated as

$$|\tilde{N}| \ge 2\sqrt{1+|p|^4}. \tag{13}$$

Next, observe that

$$|e^{\mathrm{i}\,\vartheta}E_1(p;\mathrm{i}\,\vartheta) - e^{-\mathrm{i}\,\vartheta}E_1(p;-\mathrm{i}\,\vartheta)| \le \frac{|\sin(2\vartheta)|}{\sqrt{p^2+\cos(2\vartheta)}}, \tag{14}$$

where we used the estimate $|w| \ge |\operatorname{Re} w|$ and (5). From (14) it follows that

$$\|\beta\boldsymbol{\alpha}\cdot p(e^{\mathrm{i}\,\vartheta}(1+E_1(p;\mathrm{i}\,\vartheta)) - e^{-\mathrm{i}\,\vartheta}(1+E_1(p;-\mathrm{i}\,\vartheta)))\| \le 2|p||\sin(\vartheta)| + |\sin(2\vartheta)|. \tag{15}$$

Moreover, we have

$$1-\frac{(1+E_1(p;-\mathrm{i}\,\vartheta))\,(1+E_1(p;\mathrm{i}\,\vartheta))+p^2}{\tilde{N}}$$
$$=\frac{\tilde{N}^2+\big((1+E_1(p;-\mathrm{i}\,\vartheta))\,(1+E_1(p;\mathrm{i}\,\vartheta))+p^2\big)^2}{\tilde{N}\,\Big(\tilde{N}+((1+E_1(p;-\mathrm{i}\,\vartheta))\,(1+E_1(p;\mathrm{i}\,\vartheta))+p^2)\Big)}. \quad (16)$$

Using $((1+E_1(p;-\mathrm{i}\,\vartheta))(1+E_1(p;\mathrm{i}\,\vartheta))+p^2)>0$ and (13) we estimate the denominator by

$$|\tilde{N}(\tilde{N}+((1+E_1(p;-\mathrm{i}\,\vartheta))(1+E_1(p;\mathrm{i}\,\vartheta))+p^2))|\geq 4(1+|p|^4). \quad (17)$$

In order to estimate the enumerator we find after some calculations

$$\begin{aligned}&4E_1(p;-\mathrm{i}\,\vartheta)E_1(p;\mathrm{i}\,\vartheta)(1+E_1(p;-\mathrm{i}\,\vartheta))(1+E_1(p;\mathrm{i}\,\vartheta)) \quad (18)\\ &\quad-((1+E_1(p;-\mathrm{i}\,\vartheta))(1+E_1(p;\mathrm{i}\,\vartheta))+p^2)^2\\ &=2p^4+2(e^{2\mathrm{i}\,\vartheta}+e^{-2\mathrm{i}\,\vartheta})p^2+2p^2(e^{-2\mathrm{i}\,\vartheta}E_1(p;-\mathrm{i}\,\vartheta)+e^{2\mathrm{i}\,\vartheta}E_1(p;\mathrm{i}\,\vartheta))\\ &\quad-2p^2-2p^2(E_1(p;-\mathrm{i}\,\vartheta)+E_1(p;\mathrm{i}\,\vartheta))-2p^2E_1(p;-\mathrm{i}\,\vartheta)E_1(p;\mathrm{i}\,\vartheta).\end{aligned}$$

We combine suitable terms in (18): We have

$$(e^{2\mathrm{i}\,\vartheta}+e^{-2\mathrm{i}\,\vartheta})p^2-2p^2=2(\cos(2\vartheta)-1)p^2, \quad (19)$$

$$\begin{aligned}&|2p^2(e^{-2\mathrm{i}\,\vartheta}E_1(p;-\mathrm{i}\,\vartheta)+e^{2\mathrm{i}\,\vartheta}E_1(p;\mathrm{i}\,\vartheta))-2p^2(E_1(p;-\mathrm{i}\,\vartheta)+E_1(p;\mathrm{i}\,\vartheta))|\leq 4p^2 \quad (20)\\ &\times|\sqrt{p^2+e^{2\mathrm{i}\,\vartheta}}-\sqrt{p^2+e^{-2\mathrm{i}\,\vartheta}}|\leq 4p^2\frac{2\sin(2\vartheta)}{|\sqrt{p^2+e^{2\mathrm{i}\,\vartheta}}+\sqrt{p^2+e^{-2\mathrm{i}\,\vartheta}}|}\leq 4|p|\sin(2\vartheta),\end{aligned}$$

and

$$|2p^4+2\cos(2\vartheta)p^2-2p^2E_1(p;-\mathrm{i}\,\vartheta)E_1(p;\mathrm{i}\,\vartheta)|\leq 2|\sin(2\vartheta)|^2. \quad (21)$$

Summarizing the estimates (13) and (15) through (21), we finally obtain

$$\|\hat{U}_{\mathrm{FW}}(\mathrm{i}\,\vartheta,p)^*\hat{U}_{\mathrm{FW}}(\mathrm{i}\,\vartheta,p)-1\|\leq\left[\frac{|p|+1}{\sqrt{1+|p|^4}}+\frac{p^2+2|p|+1}{1+|p|^4}\right]|\sin(\vartheta)|, \quad (22)$$

where we used that $|\sin(2\vartheta)|\leq 2|\sin\vartheta|$ for $|\vartheta|\leq\pi/4$. If we set $C_{\mathrm{FW}}:=\sup_{t\in\mathbb{R}_0^+}\left[\frac{t+1}{\sqrt{1+t^4}}+\frac{t^2+2t+1}{1+t^4}\right]<\infty$, equation (22) shows the claim on $U_{\mathrm{FW}}(c;\theta)$. The claim on the inverse operator $V_{\mathrm{FW}}(c;\theta)$ can be proven analogously.
b) We have $\hat{U}_{\mathrm{FW}}(c,p;\theta)D_{c,0}(p;\theta)\hat{V}_{\mathrm{FW}}(c,p;\theta)=D_{c,0}(p;\theta)\hat{V}_{\mathrm{FW}}(c,p;\theta)^2$ as well as $\hat{V}_{\mathrm{FW}}(c,p;\theta)=\hat{U}_{\mathrm{FW}}(c,p;\theta)-2ce^{-\theta}\beta\boldsymbol{\alpha}\cdot p/N_c(p;\theta)$. From this it follows that

$\hat{U}_{\rm FW}(c,p;\theta)D_{c,0}(p;\theta)\hat{V}_{\rm FW}(c,p;\theta) = D_{c,0}(p;\theta) - A$, where $A := \frac{1}{N_c(p;\theta)^2}$ $D_{c,0}(p;\theta)[2ce^{-\theta}\beta\boldsymbol{\alpha}\cdot p][c^2+E_c(p;\theta) - ce^{-\theta}\beta\boldsymbol{\alpha}\cdot p]$. A little calculation shows $A = -\frac{2c^2e^{-2\theta}p^2E_c(p;\theta)\beta}{N_c(p;\theta)^2} + ce^{-\theta}\boldsymbol{\alpha}\cdot p$, which implies

$$\hat{U}_{\rm FW}(c,p;\theta)D_{c,0}(p;\theta)\hat{V}_{\rm FW}(c,p;\theta) = E_c(p;\theta)\beta \tag{23}$$

and thus proves (10). □

4 Dilation Analyticity and Spectrum

We show that the operators in equations (3) and (4) define holomorphic families of closed operators. Since we will be interested in the non-relativistic limit later on, we consider only such values of c and γ which can be dealt with using Hardy's inequality. For $\theta \in S_{\pi/2}$ we define the set $M_{\gamma/c} := \{\theta \in \mathbb{C} | \frac{2\gamma}{c} < \cos(\operatorname{Im}\theta)\}$. We define $V_1(\theta) := e^{-\theta/2}\chi_\theta\sqrt{V_C}$ and $V_2(\theta) := e^{-\theta/2}\sqrt{V_C}$. Note that $V(\theta) = V_1(\theta)V_2(\theta)$.

Theorem 2. *Let $\theta \in S_{\min\{\Theta,\pi/2\}}$ and suppose that (H1) holds. Then the operator $D_{c,\gamma}(\theta)$ is closed for $\frac{2\gamma}{c} < \cos(\operatorname{Im}\theta)$ on $\operatorname{Dom}(D_{c,\gamma}(\theta)) = H^1(\mathbb{R}^3;\mathbb{C}^4)$, and we have $D_{c,\gamma}(\theta)^* = D_{c,\gamma}(\bar\theta)$. $D_{c,\gamma}(\theta)$ is a holomorphic family of type (A) in the sense of Kato for $\theta \in M_{\gamma/c}$. $D_{c,0}(\theta)$ is an entire family of type (A).*

Proof. For $f \in H^1(\mathbb{R}^3;\mathbb{C}^4)$ the estimate $\|D_{c,0}(\theta)f\|^2 \geq |\operatorname{Re} e^{-\theta}|^2c^2\|\nabla f\|^2$ holds. Hardy's inequality implies $\|\gamma V(\theta)f\|^2 \leq 4\gamma^2|e^{-\theta}|^2\|\nabla f\|^2$ and thus $\|\gamma V(\theta)f\| \leq \frac{2\gamma}{c\cos(\operatorname{Im}\theta)}\|D_{c,0}(\theta)f\|$, which proves that the operator $D_{c,\gamma}(\theta)$ is closed and has a bounded inverse. Thus, the domain $\operatorname{Dom}(D_{c,\gamma}(\theta)) = H^1(\mathbb{R}^3;\mathbb{C}^4)$ is independent of $\theta \in M_{\gamma/c}$. It is clear that for $f \in \operatorname{Dom}(D_{c,\gamma}(\theta))$ the mapping $M_{\gamma/c} \to L^2(\mathbb{R}^3;\mathbb{C}^4)$, $\theta \mapsto D_{c,\gamma}(\bar\theta)f$ is holomorphic, which implies that $D_{c,\gamma}(\theta)$ is a holomorphic family of type (A) [33, Chapter VII-2.1].
Moreover, obviously $D_{c,\gamma}(\bar\theta)^* \supset D_{c,\gamma}(\theta)$ holds. Thus, it suffices to prove the inclusion $\operatorname{Dom}(D_{c,\gamma}(\bar\theta)^*) \subset \operatorname{Dom}(D_{c,\gamma}(\theta)) = \operatorname{Ran}(D_{c,\gamma}(\theta)^{-1})$. We adapt a well known strategy from the case of self-adjoint operators (cf. [52, Satz 5.14]). We have $\operatorname{Dom}(D_{c,\gamma}(\theta)^{-1}) = \operatorname{Ran}(D_{c,\gamma}(\bar\theta)) = L^2(\mathbb{R}^3;\mathbb{C}^4)$. For $f \in \operatorname{Dom}(D_{c,\gamma}(\bar\theta)^*)$ we find $f_0 := D_{c,\gamma}(\theta)^{-1}D_{c,\gamma}(\bar\theta)^*f \in \operatorname{Dom}(D_{c,\gamma}(\theta)) \subset \operatorname{Dom}(D_{c,\gamma}(\bar\theta)^*)$. Thus $D_{c,\gamma}(\theta)f_0 = D_{c,\gamma}(\bar\theta)^*f_0$, and the definition of f_0 implies $D_{c,\gamma}(\bar\theta)^*f = D_{c,\gamma}(\theta)f_0$. From this it follows that $D_{c,\gamma}(\bar\theta)^*(f-f_0) = 0$ and thus $f - f_0 \in \operatorname{N}(D_{c,\gamma}(\bar\theta)^*) = \operatorname{Ran}(D_{c,\gamma}(\bar\theta))^\perp = \{0\}$, implying $f = f_0 \in \operatorname{Dom}(D_{c,\gamma}(\theta))$. □

Remark 1. *Note that if V is the Coulomb potential or the Yukawa potential, then $D_{c,\gamma}(\theta)$ is equal to a multiple of the self-adjoint operator $-ic\boldsymbol{\alpha}\cdot\nabla + V_C$ up to a bounded operator so that the proof of the above theorem is trivial. Note moreover, that for $V = V_C$, the operator $D_{c,\gamma}(\theta)$ is entire.*

Remark 2. *Theorem 2 and its proof imply that $H^1(\mathbb{R}^3;\mathbb{C}^4)$ is the maximal domain of the operator on $L^2(\mathbb{R}^3;\mathbb{C}^4)$ generated by the differential expression*

$\tilde{D}_{c,\gamma}(\theta) := -e^{-\theta} ic\boldsymbol{\alpha}\cdot\nabla + c^2\beta - \gamma V(\theta)$. To see this set

$$M_{\max} := \{f \in L^2(\mathbb{R}^3;\mathbb{C}^4) | \tilde{D}_{c,\gamma}(\theta) f \in L^2(\mathbb{R}^3;\mathbb{C}^4)\},$$

where the gradient is to be understood in distributional sense. Note that $f \in M_{\max}$ implies $\nabla f \in L^1_{loc}(\mathbb{R}^3;\mathbb{C}^4)$, since $V(\theta) \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. If $M_{\max} \supsetneq H^1(\mathbb{R}^3;\mathbb{C}^4)$, then the operator $D'_{c,\gamma}(\theta)$ defined by the differential expression $\tilde{D}_{c,\gamma}(\theta)$ on the domain $\mathcal{D}(D'_{c,\gamma}(\theta)) := M_{\max}$ is a strict extension of the operator $D_{c,\gamma}(\theta)$. As in the proof of Theorem 2 it would follow that there was a $0 \neq g \in M_{\max}$ such that $D'_{c,\gamma}(\theta) g = 0$. It follows by partial integration from $\nabla g \in L^1_{loc}(\mathbb{R}^3;\mathbb{C}^4)$ that $(\tilde{D}_{c,\gamma}(\bar{\theta}) f, g) = 0$ for all $f \in C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$. By density of $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ in $H^1(\mathbb{R}^3;\mathbb{C}^4)$ this equality extends to $(D_{c,\gamma}(\bar{\theta}) f, g) = 0$ for all $f \in H^1(\mathbb{R}^3;\mathbb{C}^4) = \mathcal{D}(D_{c,\gamma}(\bar{\theta}))$. Since $D_{c,\gamma}(\bar{\theta})$ is onto, it follows $g = 0$, a contradiction, which implies $H^1(\mathbb{R}^3;\mathbb{C}^4) = M_{\max}$.

The following lemma, whose simple proof we omit, contains a useful fact:

Lemma 1. *Let $a, b > 0$. Then $\sup_{p\in\mathbb{R}^3} \frac{\sqrt{a^2c^2p^2+c^4}}{\sqrt{b^2c^2p^2+c^4}} \leq \max\{1, \frac{a}{b}\}$.*

Now we need the spectrum of the operator $D_{c,\gamma}(\theta)$. Theorem 1 shows (see Figure 1) $\sigma(D_{c,0}(\theta)) = \Sigma_c^-(\theta) \cup \Sigma_c^+(\theta)$, where $\Sigma_c^\pm(\theta) = \pm E_c(\mathbb{R};\theta)$.
In the case of self-adjoint operators the compactness of the difference of free and interacting resolvent would imply that $D_{c,0}(\theta)$ and $D_{c,\gamma}(\theta)$ with $\gamma \neq 0$ have the same essential spectrum. This is however not true for non-self-adjoint operators in general. In particular there exist several different definitions of the essential spectrum, which do not coincide in general and have different invariance properties.
In the case of relatively compact perturbations this difficulty can be mastered using the analytic Fredholm theorem [50]. Since Coulomb type potentials are not relatively compact, we adapt a strategy invented by Nenciu [40] for the self-adjoint case. We need the following lemma:

Lemma 2. *Let $\theta \in S_{\pi/4}$ and $z \notin \sigma(D_{c,0}(\theta))$. Then the operator $V_C^{1/2}(D_{c,0}(\theta) - z)^{-1}$ is compact.*

Proof. It suffices to consider the case $z = 0$. We write $V_C^{1/2} D_{c,0}(\theta)^{-1} = V_C^{1/2}\left(\sqrt{-c^2e^{-2\theta}\Delta + c^4}\beta\right)^{-1}\left(\sqrt{-c^2e^{-2\theta}\Delta + c^4}\beta\right) D_{c,0}(\theta)^{-1}$. Because of $V_C^{1/2} \in L^6_w(\mathbb{R}^3)$ and $1/(\pm\sqrt{c^2e^{-2\theta}(\cdot)^2 + c^4} - z) \in L^6(\mathbb{R}^3)$, the operator $V_C^{1/2}(\sqrt{-c^2e^{-2\theta}\Delta + c^4}\beta - z)^{-1}$ is compact [44]. Moreover, Theorem 1 implies $(\sqrt{-c^2e^{-2\theta}\Delta + c^4}\beta) D_{c,0}(\theta)^{-1}\| \leq 1 + C_{\mathrm{FW}}|\mathrm{Im}\,\theta|$. This shows the claim. □

For $z \notin \sigma(D_{c,0}(\theta))$ we define the operator $M_{c;\theta}(z) := V_2(\theta)(D_{c,0}(\theta) - z)^{-1} V_1(\theta)$. Moreover, let $B_{c;\theta;+}$ and $B_{c;\theta;-}$ (see Figure 1) the closed subsets of $\{z \in \mathbb{C} | \mathrm{Re}\, z > 0\}$ and $\{z \in \mathbb{C} | \mathrm{Re}\, z < 0\}$ respectively, which are enclosed be the

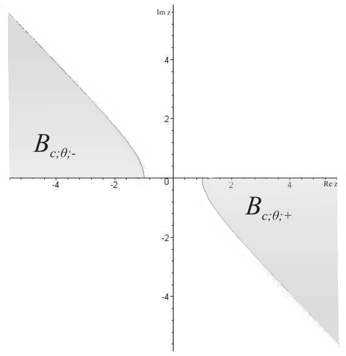

Figure 1: The spectrum of the operator $D_{c,0}(\theta)$ and sets $B_{c;\theta;\pm}$ for $c = 1$ and $\theta = \mathrm{i}\,\pi/4$.

curves $[c^2, \infty)$ and $E_c(\mathbb{R};\theta)$ ($(-\infty, -c^2]$ and $-E_c(\mathbb{R};\theta)$ respectively). We set $B_{c;\theta} = B_{c;\theta;+} \cup B_{c;\theta;-}$.
Furthermore, for $\theta \in S_{\pi/4}$ we define the constants

$$C(\operatorname{Im}\theta) := \frac{1 + C_{\mathrm{FW}}|\operatorname{Im}\theta|}{\sqrt{\cos(2\operatorname{Im}\theta)}}, \quad C_1(\operatorname{Im}\theta) := C(\operatorname{Im}\theta) + \frac{1 + C_{\mathrm{FW}}|\operatorname{Im}\theta|}{\cos(\operatorname{Im}\theta)}. \quad (24)$$

Note the inequality $1/\cos(\operatorname{Im}\theta) \leq C(\operatorname{Im}\theta)$.
The following theorem yields a precise description of the spectrum of the operator $D_{c,\gamma}(\theta)$. In particular, outside the set $B_{c,\theta}$ the spectra of $D_{c,\gamma}(\theta)$ and $D_{c,\gamma}(0)$ coincide so that one particle resonances – if any exist – can be located only within the set $B_{c,\theta}$.
Let $\mathcal{B}(L^2(\mathbb{R}^3;\mathbb{C}^4))$ be the set of bounded and everywhere defined operators on $L^2(\mathbb{R}^3;\mathbb{C}^4)$. Moreover, we set $B_a(x_0) := \{x \in \mathbb{R}^3 | |x - x_0| < a\}$ for $a > 0$ and $x_0 \in \mathbb{R}^3$

Theorem 3. *Let $\theta \in S_{\min\{\pi/4,\Theta\}}$ and $\frac{2\gamma}{c} C(\operatorname{Im}\theta) < 1$. Suppose that (H1) holds. Then $\sigma(D_{c,\gamma}(\theta)) = \sigma(D_{c,0}(\theta)) \cup A_{c,\gamma;\theta}$, where $A_{c,\gamma;\theta}$ is a discrete subset of $\mathbb{C} \setminus \sigma(D_{c,0}(\theta))$, and we have $A_{c,\gamma;\theta} \cap (\mathbb{C} \setminus B_{c;\theta}) = \sigma_{\mathrm{disc}}(D_{c,\gamma}(0))$. The set $A_{c,\gamma;\theta}$ has at most the accumulation points $\pm c^2$. For $z \notin \sigma(D_{c,\gamma}(\theta))$ the resolvent identity*

$$\begin{aligned}(D_{c,\gamma}(\theta) - z)^{-1} &= (D_{c,0}(\theta) - z)^{-1} + \\ &+ \gamma (D_{c,0}(\theta) - z)^{-1} V_1(\theta)(1 - e^{-\theta}\gamma M_{c;\theta}(z))^{-1} V_2(\theta)(D_{c,0}(\theta) - z)^{-1} \quad (25)\end{aligned}$$

holds.

Proof. We denote the r.h.s. of (25) by $R_{c,\gamma;\theta}(z)$.

Step 1: Proof of (25) *for $z = \mathrm{i}\,\eta$, $\eta \in \mathbb{R}$.* Using Kato's inequality and Theorem 1 we obtain

$$\|\gamma M_{c;\theta}(\mathrm{i}\,\eta)\| = \|\gamma V_2(\theta)(D_{c,0}(\theta) - \mathrm{i}\,\eta)^{-1}V_1(\theta)\| \leq \frac{\gamma\pi e^{-\mathrm{Re}\,\theta}(1 + C_{\mathrm{FW}}|\mathrm{Im}\,\theta|)}{2} \times \|\frac{|\nabla|}{\sqrt{-\cos(2\mathrm{Im}\,\theta)c^2 e^{-2\mathrm{Re}\,\theta}\Delta + c^4}}\| \leq \frac{\gamma}{c}\frac{\pi}{2}C(\mathrm{Im}\,\theta), \quad (26)$$

where we used additionally (5) and Lemma 1. Equation (26) shows that (25) holds for $z = \mathrm{i}\,\eta$, $\eta \in \mathbb{R}$.
Step 2: Proof of (25)*, general case.* We have

$$1-\gamma M_{c;\theta}(z) = 1-\gamma M_{c;\theta}(0)-\gamma(M_{c;\theta}(z)-M_{c;\theta}(0)) = (1-\gamma M_{c;\theta}(0))(1-N(z)),$$

where $N(z) := z\,(1 - \gamma M_{c;\theta}(0))^{-1}\left[V_2(\theta)D_{c,0}(\theta)^{-1}(D_{c,0}(\theta) - z)^{-1}V_1(\theta)\right]$. Using Step 1 and Lemma 2 we see that $N(z)$ is compact and a holomorphic function of z for $z \in \mathbb{C} \setminus \sigma(D_{c,0}(\theta))$. Applying the analytic Fredholm theorem [41, Theorem VI.14] yields that $(1 - N(z))^{-1}$ is a meromorphic function on $\mathbb{C} \setminus \sigma(D_{c,0}(\theta))$ with values in $\mathcal{B}(L^2(\mathbb{R}^3;\mathbb{C}^4))$, whose residues are operators of finite rank. Using Step 1 once more, we see that this also holds for $(1 - e^{-\theta}\gamma M_{c;\theta}(z))^{-1}$. In particular, there is a set $A_{c,\gamma;\theta} \subset \mathbb{C} \setminus \sigma(D_{c,0}(\theta))$ which has no accumulation point in $\mathbb{C} \setminus \sigma(D_{c,0}(\theta))$ such that $z \mapsto R_{c,\gamma;\theta}(z)$ is holomorphic in $\mathbb{C} \setminus (\sigma(D_{c,0}(\theta)) \cup A_{c,\gamma;\theta})$.
Step 3: The mapping $z \mapsto R_{c,\gamma;\theta}(z)\,(D_{c,\gamma}(\theta) - z)\,f$ with $f \in \mathrm{Dom}(D_{c,\gamma}(\theta))$ is holomorphic on $\mathbb{C} \setminus (\sigma(D_{c,0}(\theta)) \cup A_{c,\gamma;\theta})$. Because of Step 1 the operator $R_{c,\gamma;\theta}(z)$ equals the resolvent of $D_{c,\gamma}(\theta)$ for $z = \mathrm{i}\,\eta$, $\eta \in \mathbb{R}$. It follows that $R_{c,\gamma;\theta}(z)\,(D_{c,\gamma}(\theta) - z)\,f = f$ for all $z \in \mathbb{C} \setminus (\sigma(D_{c,0}(\theta)) \cup A_{c,\gamma;\theta})$ and $f \in \mathrm{Dom}(D_{c,\gamma}(\theta))$.
Moreover, it is easy to see that $\mathrm{Ran}\,R_{c,\gamma;\theta}(z) \subset H^{1/2}(\mathbb{R}^3;\mathbb{C}^4)$. Thus, we obtain as before $(g, (D_{c,\gamma}(\theta) - z)\,R_{c,\gamma;\theta}(z)f) = (g, f)$ for all $f \in L^2(\mathbb{R}^3;\mathbb{C}^4)$, $g \in H^{1/2}(\mathbb{R}^3;\mathbb{C}^4)$ and $z \in \mathbb{C} \setminus (\sigma(D_{c,0}(\theta)) \cup A_{c,\gamma;\theta})$. It follows that $\mathrm{Ran}\,R_{c,\gamma;\theta}(z) \subset H^1(\mathbb{R}^3;\mathbb{C}^4)$ and $(D_{c,\gamma}(\theta) - z)R_{c,\gamma;\theta}(z)f = f$ for $f \in L^2(\mathbb{R}^3;\mathbb{C}^4)$ and $z \in \mathbb{C} \setminus (\sigma(D_{c,0}(\theta)) \cup A_{c,\gamma;\theta})$. Summarizing, we find $R_{c,\gamma;\theta}(z) = (D_{c,\gamma}(\theta) - z)^{-1}$ for all $z \in \mathbb{C} \setminus (\sigma(D_{c,0}(\theta)) \cup A_{c,\gamma;\theta})$. In particular, it follows that $\sigma(D_{c,\gamma}(\theta)) \subset \sigma(D_{c,0}(\theta)) \cup A_{c,\gamma;\theta}$.
Let now $z_0 \in A_{c,\gamma;\theta}$. Then the analytic Fredholm theorem implies the existence of $f \in L^2(\mathbb{R}^3;\mathbb{C}^4)$ with $(1 - N(z_0))f = 0$, and thus also $(1 - \gamma M_{c;\theta}(z_0))f = 0$. We proceed as follows: Since $(D_{c,0}(\theta) - z)^{-1}V_1(\theta)$ is bounded, we find $f \in \mathrm{Ran}(V_2(\theta))$, i.e. $f = V_2(\theta)g$ for $g = (D_{c,0}(\theta) - z)^{-1}V_1(\theta)f \in L^2(\mathbb{R}^3;\mathbb{C}^4)$. It follows that $(D_{c,0}(\theta) - z_0)g = \gamma V_1(\theta)f = \gamma V(\theta)g$ in $H^{-1/2}(\mathbb{R}^3;\mathbb{C}^4)$. Rewriting this equality (in the sense of $H^{-1/2}(\mathbb{R}^3;\mathbb{C}^4)$) we find $-\mathrm{i}\,ce^{-\theta}\boldsymbol{\alpha}\cdot\nabla g - \beta c^2 g - \gamma V(\theta)g = z_0 g$. Since the r.h.s. of this equality is a (regular distribution generated by a) function in $L^2(\mathbb{R}^3;\mathbb{C}^4)$, the l.h.s. is. This implies that $g \in H^1(\mathbb{R}^3;\mathbb{C}^4) = \mathcal{D}(D_{c,\gamma}(\theta))$ by Remark 2, i.e. $z_0 \in \sigma(D_{c,\gamma}(\theta))$ which in turn proves $\sigma(D_{c,\gamma}(\theta)) \cap (\mathbb{C} \setminus \sigma(D_{c,0}(\theta))) = A_{c,\gamma;\theta}$.

Step 4: It remains to show that $\sigma(D_{c,\gamma}(\theta)) \cap \sigma(D_{c,0}(\theta)) = \sigma(D_{c,0}(\theta))$ holds. To show this, we pick $E \in \sigma(D_{c,0}(\theta))$ and $p \in \mathbb{R}^3$ with $E = E_c(p;\theta)$ in order to construct a suitable Weyl sequence. Let us define $\psi_{p,c;\theta} \in C^\infty(\mathbb{R}^3;\mathbb{C}^4)$ by

$$\psi_{p,c;\theta}(x) := N_c(p;\theta)^{-1}(c^2 + E_c(p;\theta)\xi, ce^{-\theta}\boldsymbol{\sigma}\cdot p\xi)^T e^{-ipx} \tag{27}$$

with $\xi = (1,0)^T$. Equations (7) and (23) imply

$$(-ic\boldsymbol{\alpha}\cdot\nabla + \beta c^2)\psi_{p,c;\theta}(x) = E_c(p;\theta)\psi_{p,c;\theta}(x). \tag{28}$$

We pick a function $0 \neq \phi \in C_0^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\phi \subset B_1(0)$ and set for $n \in \mathbb{N}$ $\phi_n(x) := \phi(\frac{1}{n}x - ne_1)$ with $e_1 = (1,0,0)^T$ as well as $f_n := \phi_n\psi_{p,c;\theta}$. Obviously, we have $f_n \in \mathrm{Dom}(D_{c,\gamma}(\theta))$. First, we calculate

$$\|f_n\| \geq (1 + C_{\mathrm{FW}})^{-1/2}\|\phi_n\| = n^{3/2}(1 + C_{\mathrm{FW}})^{-1/2}\|\phi\|, \tag{29}$$

where we used the definition (27) of $\psi_{p,c;\theta}$, Equation (7), Equation (11), Equation (8) and the identity $\int \mathrm{d}x\, \phi_n(x)^2 = \int \mathrm{d}x\, \phi(\frac{1}{n}x - ne_1) = n^3 \int \mathrm{d}x\, \phi(x)^2$. Furthermore, we find for $n \geq 2$

$$\begin{aligned}\|V_C f_n\|^2 =& \int \mathrm{d}x\, \frac{1}{|x|^2}\phi_n(x)^2 \|\psi_{p,c;\theta}(0)\|^2 \\ \leq& (1 + C_{\mathrm{FW}}|\mathrm{Im}\,\theta|)\frac{4}{n^4}\int \mathrm{d}x\, \phi_n(x)^2 \frac{4(1 + C_{\mathrm{FW}}|\mathrm{Im}\,\theta|)}{n^4} n^3\|\phi\|^2,\end{aligned} \tag{30}$$

since $\operatorname{supp}\phi_n \subset B_n(n^2e_1)$ and $\|\psi_{p,c;\theta}(0)\| \leq \sqrt{1 + C_{\mathrm{FW}}|\mathrm{Im}\,\theta|}$ because of Formula (9). Moreover, we obtain

$$\|(c\boldsymbol{\alpha}\cdot\nabla\phi_n)\psi_{p,c;\theta}(\cdot)\| \leq \frac{c\sqrt{1 + C_{\mathrm{FW}}|\mathrm{Im}\,\theta|}}{n} n^{3/2}\|\nabla\phi\|. \tag{31}$$

Formulas (28) through (31) imply

$$\frac{\|(D_{c,\gamma}(\theta) - E_c(p;\theta))f_n\|}{\|f_n\|} \leq \sqrt{1 + C_{\mathrm{FW}}|\mathrm{Im}\,\theta|}\frac{\frac{2n^{3/2}}{n^2}\|\phi\| + \frac{cn^{3/2}}{n}\|\nabla\phi\|}{\frac{n^{3/2}}{\sqrt{1+C_{\mathrm{FW}}}}\|\phi\|} \underset{n\to\infty}{\longrightarrow} 0.$$

Thus $D_{c,\gamma}(\theta) - E_c(p;\theta)$ does not have a bounded inverse and $E_c(p;\theta) \in \sigma(D_{c,\gamma}(\theta))$.

Step 5: The proof of $A_{c,\gamma;\theta} \cap (\mathbb{C} \setminus B_{c;\theta}) = \sigma_{\mathrm{disc}}(D_{c,\gamma}(0))$ is a standard argument, which uses the dilation analyticity of the operators $D_{c,\gamma}(\theta)$ (see [43, Chapter XII.6] or [46]). The same holds for the claim on the accumulation points. □

Remark 3. *Note that for $V = V_C$ the set of resonances is empty. This follows similarly as for the Schrödinger case (see [8]): If there was a resonance, then $D_{c,\gamma}(\pi)$ would have a non-real eigenvalue.*

5 Spectral Projections

In this section we extend the notion of positive and negative spectral projections to dilated Dirac operators. We define for $p \in \mathbb{R}^3$ the matrices $\Lambda_{c,0}^{(\pm)}(p;\theta) := \frac{1}{2}(\mathbf{1} \pm \frac{cp\cdot\boldsymbol{\alpha}+c^2\beta}{E_c(p;\theta)})$. A calculation shows that $\Lambda_{c,0}^{(\pm)}(p;\theta)^2 = \Lambda_{c,0}^{(\pm)}(p;\theta)$ and $\Lambda_{c,0}^{(\pm)}(p;\theta)D_{c,0}(p;\theta) = \pm E_c(p;\theta)\Lambda_{c,0}^{(\pm)}(p;\theta)$. Moreover, one verifies the identity $\Lambda_{c,0}^{(\pm)}(p;\theta) = \frac{1}{2} \pm \frac{1}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} \mathrm{d}\eta \, \frac{1}{D_{c,0}(p;\theta)-\mathrm{i}\,\eta}$. These observations motivate the following definition for the dilated interacting operators:

$$\Lambda_{c,\gamma}^{(\pm)}(\theta) := \frac{1}{2} \pm \frac{1}{2\pi} \operatorname*{s-lim}_{R\to\infty} \int_{-R}^{R} \mathrm{d}\eta \, \frac{1}{D_{c,\gamma}(\theta) - \mathrm{i}\,\eta} \tag{32}$$

It is well known [33, Chapter VI-5.2, Lemma 5.6] that Equation (32) yields the positive and negative spectral projections for real θ. Note that similar formulas for not necessarily self-adjoint operators are known (see [16, Chapter VX]). These authors use a different definition for the spectral projections, however. First, we show in Theorem 4 that these operators are well defined and bounded projections even if $\theta \notin \mathbb{R}$. We need the following technical lemma:

Lemma 3. *Let $\theta \in S_{\pi/4}$. Then for all $\eta \in \mathbb{R}$*

$$\left\| \frac{|D_{c,0}(\operatorname{Re}\theta)| - \mathrm{i}\,\eta}{D_{c,0}(\theta) - \mathrm{i}\,\eta} \right\| \le C_1(\operatorname{Im}\theta), \tag{33}$$

where $C_1(\operatorname{Im}\theta)$ is defined in (24).

Proof. We prove the estimate

$$\begin{aligned} &\left\| \frac{|D_{c,0}(\operatorname{Re}\theta)| - \mathrm{i}\,\eta}{\sqrt{-e^{-2\theta}c^2\Delta + c^4\beta} - \mathrm{i}\,\eta} \right\| \le \left\| \frac{|D_{c,0}(\operatorname{Re}\theta)|}{\sqrt{-e^{-2\theta}c^2\Delta + c^4\beta} - \mathrm{i}\,\eta} \right\| \\ &+ \left\| \frac{\eta}{\sqrt{-e^{-2\theta}c^2\Delta + c^4\beta} - \mathrm{i}\,\eta} \right\| \le \frac{1}{\sqrt{\cos(2\operatorname{Im}\theta)}} + \frac{1}{\cos \operatorname{Im}\theta}. \end{aligned} \tag{34}$$

We estimate the first summand using inequality (5) and Lemma 1. For the second summand we restrict ourselves to the case $\operatorname{Im}\theta < 0$. The proof for $\operatorname{Im}\theta > 0$ works analogously, and (33) holds obviously if $\operatorname{Im}\theta = 0$. Moreover, it suffices to consider $\operatorname{Re}\theta = 0$. We investigate the term $|\sqrt{e^{-2\theta}c^2p^2 + c^4} - \mathrm{i}\,\eta|$. For $\eta > 0$ the inequality $\operatorname{Im}\sqrt{e^{-2\theta}c^2p^2 + c^4} < 0$ yields $|-\sqrt{e^{-2\theta}c^2p^2 + c^4} + \mathrm{i}\,\eta| \ge |\eta|$. For $\eta < 0$ the inequality $\operatorname{Im}\sqrt{c^2p^2 + e^{+2\theta}c^4} > 0$ implies $|\sqrt{c^2p^2 + e^{+2\theta}c^4} - ie^{+\theta}\eta| \ge -\cos(\operatorname{Im}\theta)\eta = \cos(\operatorname{Im}\theta)|\eta|$, which proves (34). The claim follows using Theorem 1. ☐

Theorem 4. *Let $\theta \in S_{\min\{\pi/4,\Theta\}}$ and $\frac{2\gamma}{c}C(\operatorname{Im}\theta) < 1$. Suppose that (H1) holds. Then the following statements hold: $\Lambda_{c,\gamma}^{(\pm)}(\theta) \in \mathcal{B}(L^2(\mathbb{R}^3;\mathbb{C}^4))$, $\Lambda_{c,\gamma}^{(\pm)}(\theta) = \Lambda_{c,\gamma}^{(\pm)}(\theta)^2$ and $\Lambda_{c,\gamma}^{(+)}(\theta) + \Lambda_{c,\gamma}^{(-)}(\theta) = \mathbf{1}$. The operators $\Lambda_{c,\gamma}^{(\pm)}(\theta)$ are bounded holomorphic families in θ for $\theta \in M_{\gamma/c}$.*

Proof. The proof is inspired by similar estimates in [47].
Step 1: The resolvent equation (25) and the estimate (26) yield the convergence of the series

$$(D_{c,\gamma}(\theta) - \mathrm{i}\,\eta)^{-1} - (D_{c,0}(\theta) - \mathrm{i}\,\eta)^{-1} = \gamma \sum_{n=1}^{\infty} (D_{c,0}(\theta) - \mathrm{i}\,\eta)^{-1} V_1(\theta) \times [\gamma V_2(\theta)(D_{c,0}(\theta) - \mathrm{i}\,\eta)^{-1} V_1(\theta)]^{n-1} V_2(\theta)(D_{c,0}(\theta) - \mathrm{i}\,\eta)^{-1} \quad (35)$$

in norm.
Step 2: We show that the expression

$$\lim_{R\to\infty} \int_{-R}^{R} \mathrm{d}\eta \, \big(f, [\frac{1}{D_{c,\gamma}(\theta) - \mathrm{i}\,\eta} - \frac{1}{D_{c,0}(\theta) - \mathrm{i}\,\eta}] g\big), \quad f, g \in L^2(\mathbb{R}^3; \mathbb{C}^4) \quad (36)$$

defines a bounded operator on $L^2(\mathbb{R}^3; \mathbb{C}^4)$. In order to achieve this, we estimate

$$\begin{aligned} &\big|\big(f, \frac{1}{D_{c,0}(\theta) - \mathrm{i}\,\eta} V_1(\theta) \big[\gamma V_2(\theta) \frac{1}{D_{c,0}(\theta) - \mathrm{i}\,\eta} V_1(\theta)\big]^{n-1} V_2(\theta) \frac{1}{D_{c,0}(\theta) - \mathrm{i}\,\eta} g\big)\big| \\ &\leq \frac{\pi}{2} \big\| \frac{|\nabla|^{1/2}}{D_{c,0}(\bar{\theta}) + \mathrm{i}\,\eta} f \big\| \big\| \frac{|\nabla|^{1/2}}{D_{c,0}(\theta) - \mathrm{i}\,\eta} g \big\| \big(\frac{\gamma}{c} \frac{\pi}{2} C(\operatorname{Im}\theta) \big)^{n-1} \leq \frac{\pi}{2ce^{-\operatorname{Re}\theta}} \\ &\times \big\| \frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{|D_{c,0}(\operatorname{Re}\theta)| + \mathrm{i}\,\eta} f \big\| \big\| \frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{|D_{c,0}(\operatorname{Re}\theta)| - \mathrm{i}\,\eta} g \big\| C_1(\operatorname{Im}\theta)^2 \big(\frac{\gamma}{c} \frac{\pi}{2} C(\operatorname{Im}\theta) \big)^{n-1}, \end{aligned}$$

where we used (26) in the first estimate and Lemma 3 in the second estimate. $C(\operatorname{Im}\theta)$ and $C_1(\operatorname{Im}\theta)$ were defined in (24). As in [47, Proof of Lemma 1] we obtain $\int_{-\infty}^{\infty} \mathrm{d}\eta \, \big\| \frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{|D_{c,0}(\operatorname{Re}\theta)| + \mathrm{i}\,\eta} f \big\| \big\| \frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{|D_{c,0}(\operatorname{Re}\theta)| - \mathrm{i}\,\eta} g \big\| \leq \pi \|f\| \|g\|$ and thus

$$\int_{-\infty}^{\infty} \mathrm{d}\eta \, \big|\big(f, [\frac{1}{D_{c,\gamma}(\theta) - \mathrm{i}\,\eta} - \frac{1}{D_{c,0}(\theta) - \mathrm{i}\,\eta}] g\big)\big| \leq \leq \pi \frac{\gamma}{c} \frac{\pi}{2} \|f\| \|g\| C_1(\operatorname{Im}\theta)^2 \frac{1}{1 - \big(\frac{\gamma}{c} \frac{\pi}{2} C(\operatorname{Im}\theta)\big)} \quad (37)$$

Step 3: The expressions

$$\Big(f, \frac{1}{D_{c,0}(\theta) - \mathrm{i}\,\eta} V_1(\theta) \big[\gamma V_2(\theta) \frac{1}{D_{c,0}(\theta) - \mathrm{i}\,\eta} V_1(\theta)\big]^{n-1} V_2(\theta) \frac{1}{D_{c,0}(\theta) - \mathrm{i}\,\eta} g\Big)$$

are holomorphic functions of $\theta \in S_{\min\{\pi/4, \Theta\}}$. These estimates show the existence of an integrable and summable majorant, independent of θ for $\theta \in M_{\gamma/c}$. Thus, the operator in Equation (36) is a holomorphic function of θ [33, Chapter VII-1.1], and the identity $\Lambda_{c,\gamma}^{(+)}(\theta) = \Lambda_{c,\gamma}^{(+)}(\theta)^2$, which is obviously true for $\theta \in \mathbb{R}$, extends to $\theta \in M_{\gamma/c}$, i.e. $\Lambda_{c,\gamma}^{(+)}(\theta)$ is a projection.

Step 4: We show that the limit exists as a strong limit and estimate for $g \in H^{1/2}(\mathbb{R}^3;\mathbb{C}^4)$ as follows:

$$\begin{aligned}&\big|\big(f, \frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big[\gamma V(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big]^{n-1}V(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}g\big)\big|\\ &\quad\le \frac{2}{ce^{-\operatorname{Re}\theta}}\Big\|\frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{|D_{c,0}(\operatorname{Re}\theta)|+\mathrm{i}\,\eta}\Big\|\|f\|\Big\|\frac{1}{|D_{c,0}(\operatorname{Re}\theta)|-\mathrm{i}\,\eta}\Big\|\big\||D_{c,0}(\operatorname{Re}\theta)|^{1/2}g\big\|\\ &\qquad\qquad\times C_1(\operatorname{Im}\theta)^2\big(\frac{2\gamma}{c}C(\operatorname{Im}\theta)\big)^{n-1}\end{aligned}$$

Here we estimated the expression in the square brackets similarly to (26), but used Hardy's inequality instead of Kato's inequality. Moreover, we used the estimate (33) twice. Since $\sigma(D_{c,0}(\operatorname{Re}\theta)) = (-\infty, c^2] \cup [c^2, \infty)$, we have

$$\Big\|\frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{|D_{c,0}(\operatorname{Re}\theta)|+\mathrm{i}\,\eta}\Big\| = \sup_{|\lambda|\ge c^2}\frac{\sqrt{|\lambda|}}{\sqrt{\lambda^2+\eta^2}} \le \min\{\frac{1}{c},\frac{1}{\sqrt{|\eta|}}\}.$$

This estimate shows that the convergence in formula (36) is uniform in $f \in L^2(\mathbb{R}^3;\mathbb{C}^4)$, which implies the strong convergence [33, Theorem III.1.32 and Lemma III.3.5], since $H^{1/2}(\mathbb{R}^3;\mathbb{C}^4)$ is dense in $L^2(\mathbb{R}^3;\mathbb{C}^4)$. □

Obviously, the identity $\Lambda^{(+)}_{c,\gamma}(\theta) + \Lambda^{(-)}_{c,\gamma}(\theta) = \mathbf{1}$ holds. We set $\mathcal{H}^{(\pm)}_{c,\gamma}(\theta) := \Lambda^{(\pm)}_{c,\gamma}(\theta)L^2(\mathbb{R}^3;\mathbb{C}^4)$ and find $L^2(\mathbb{R}^3;\mathbb{C}^4) = \mathcal{H}^{(+)}_{c,\gamma}(\theta) \dotplus \mathcal{H}^{(-)}_{c,\gamma}(\theta)$, wehre $\dotplus$ denotes the direct sum. We call the $\Lambda^{(\pm)}_{c,\gamma}(\theta)$ positive and negative spectral projections and $\mathcal{H}^{(\pm)}_{c,\gamma}(\theta)$ positive and negative spectral subspaces, respectively. This is justified because of Theorem 5.
The following corollary generalizes [47, Lemma 1] to dilated spectral projections.

COROLLARY 1. *Let $\theta \in S_{\min\{\pi/4,\Theta\}}$ and suppose that (H1) holds. Then there exists a constant $C_{\mathrm{NR}} > 0$ such that for $\frac{2\gamma}{c}C(\operatorname{Im}\theta) < 1$ the estimate*

$$\|\Lambda^{(\pm)}_{c,\gamma}(\theta) - \Lambda^{(\pm)}_{c,0}(\theta)\| \le C_{\mathrm{NR}}\frac{\gamma}{c}$$

holds.

Proof. This follows directly from Equation (37) in the proof of Theorem 4. □

The next theorem shows that the spaces $\mathcal{H}^{(\pm)}_{c,\gamma}(\theta)$ are invariant under $D_{c,\gamma}(\theta)$ and describes the spectrum of the restriction of the operator to these spaces. If a part of the spectrum is contained in a Jordan curve, analogous statements can be found in [33, Theorem III-6.17]. The following theorem describes a more general situation, but the essential elements of the proof of [33, Theorem III-6.17] can be adapted.
For a closed operator A we denote its resolvent set by $\rho(A)$.

Theorem 5. *Let* $\theta \in S_{\min\{\pi/4,\Theta\}}$ *and* $\frac{2\gamma}{c}C(\operatorname{Im}\theta) < 1$. *Suppose that (H1) holds. Then the identity*

$$\Lambda_{c,\gamma}^{(\pm)}(\theta)(D_{c,\gamma}(\theta) - z)^{-1} = (D_{c,\gamma}(\theta) - z)^{-1}\Lambda_{c,\gamma}^{(\pm)}(\theta) \tag{38}$$

holds for all $z \in \rho(D_{c,\gamma}(\theta))$. *The subspaces* $\operatorname{Ran}\Lambda_{c,\gamma}^{(+)}(\theta)$ *and* $\operatorname{Ran}\Lambda_{c,\gamma}^{(-)}(\theta)$ *are invariant subspaces for* $D_{c,\gamma}(\theta)$. *In particular,*

$$\sigma(D_{c,\gamma}(\theta)|_{\operatorname{Ran}\Lambda_{c,\gamma}^{(+)}(\theta)}) = \sigma(D_{c,\gamma}(\theta)) \cap \{z \in \mathbb{C} | \operatorname{Re} z > 0\} \tag{39}$$

and

$$\sigma(D_{c,\gamma}(\theta)|_{\operatorname{Ran}\Lambda_{c,\gamma}^{(-)}(\theta)}) = \sigma(D_{c,\gamma}(\theta)) \cap \{z \in \mathbb{C} | \operatorname{Re} z < 0\} \tag{40}$$

hold.

Proof. Obviously, for all $z \notin \sigma(D_{c,\gamma}(\theta))$, all $\eta \in \mathbb{R}$ and all $f \in L^2(\mathbb{R}^3;\mathbb{C}^4)$ the equation $(D_{c,\gamma}(\theta)-z)^{-1}(D_{c,\gamma}(\theta)-\mathrm{i}\,\eta)^{-1}f = (D_{c,\gamma}(\theta)-\mathrm{i}\,\eta)^{-1}(D_{c,\gamma}(\theta)-z)^{-1}f$ is true. This immediately implies

$$\begin{aligned}(D_{c,\gamma}(\theta) - z)^{-1} \lim_{R\to\infty}\int_{-R}^{R} \mathrm{d}\eta\ (D_{c,\gamma}(\theta) - \mathrm{i}\,\eta)^{-1} f =& \\ = \lim_{R\to\infty}\int_{-R}^{R} \mathrm{d}\eta\ (D_{c,\gamma}(\theta) - \mathrm{i}\,\eta)^{-1} (D_{c,\gamma}(\theta) - z)^{-1} f&\end{aligned}$$

and thus (38). It follows that [33, Chapter III-5.6 and Theorem III.6.5] $(D_{c,\gamma}(\theta) - z)^{-1}\operatorname{Ran}\Lambda_{c,\gamma}^{(\pm)}(\theta) \subset \operatorname{Ran}\Lambda_{c,\gamma}^{(\pm)}(\theta)$ and $\Lambda_{c,\gamma}^{(\pm)}(\theta)\operatorname{Dom}(D_{c,\gamma}(\theta)) \subset \operatorname{Dom}(D_{c,\gamma}(\theta))$ as well as $D_{c,\gamma}(\theta)\mathcal{H}_{c,\gamma}^{(\pm)}(\theta) \subset \mathcal{H}_{c,\gamma}^{(\pm)}(\theta)$. We define the operators $D_{c,\gamma}^{(\pm)}(\theta) := D_{c,\gamma}(\theta)|_{\mathcal{H}_{c,\gamma}^{(\pm)}(\theta)}$ and (for $z \notin \sigma(D_{c,\gamma}(\theta))$ at the moment) the resolvents $R_{c,\gamma;\theta}^{(\pm)}(z) := (D_{c,\gamma}^{(\pm)}(\theta) - z)^{-1} = (D_{c,\gamma}(\theta) - z)^{-1}|_{\mathcal{H}_{c,\gamma}^{(\pm)}(\theta)}$. In particular, $\sigma(D_{c,\gamma}^{(\pm)}(\theta)) \subset \sigma(D_{c,\gamma}(\theta))$.

On the other side, we have $f \in \mathcal{H}_{c,\gamma}^{(\pm)}(\theta)$ and $z \notin \sigma(D_{c,\gamma}(\theta))$ $R_{c,\gamma;\theta}^{(\pm)}(z)f = (D_{c,\gamma}(\theta) - z)^{-1}f = (D_{c,\gamma}(\theta) - z)^{-1}\Lambda_{c,\gamma}^{(\pm)}(\theta)f$. Using the first resolvent identity, we find for $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$ respectively $\operatorname{Re} z > 0$

$$(D_{c,\gamma}(\theta) - z)^{-1}\Lambda_{c,\gamma}^{(\pm)}(\theta)f = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\eta\frac{1}{z-\mathrm{i}\,\eta}(D_{c,\gamma}(\theta)-\mathrm{i}\,\eta)^{-1}f, \tag{41}$$

since for $z \in \mathbb{C}$ with $\operatorname{Re} z < 0$ respectively $\operatorname{Re} z > 0$ the residue theorem implies $\lim_{R\to\infty}\int_{-R}^{R}\mathrm{d}\eta\,\frac{1}{z-\mathrm{i}\,\eta} = \lim_{R\to\infty}\int_{-R}^{R}\mathrm{d}\eta\,\frac{z}{z^2+\eta^2} = \mp\pi$.

The r.h.s. of equation (41) is holomorphic in $z \notin \mathrm{i}\,\mathbb{R}$. Thus, $R_{c,\gamma;\theta}^{(+)}(z)$ has a holomorphic continuation to $\{z \in \mathbb{C}|\operatorname{Re} z < 0\}$, and $R_{c,\gamma;\theta}^{(-)}(z)$ has a holomorphic continuation to $\{z \in \mathbb{C}|\operatorname{Re} z > 0\}$. The holomorphicity of the resolvent implies $\{z \in \mathbb{C}|\operatorname{Re} z < 0\} \subset \rho(D_{c,\gamma}^{(+)}(\theta))$ and $\{z \in \mathbb{C}|\operatorname{Re} z >$

$0\} \subset \rho(D^{(-)}_{c,\gamma}(\theta))$. This proves $\sigma(D^{(-)}_{c,\gamma}(\theta)) \subset \{z \in \mathbb{C} | \mathrm{Re}\, z < 0\}$ and $\sigma(D^{(+)}_{c,\gamma}(\theta)) \subset \{z \in \mathbb{C} | \mathrm{Re}\, z > 0\}$. On the other side, $z \in \sigma(D_{c,\gamma}(\theta))$ cannot fulfill both $z \in \rho(D^{(-)}_{c,\gamma}(\theta))$ and $z \in \rho(D^{(+)}_{c,\gamma}(\theta))$, because otherwise the identity $(D_{c,\gamma}(\theta) - z)^{-1} = (D^{(+)}_{c,\gamma}(\theta) - z)^{-1} \Lambda^{(+)}_{c,\gamma}(\theta) + (D^{(-)}_{c,\gamma}(\theta) - z)^{-1} \Lambda^{(-)}_{c,\gamma}(\theta)$ would imply the contradiction $z \in \rho(D_{c,\gamma}(\theta))$. This shows (39) and (40). □

Next, we need spectral projections for the eigenvalues: We define for all $n \geq 1$ (and $n \leq N_{\max}$ if there only finitely many eigenvalues) the spectral projections

$$P_n(c,\gamma;\theta) := -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_n(c,\gamma)} \frac{1}{D_{c,\gamma}(\theta) - z} \mathrm{d}z\,, \tag{42}$$

where z runs through $\Gamma_n(c,\gamma)$ in the positive sense. $\Gamma_n(c,\gamma)$ is chosen such that for all $1 \leq l \leq N_n$ the eigenvalues $\tilde{E}_{n,l}(c,\gamma)$ are located within the contour, but no other elements of the spectrum $D_{c,\gamma}(\theta)$.

For later, we need spectral projections for the fine structure components. We set for $n \geq 1$ and $1 \leq l \leq N_n$

$$P_{n,l}(c,\gamma;\theta) := -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{n,l}(c,\gamma)} \frac{1}{D_{c,\gamma}(\theta) - z} \mathrm{d}z\,, \tag{43}$$

where z runs through $\Gamma_{n,l}(c,\gamma)$ in the positive sense, and $\Gamma_{n,l}(c,\gamma)$ is chosen such that only the eigenvalue $\tilde{E}_{n,l}(c,\gamma)$ lies within the contour. We denote the corresponding normed eigenfunctions by $\phi_{n,l}(c,\gamma;\theta)$.

6 Transformation Functions

We need transformation functions between the spectral subspaces of dilated and not dilated operators for the resolvent estimate in Section 7 and in order to establish the dilation analyticity of a relativistic Pauli-Fierz model in [30]. Another example for a transformation function is the Douglas-Kroll transformation, which was investigated by Siedentop and Stockmeyer [47] (see also Huber and Stockmeyer [31]). Contrary to the situation there, our spectral projections are not self-adjoint and thus the transformation function is a non-unitary similarity transformation. The estimates in this section can be used to generalize the Douglas-Kroll transformation to complex dilated operators.

In order to prove the existence of the transformation function, we need norm estimates on the difference between the spectral projections.

LEMMA 4. *Let $\theta \in S_{\min\{\pi/4,\Theta\}}$. Suppose that (H1) and (H2) hold. Then the following statements hold:*

a) There is a constant $C_{\mathrm{DL}} > 0$ (independent of c, γ and θ) such that for $\frac{2\gamma}{c} C(\mathrm{Im}\,\theta) < 1$ the estimate

$$\|\Lambda^{(\pm)}_{c,\gamma}(0) - \Lambda^{(\pm)}_{c,\gamma}(\theta)\| \leq C_{\mathrm{DL}} |\theta| \tag{44}$$

holds. The operator $|D_{c,0}(0)|^{1/2}[\Lambda^{(\pm)}_{c,\gamma}(0)-\Lambda^{(\pm)}_{c,\gamma}(\theta)]|D_{c,0}(0)|^{-1/2}$ *is a holomorphic function of* $\theta \in M_{\gamma/c}$.

b) *Let moreover* $0 < q < 1$. *Then there is a constant* $C_{\rm DLS} > 0$ *(independent of* c, γ *and* θ*) such that for* $\frac{2\gamma}{c}C(\operatorname{Im}\theta) < q$ *the estimate*

$$\||D_{c,0}(0)|^{1/2}[\Lambda^{(\pm)}_{c,\gamma}(0) - \Lambda^{(\pm)}_{c,\gamma}(\theta)]|D_{c,0}(0)|^{-1/2}\| \leq C_{\rm DLS}|\theta| \tag{45}$$

holds.

Proof. We adapt method which was used by Siedentop and Stockmeyer [47] and by Griesemer, Lewis and Siedentop [19] for other choices of projections. We start with the difference of resolvents

$$\begin{aligned}(D_{c,0}(\theta) - \mathrm{i}\,\eta)^{-1} &- (D_{c,0}(0) - \mathrm{i}\,\eta)^{-1}\\ &= ic[e^{-\theta} - 1](D_{c,0}(\theta) - \mathrm{i}\,\eta)^{-1}\boldsymbol{\alpha}\cdot\nabla(D_{c,0}(0) - \mathrm{i}\,\eta)^{-1}\end{aligned} \tag{46}$$

and note that $|e^{-\theta} - 1| \leq B|\theta|$ holds with $B = e^{\pi/4}$ for all $|\theta| \leq \pi/4$.
Step 1: Proof for the free projections. Equation (46) it and Lemma 3 imply that

$$\begin{aligned}&|(f, [(D_{c,0}(\theta) - \mathrm{i}\,\eta)^{-1} - (D_{c,0}(0) - \mathrm{i}\,\eta)^{-1}]g)|\\ \leq& B|\theta|\||D_{c,0}(\operatorname{Re}\theta)|^{1/2}(|D_{c,0}(\operatorname{Re}\theta)| + \mathrm{i}\,\eta)^{-1}f\|\||D_{c,0}(0)|^{1/2}(D_{c,0}(0) - \mathrm{i}\,\eta)^{-1}g\|\\ &\times \||D_{c,0}(\operatorname{Re}\theta)|^{-1/2}c\boldsymbol{\alpha}\cdot\nabla|D_{c,0}(0)|^{-1/2}\|\|\frac{|D_{c,0}(\operatorname{Re}\theta)| - \mathrm{i}\,\eta}{D_{c,0}(\theta) - \mathrm{i}\,\eta}\|\\ \leq& \frac{B|\theta|}{e^{-\operatorname{Re}\theta/2}}C_1(\operatorname{Im}\theta)\|\frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{|D_{c,0}(\operatorname{Re}\theta)| + \mathrm{i}\,\eta}f\|\|\frac{|D_{c,0}(0)|^{1/2}}{D_{c,0}(0) - \mathrm{i}\,\eta}g\|,\end{aligned}$$

where we used the estimate $\|c|\nabla||D_{c,0}(\operatorname{Re}\theta)|^{-1}\| \leq 1/e^{-\operatorname{Re}\theta}$.
This proves (cf. [47, Proof of Lemma 1] and proof of Corollary 1) $\|\Lambda^{(\pm)}_{c,0}(0) - \Lambda^{(\pm)}_{c,0}(\theta)\| \leq \tilde{C}_{\rm DL}|\theta|$ with a $\tilde{C}_{\rm DL} > 0$ and analogously $\||D_{c,0}(0)|^{1/2}[\Lambda^{(\pm)}_{c,0}(0) - \Lambda^{(\pm)}_{c,0}(\theta)]|D_{c,0}(0)|^{-1/2}\| \leq \tilde{C}_{\rm DL}|\theta|$, since $|D_{c,0}(0)|^{1/2}$ commutes with all operators in (46).
Step 2: Proof of (44). We write

$$\begin{aligned}&\big\|\big[V_2(\theta)\frac{1}{D_{c,0}(\theta) - \mathrm{i}\,\eta}V_1(\theta)\big] - \big[V_2(0)\frac{1}{D_{c,0}(0) - \mathrm{i}\,\eta}V_1(0)\big]\big\| \qquad (47)\\ \leq& \big\|\big[V_1(\theta)\frac{e^{-\theta}}{D_{c,0}(\theta) - \mathrm{i}\,\eta}\chi_\theta V_2(\theta)\big] - \big[V_1(\theta)\frac{e^{-\theta}}{D_{c,0}(0) - \mathrm{i}\,\eta}\chi_\theta V_2(\theta)\big]\big\|\\ &+ \big\|\big[V_C^{1/2}\frac{1}{D_{c,0}(0) - \mathrm{i}\,\eta}(\chi_\theta e^{-\theta} - 1)V_C^{1/2}\big]\big\| \leq \frac{B|\theta|\pi}{2c}(C(\operatorname{Im}\theta) + 1 + \tilde{C}),\end{aligned}$$

where we estimated the second summand by $B(1+\tilde{C})|\theta|\pi/(2c)$ from above, and the second summand – similarly as in (26) – according to

$$\begin{aligned}&\big\|(e^{-\theta}-1)\big[V_2(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}c\boldsymbol{\alpha}\cdot\nabla\frac{1}{D_{c,0}(0)-\mathrm{i}\,\eta}V_1(\theta)\big]\big\|\leq\\&\qquad\leq\frac{B|\theta|\pi}{2c}\big\|\frac{|D_{c,0}(\operatorname{Re}\theta)|}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big\|\big\|\frac{|D_{c,0}(0)|}{D_{c,0}(0)-\mathrm{i}\,\eta}\big\|\leq\frac{B|\theta|\pi}{2c}C(\operatorname{Im}\theta).\end{aligned}$$

In the same way we obtain

$$\begin{aligned}&\big\|\big[e^{-\theta/2}V_C^{1/2}(D_{c,0}(\theta)-\mathrm{i}\,\eta)^{-1}-V_C^{1/2}(D_{c,0}(0)-\mathrm{i}\,\eta)^{-1}\big]g\big\| \\&\leq|e^{-\theta/2}-e^{\theta/2}|\big\|e^{-\theta}V_C^{1/2}(D_{c,0}(\theta)-\mathrm{i}\,\eta)^{-1}c\boldsymbol{\alpha}\cdot\nabla(D_{c,0}(0)-\mathrm{i}\,\eta)^{-1}g\big\|\\&+|e^{-\theta/2}-1|\big\|V_C^{1/2}\frac{1}{D_{c,0}(0)-\mathrm{i}\,\eta}g\big\|\leq B|\theta|\sqrt{\frac{\pi}{2c}}\big(C(\operatorname{Im}\theta)+1/2\big)\big\|\frac{|D_{c,0}(0)|^{1/2}}{D_{c,0}(0)-\mathrm{i}\,\eta}g\big\|.\end{aligned}\tag{48}$$

Lemma 3 implies

$$\big\|V_C^{1/2}\frac{e^{-\theta/2}}{D_{c,0}(\theta)-\mathrm{i}\,\eta}g\big\|\leq C_1(\operatorname{Im}\theta)\sqrt{\frac{\pi}{2c}}\big\|\frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{|D_{c,0}(\operatorname{Re}\theta)|-\mathrm{i}\,\eta}g\big\|\tag{49}$$

and (see Formula (26))

$$\big\|V_C^{1/2}e^{-\theta}(D_{c,0}(\theta)-\mathrm{i}\,\eta)^{-1}V_C^{1/2}\big\|\leq\frac{\pi C(\operatorname{Im}\theta)}{2c}.\tag{50}$$

Formulas (47) through (50) show

$$\begin{aligned}&\big|\gamma^n\big(f,\frac{e^{-\theta/2}}{D_{c,0}(\theta)-\mathrm{i}\,\eta}V_1(\theta)\big[V_2(\theta)\frac{e^{-\theta}}{D_{c,0}(\theta)-\mathrm{i}\,\eta}V_1(\theta)\big]^{n-1}V_2(\theta)\frac{e^{-\theta/2}}{D_{c,0}(\theta)-\mathrm{i}\,\eta}g\big)-\\&-\gamma^n\big(f,\frac{1}{D_{c,0}(0)-\mathrm{i}\,\eta}V_1(0)\big[V_2(0)\frac{1}{D_{c,0}(0)-\mathrm{i}\,\eta}V_1(0)\big]^{n-1}V_2(0)\frac{1}{D_{c,0}(0)-\mathrm{i}\,\eta}g\big)\big|\\&\qquad\leq B|\theta|\big(\frac{\pi\gamma C(\operatorname{Im}\theta)}{2c}\big)^{n-1}\big(\frac{\pi\gamma C_1(\operatorname{Im}\theta)}{2c}\big)\big(C(\operatorname{Im}\theta)+1+\tilde{C}\big)\\&\qquad\times\big\|\frac{|D_{c,0}(0)|^{1/2}}{|D_{c,0}(0)|-\mathrm{i}\,\eta}f\big\|\big[n\big\|\frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{|D_{c,0}(\operatorname{Re}\theta)|+\mathrm{i}\,\eta}g\big\|+\big\|\frac{|D_{c,0}(0)|^{1/2}}{D_{c,0}(0)+\mathrm{i}\,\eta}g\big\|\big],\end{aligned}$$

which implies (44).
Step 3: Proof of (45). We use the expansion

$$\begin{aligned}&(D_{c,\gamma}(\theta)-\mathrm{i}\,\eta)^{-1}-(D_{c,0}(\theta)-\mathrm{i}\,\eta)^{-1}\\&\qquad=\sum_{n=1}^{\infty}\gamma^n\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big[V(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big]^{n-1}V(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\end{aligned}\tag{51}$$

and start with the necessary estimates on the differences of the resolvents: Using Hardy's inequality, we obtain as in (26)

$$\begin{aligned}&\|[V(\theta)(D_{c,0}(\theta)-\mathrm{i}\,\eta)^{-1}-V(0)(D_{c,0}(0)-\mathrm{i}\,\eta)^{-1}]|D_{c,0}(0)|^{-1/2}g\|\\&\le\frac{2B|\theta|}{c}(C(\operatorname{Im}\theta)+1+\tilde{C})\big\|\frac{|D_{c,0}(0)|^{1/2}}{D_{c,0}(0)-\mathrm{i}\,\eta}g\big\|,\end{aligned}\tag{52}$$

and we find analogously

$$\big\|V_C\big[\chi_\theta\frac{e^{-\theta}}{D_{c,0}(\theta)-\mathrm{i}\,\eta}-\chi\frac{1}{D_{c,0}(0)-\mathrm{i}\,\eta}\big]\big\|\le\frac{2B|\theta|}{c}(C(\operatorname{Im}\theta)+1+\tilde{C})\tag{53}$$

as well as

$$\begin{aligned}&\|[(D_{c,0}(\bar\theta)+\mathrm{i}\,\eta)^{-1}-(D_{c,0}(0)+\mathrm{i}\,\eta)^{-1}]|D_{c,0}(0)|^{1/2}f\|\\&\le\|[e^{\bar\theta}-1]\frac{e^{-\bar\theta}}{D_{c,0}(\bar\theta)+\mathrm{i}\,\eta}c\boldsymbol{\alpha}\cdot\nabla\frac{|D_{c,0}(0)|^{1/2}}{D_{c,0}(0)+\mathrm{i}\,\eta}f\|\le2B|\theta|C(\operatorname{Im}\theta)\|\frac{|D_{c,0}(0)|^{1/2}}{D_{c,0}(0)+\mathrm{i}\,\eta}f\|.\end{aligned}\tag{54}$$

For the terms with the resolvents we use Lemma 3 and Lemma 1 to estimate

$$\begin{aligned}&\big\|V(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}|D_{c,0}(0)|^{-1/2}g\big\|\le\frac{2}{c}\big\|\frac{|D_{c,0}(\operatorname{Re}\theta)|}{D_{c,0}(\theta)-\mathrm{i}\,\eta}|D_{c,0}(0)|^{-1/2}g\big\|\\&\le\frac{2C_1(\operatorname{Im}\theta)e^{\pi/8}}{c}\big\|\frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{|D_{c,0}(\operatorname{Re}\theta)|-\mathrm{i}\,\eta}g\big\|,\end{aligned}\tag{55}$$

and (cf. Formula (26))

$$\big\|V(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big\|\le\frac{2}{c}\big\||D_{c,0}(\operatorname{Re}\theta)|\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big\|\le\frac{2C(\operatorname{Im}\theta)}{c}.\tag{56}$$

Formulas (52) through (56) show

$$\begin{aligned}&\gamma^n\big|\big(f,|D_{c,0}(0)|^{1/2}\Big[\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big[V(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big]^{n-1}V(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\\&-\frac{1}{D_{c,0}(0)-\mathrm{i}\,\eta}\big[V\frac{1}{D_{c,0}(0)-\mathrm{i}\,\eta}\big]^{n-1}V\frac{1}{D_{c,0}(0)-\mathrm{i}\,\eta}\Big]|D_{c,0}(0)|^{-1/2}g\big)\big|\\&\le e^{\pi/4}B|\theta|\big(\frac{2\gamma C(\operatorname{Im}\theta)}{c}\big)^{n-1}\big(\frac{2\gamma C_1(\operatorname{Im}\theta)}{c}\big)(C(\operatorname{Im}\theta)+1+\tilde{C})\big\|\frac{|D_{c,0}(0)|^{1/2}}{D_{c,0}(0)+\mathrm{i}\,\eta}f\big\|\\&\times\big[n\big\|\frac{|D_{c,0}(\operatorname{Re}\theta)|^{1/2}}{D_{c,0}(\operatorname{Re}\theta)-\mathrm{i}\,\eta}g\big\|+\big\|\frac{|D_{c,0}(0)|^{1/2}}{D_{c,0}(0)-\mathrm{i}\,\eta}g\big\|\big],\end{aligned}$$

which in turn proves (45).
Step 4: Holomorphicity. This follows as in the proof of Theorem 4, since $\big(f,|D_{c,0}(0)|^{1/2}\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big[V(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}\big]^{n-1}V(\theta)\frac{1}{D_{c,0}(\theta)-\mathrm{i}\,\eta}|D_{c,0}(0)|^{-1/2}g\big)$ are holomorphic functions of θ and the above estimates imply the existence of summable and integrable majorant which does not depend on θ. ☐

Before we turn to the existence of a transformation function in Theorem 6, we need two operator inequalities, one of which was proven in [19]. Since the other inequality can be proven completely analogously, we omit the proof. Let us mention that there exits an improved version of one of these inequalities (see [39]). But since we will be interested in the non-relativistic limit only, it is sufficient to use the original version.

LEMMA 5 ([19], Lemma 2). *Suppose that $\vartheta \in \mathbb{R}$ and $\frac{\gamma}{c} < \frac{1}{2}$. Then the operator inequalities*

$$(1 - \frac{2\gamma}{c})|D_{c,0}(\vartheta)\| \leq |D_{c,\gamma}(\vartheta)| \leq (1 + \frac{2\gamma}{c})|D_{c,0}(\vartheta)|$$

hold.

Now we can turn to the transformation function $\mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta)$ defined below. It enables us to consider the operator $\mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta) D^{(\pm)}_{c,\gamma}(\theta) \mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta)^{-1}$ instead of the operator $D^{(\pm)}_{c,\gamma}(\theta)$. This is necessary for technical reasons, since the latter operates on a fixed space (i.e. $\operatorname{Ran} \Lambda^{(+)}_{c,\gamma}(0)$). We will prove in [30] that this operator defines a holomorphic family of operators. Moreover, we will need the transformation function in the proof of the resolvent estimate in Theorem 7.

THEOREM 6. *Suppose that $\theta \in S_{\min\{\pi/4,\Theta\}}$, $\frac{2\gamma}{c} C(\operatorname{Im} \theta) < 1$ and $C_{\mathrm{DL}}|\theta| < q$ for some $0 < q < 1$. Suppose moreover that (H1) and (H2) hold. Then the following statements hold:*

a) There is a bounded mapping $\mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta) : L^2(\mathbb{R}^3; \mathbb{C}^4) \to L^2(\mathbb{R}^3; \mathbb{C}^4)$ with the property

$$\mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta) \Lambda^{(+)}_{c,\gamma}(\theta) \mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta)^{-1} = \Lambda^{(+)}_{c,\gamma}(0) \tag{57}$$

and bounded inverse $\mathcal{V}_{\mathrm{DL}}(c, \gamma; \theta) := \mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta)^{-1}$. There is a constant $C_{UDL} > 0$, independent of c, γ and θ, such that

$$\|\mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta) - \mathbf{1}\| \leq C_{UDL}|\theta| \tag{58}$$

holds.

b) Suppose that additionally $C_{\mathrm{DLS}}|\theta| < q$ holds. Then there is a constant C_{UDLS}, independent of c, γ and θ, such that

$$\||D_{c,0}(0)|^{1/2} \mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta) |D_{c,0}(0)|^{-1/2} - \mathbf{1}\| \leq C_{UDLS}|\theta| \tag{59}$$

is true. The same estimates hold for $\mathcal{V}_{\mathrm{DL}}(c, \gamma; \theta)$.

c) The operator $\mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta)$, and for $C_{\mathrm{DLS}}|\theta| < q$ the operator

$$|D_{c,0}(0)|^{1/2} \mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta) |D_{c,0}(0)|^{-1/2}$$

and the operator $|D_{c,0}(0)|^{-1/2} \mathcal{U}_{\mathrm{DL}}(c, \gamma; \theta) |D_{c,0}(0)|^{1/2}$, are holomorphic functions of θ. The same statements hold for $\mathcal{V}_{\mathrm{DL}}(c, \gamma; \theta)$.

Proof. We follow [47, Theorem 1] and [33, Chapter I-4.6.] and define

$$\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta) := [\Lambda^{(+)}_{c,\gamma}(0)\Lambda^{(+)}_{c,\gamma}(\theta)+\Lambda^{(-)}_{c,\gamma}(0)\Lambda^{(-)}_{c,\gamma}(\theta)][\mathbf{1}-(\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0))^2]^{-1/2}. \tag{60}$$

It is easy to see that $(\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0))^2$ commutes with $\Lambda^{(+)}_{c,\gamma}(\theta)$ and $\Lambda^{(+)}_{c,\gamma}(0)$ and that $\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)$ is invertible with inverse $\mathcal{V}_{\mathrm{DL}}(c,\gamma;\theta) := [\Lambda^{(+)}_{c,\gamma}(\theta)\Lambda^{(+)}_{c,\gamma}(0)+$ $+\Lambda^{(-)}_{c,\gamma}(\theta)\Lambda^{(-)}_{c,\gamma}(0)t][\mathbf{1}-(\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0))^2]^{-1/2}$, and that Equation (57) holds. Lemma 4 implies that $\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)$ is a holomorphic function θ, since $(1-A)^{-1/2}$ has a norm convergent series expansion for bounded operators A with $\|A\|<1$.
Proof of (58)*:* We follow [47, Proof of Lemma 5]. We have $\Lambda^{(+)}_{c,\gamma}(0)\Lambda^{(+)}_{c,\gamma}(\theta)+$ $\Lambda^{(-)}_{c,\gamma}(0)\Lambda^{(-)}_{c,\gamma}(\theta)=\mathbf{1}-\left[\Lambda^{(-)}_{c,\gamma}(0)-\Lambda^{(+)}_{c,\gamma}(0)\right]\left[\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0)\right]$ and thus

$$\begin{aligned}\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta) := \Big\{\mathbf{1}-\left[\Lambda^{(-)}_{c,\gamma}(0)-\Lambda^{(+)}_{c,\gamma}(0)\right]\left[\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0)\right]\Big\}&\\ \times\left[\mathbf{1}-(\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0))^2\right]^{-1/2}&.\end{aligned}$$

Using the representation $(1-a^2)^{-1/2}=\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-y^2}}\frac{1}{1-ya}\mathrm{d}y$ (see [17, Formula 3.197.4]) we obtain

$$\begin{aligned}\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta) = \Big\{\mathbf{1}-\left[\Lambda^{(-)}_{c,\gamma}(0)-\Lambda^{(+)}_{c,\gamma}(0)\right]\left[\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0)\right]\Big\}&\\ \times\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-y^2}}\frac{1}{1-y(\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0))}\mathrm{d}y&.\end{aligned}$$

Lemma 4 implies that the estimates $\|\left[\Lambda^{(-)}_{c,\gamma}(0)-\Lambda^{(+)}_{c,\gamma}(0)\right]\left[\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0)\right]\|\le$ $2C_{\mathrm{DL}}|\theta|$ and

$$\begin{aligned}&\Big\|\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-y^2}}\frac{1}{1-y(\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0))^2}\mathrm{d}y\ -\mathbf{1}\Big\| =\\ &\qquad=\Big\|\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-y^2}}\frac{y(\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0))}{1-y(\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0))}\mathrm{d}y\Big\|\le C'C_{\mathrm{DL}}|\theta|\end{aligned}$$

hold for some $C'>0$.
Proof of (59)*:* The strategy is similar to the proof of (58). We write

$$\begin{aligned}&|D_{c,0}(0)|^{1/2}\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)|D_{c,0}(0)|^{-1/2} =\\ &=\Big\{\mathbf{1}-|D_{c,0}(0)|^{1/2}|D_{c,\gamma}(0)|^{-1/2}\left[\Lambda^{(-)}_{c,\gamma}(0)-\Lambda^{(+)}_{c,\gamma}(0)\right]|D_{c,\gamma}(0)|^{1/2}|D_{c,0}(0)|^{-1/2}\\ &\quad\times|D_{c,0}(0)|^{1/2}\left[\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0)\right]|D_{c,0}(0)|^{-1/2}\Big\}\\ &\quad\times\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-y^2}}\frac{1}{1-y|D_{c,0}(0)|^{1/2}\left(\Lambda^{(+)}_{c,\gamma}(\theta)-\Lambda^{(+)}_{c,\gamma}(0)\right)|D_{c,0}(0)|^{-1/2}}\mathrm{d}y,\end{aligned}$$

where we used that $|D_{c,\gamma}(0)|^{-1/2}$ commutes with $\Lambda^{(\pm)}_{c,\gamma}(0)$. Using Lemma 5 and Lemma 4 we obtain the claim as before. □

A first application of the transformation function $\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)$ is the following lemma, which estimates the difference between the dilated Dirac operator and its original version.

LEMMA 6. *Under the assumptions of Theorem 6 b) there is a constant $C_{UD} > 0$, independent of γ, c and θ, such that*

$$\Big\| |D_{c,0}(0)|^{-1/2} \big[\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta) D_{c,\gamma}(\theta) \mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)^{-1} - D_{c,\gamma}(0)\big] |D_{c,0}(0)|^{-1/2} \Big\| \le C_{UD}|\theta| \quad (61)$$

holds.

Proof. We have

$$\begin{aligned}
&|D_{c,0}(0)|^{-1/2}[\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta) D_{c,\gamma}(\theta) \mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)^{-1} - D_{c,\gamma}(0)]|D_{c,0}(0)|^{-1/2} \\
&=|D_{c,0}(0)|^{-1/2}[\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta) - \mathbf{1}]|D_{c,0}(0)|^{1/2}|D_{c,0}(0)|^{-1/2} D_{c,\gamma}(\theta)|D_{c,0}(0)|^{-1/2} \\
&\times|D_{c,0}(0)|^{1/2}\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)^{-1}|D_{c,0}(0)|^{-1/2}+ \\
&+|D_{c,0}(0)|^{-1/2}[D_{c,\gamma}(\theta) - D_{c,\gamma}(0)]|D_{c,0}(0)|^{-1/2} \\
&\times|D_{c,0}(0)|^{1/2}\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)^{-1}|D_{c,0}(0)|^{-1/2} + |D_{c,0}(0)|^{-1/2} D_{c,\gamma}(0)|D_{c,0}(0)|^{-1/2} \\
&\times|D_{c,0}(0)|^{1/2}[\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)^{-1} - \mathbf{1}]|D_{c,0}(0)|^{-1/2},
\end{aligned}$$

which implies the claim, if we use additionally

$$\begin{aligned}
&\||D_{c,0}(0)|^{-1/2}[D_{c,\gamma}(\theta) - D_{c,\gamma}(0)]|D_{c,0}(0)|^{-1/2}\| = \||D_{c,0}(0)|^{-1/2} \\
&\quad\times[-(e^{-\theta}-1)ic\boldsymbol{\alpha}\cdot\nabla - \gamma(V(\theta)-V)]|D_{c,0}(0)|^{-1/2}\| \le (B+\tilde{C})|\theta|(1+\frac{\pi\gamma}{2c})
\end{aligned} \quad (62)$$

and Theorem 6. Moreover, we used the inequality $|e^{-\theta}-1| \le B|\theta|$ with $B = e^{\pi/4}$ and Kato's inequality in the proof of (62).

□

7 A RESOLVENT ESTIMATE FOR THE DIRAC OPERATOR

In the following, we choose an $\eta > 0$ such that for some $\tilde{n} > 1$ and all $c \ge 1$ the inequalities $\tilde{E}_{\tilde{n},\tilde{n}}(c,\gamma) < c^2 - \eta$ and $\tilde{E}_{\tilde{n}+1,1}(c,\gamma) > c^2 - \eta$ hold. If $\tilde{n} = N_{\max}$, then the second condition has to be omitted.
Using the notation of Section 5 we define $P_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta) := \sum_{1\le n\le\tilde{n}} P_n(c,\gamma;\theta)$ and $\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta) := \mathbf{1} - (\Lambda^{(-)}_{c,\gamma}(\theta) + P_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta))$. Note that $\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)$ projects onto a subspace of the positive spectral subspace.

The following theorem partly generalizes [5, Lemma 3.8] for Dirac operators (see also Theorem A.1). We will slightly extend this theorem in the non-relativistic limit (see Lemma 7 and Corollary 4). This theorem and Corollary 4 enable us to control the norm of the resolvent of the non-self-adjoint operator $D_{c,\gamma}(\theta)|_{\mathrm{Ran}\,\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)}$. Note that the usual theorems about the norm of the resolvent of a self-adjoint operator fail in general, and that for the following to hold it is essential that to restrict the operator to (a subspace of) the positive spectral subspace.

Theorem 7. *Suppose that the assumptions of Theorem 6 b) hold. Assume additionally that the inequalities* $C_{\mathrm{UD}}|\theta|(1+2\gamma/c) < q$ *and* $2\gamma(1+C_{\mathrm{FW}}|\mathrm{Im}\,\theta|) < q$ *are fulfilled for some* $0 < q < 1$. *Then the following statements are true: The operator* $D_{c,\gamma}(\theta)|_{\mathrm{Ran}\,\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)} - z$ *has a bounded inverse for all* $z \in \mathbb{C}$ *with* $\mathrm{Re}\,z \le c^2 - 1$. *There is a constant* $C_R > 0$, *independent of* c, γ *and* θ, *such that for all* $z \in \mathbb{C}$ *with* $\mathrm{Re}\,z \le c^2 - 1$ *the estimate*

$$\left\| \left[D_{c,\gamma}(\theta)|_{\mathrm{Ran}\,\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)} - z \right]^{-1} \bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta) \right\| \le \frac{C_R \|\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)\|}{c^2 - \eta - \mathrm{Re}\,z}$$

holds.

Proof. We make a case distinction:
Case 1: $\mathrm{Re}\,z \le 0$. Theorem 6 implies the inclusion $\mathrm{Ran}(\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta) \times \bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)^{-1}) \subset \mathrm{Ran}(\Lambda^{(+)}_{c,\gamma}(0))$. Thus, using Theorem 6 again, it suffices to show

$$\left\| [(\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta) D_{c,\gamma}(\theta) \mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)^{-1})|_{\mathrm{Ran}\,\Lambda^{(+)}_{c,\gamma}(0)} - z]^{-1} \Lambda^{(+)}_{c,\gamma}(0) \right\| \le \frac{C}{c^2 - \eta - \mathrm{Re}\,z}.$$

As in [5, Proof of Lemma 3.8], we use a resolvent expansion:

$$\begin{aligned}
&[(\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta) D_{c,\gamma}(\theta) \mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)^{-1})|_{\mathrm{Ran}\,\Lambda^{(+)}_{c,\gamma}(0)} - z]^{-1} \Lambda^{(+)}_{c,\gamma}(0) \qquad (63)\\
&= \sum_{n=0}^{\infty} [D_{c,\gamma}(0)|_{\mathrm{Ran}\,\Lambda^{(+)}_{c,\gamma}(0)} - z]^{-1} \Lambda^{(+)}_{c,\gamma}(0) |D_{c,\gamma}(0)|^{1/2} \\
&\quad \times \Big[\Lambda^{(+)}_{c,\gamma}(0) |D_{c,\gamma}(0)|^{-1/2} |D_{c,0}(0)|^{1/2} |D_{c,0}(0)|^{-1/2} \\
&\quad \times [\mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta) D_{c,\gamma}(\theta) \mathcal{U}_{\mathrm{DL}}(c,\gamma;\theta)^{-1} - D_{c,\gamma}(0)] \\
&\quad \times |D_{c,0}(0)|^{-1/2} |D_{c,0}(0)|^{1/2} |D_{c,\gamma}(0)|^{-1/2} \Lambda^{(+)}_{c,\gamma}(0) \\
&\quad \times |D_{c,\gamma}(0)|^{1/2} [D_{c,\gamma}(0)|_{\mathrm{Ran}\,\Lambda^{(+)}_{c,\gamma}(0)} - z]^{-1} \Lambda^{(+)}_{c,\gamma}(0) |D_{c,\gamma}(0)|^{1/2} \Big]^n |D_{c,\gamma}(0)|^{-1/2}
\end{aligned}$$

In order to prove the convergence of the series, we have to estimate the terms in (63). First, we note that

$$|D_{c,\gamma}(0)| (D_{c,\gamma}(0)|_{\mathrm{Ran}\,\Lambda^{(+)}_{c,\gamma}(0)} - z)^{-1} \Lambda^{(+)}_{c,\gamma}(0) = \sup_{\lambda \ge c^2} \frac{\lambda}{|\lambda - z|} \le 1 \qquad (64)$$

holds, since $\operatorname{Re} z \leq 0$. Moreover, the spectral theorem implies

$$\begin{aligned} &\| |D_{c,\gamma}(0)|^{1/2} [D_{c,\gamma}(0)|_{\operatorname{Ran}\Lambda^{(+)}_{c,\gamma}(0)} - z]^{-1} \Lambda^{(+)}_{c,\gamma}(0)\|^2 \\ \leq &\| [D_{c,\gamma}(0)|_{\operatorname{Ran}\Lambda^{(+)}_{c,\gamma}(0)} - z]^{-1} \Lambda^{(+)}_{c,\gamma}(0)\| \\ &\quad \times \| |D_{c,\gamma}(0)| [D_{c,\gamma}(0)|_{\operatorname{Ran}\Lambda^{(+)}_{c,\gamma}(0)} - z]^{-1} \Lambda^{(+)}_{c,\gamma}(0)\| \leq \frac{C}{c^2 - \eta - \operatorname{Re} z}. \end{aligned} \tag{65}$$

Lemma 6, Lemma 5 and (64) prove the convergence of the series in (63). Using Formula (65), the claim follows for $\operatorname{Re} z \leq 0$ from (63).
Case 2: $0 \leq \operatorname{Re} z \leq c^2 - 1$. We use the resolvent expansion

$$[D_{c,\gamma}(\theta) - z]^{-1} = \sum_{n=0}^{\infty} [D_{c,0}(\theta) - z]^{-1} [\gamma V(\theta) [D_{c,0}(\theta) - z]^{-1}]^n. \tag{66}$$

Hardy's inequality and Theorem 1 yield $\|\gamma V(\theta)[D_{c,0}(\theta) - z]^{-1}\| \leq 2\gamma e^{-\operatorname{Re}\theta}(1 + C_{\mathrm{FW}}|\operatorname{Im}\theta|)\|\frac{|\nabla|}{\sqrt{-e^{-2\theta}c^2\Delta + c^4\beta - z}}\|$. In order to control this norm, we estimate as follows:

$$\sup_{p\in\mathbb{R}^3} \frac{e^{-\operatorname{Re}\theta}|p|}{|\sqrt{e^{-2\theta}c^2p^2 + c^4} \pm z|} \leq \frac{1}{\sqrt{\cos(2\operatorname{Im}\theta)}} \sup_{p\in\mathbb{R}^3} \frac{|p|}{|\sqrt{c^2p^2 + c^4} \pm \operatorname{Re} z}$$

Since $\frac{|p|}{\sqrt{c^2p^2+c^4}+\operatorname{Re} z} \leq 1/c$, it suffices to consider the case with the minus sign. We need to find the supremum of the function $f_{c,l} : [0,\infty) \to \mathbb{R}$, $f_{c,l}(r) := \frac{r}{\sqrt{c^2r^2+c^4}-l}$ for $0 \leq l \leq (c^2 - 1)$. If we differentiate this function, we find that it attains its maximum at the point $r_0 := \frac{\sqrt{c^4-l^2}c}{l}$. Now, we define the function $g_c(l) := f_{c,l}(r_0) = \frac{c}{\sqrt{c^4-l^2}}$ for $0 \leq l \leq (c^2 - 1)$. This function is obviously monotonously increasing in l and therefore attains its maximum at the point $l_0 := c^2 - 1$. We have $g_c(l_0) = \frac{c}{\sqrt{c^4-(c^2-1)^2}} = \frac{c}{\sqrt{2c^2-1}} \leq 1$.
Thus, Equation (66) and Theorem 1 yield the estimate $\|[D_{c,\gamma}(\theta) - z]^{-1}\| \leq \tilde{C}\|1/(\sqrt{e^{-2\theta}c^2p^2 + c^4}\beta - z)\|$ with some $\tilde{C} > 0$. Since $\sqrt{e^{-2\theta}c^2p^2 + c^4}\beta$ is normal, we find $\|[D_{c,\gamma}(\theta) - z]^{-1}\| \leq C_R/(c^2 - \eta - \operatorname{Re} z)$, which remains true, if we restrict the resolvent to $\operatorname{Ran}\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)$.

□

8 Non-relativistic limit

In this section we investigate the non-relativistic limit of complex dilated Dirac operators. We will use these results in [30], where we will discuss the interaction with the second quantized radiation field. Moreover, we can extend the resolvent estimate of Theorem 7 to the region close to the spectrum of the operator and control the norm of the projection occurring there.

8.1 General Theory

We extend some statements from [49] to the non-self-adjoint case. We define $\beta_\pm := \frac{1}{2}(\mathbf{1} \pm \beta)$ as well as $M := \{z \in \mathbb{C} | -1 \le \operatorname{Re} z < 0, |\operatorname{Im} z| \le 1\}$ and fix a $\gamma > 0$ such that $D_{c,\gamma}(\theta) - c^2$ has no eigenvalues E with $\operatorname{Re} E \le -1$. This is at least true for $0 \le \gamma < 1$ in the case of $V = V_C$, which can be seen, for example, using the explicit formula for the eigenvalues, see [35]. We define as operators on $L^2(\mathbb{R}^3; \mathbb{C}^4)$:

$$D_{\infty,0}(\theta) := -\frac{e^{-2\theta}}{2}\Delta, \qquad D_{\infty,\gamma}(\theta) := -\frac{e^{-2\theta}}{2}\Delta - \gamma V(\theta)\beta_+$$

$$K_{c,0}(\theta) := (D_{\infty,0}(\theta) - z - \frac{z^2}{2c^2})^{-1}, \quad K_{c,\gamma}(\theta) := (D_{\infty,\gamma}(\theta) - z - \frac{z^2}{2c^2})^{-1}$$

as well as

$$R_{\infty,0;\theta}(z) := (D_{\infty,0}(\theta) - z)^{-1}, \qquad R_{c,0;\theta}(z) := (D_{c,0}(\theta) - z)^{-1}$$

$$R_{\infty,\gamma;\theta}(z) := (D_{\infty,\gamma}(\theta) - z)^{-1}, \qquad R_{c,\gamma;\theta}(z) := (D_{c,\gamma}(\theta) - z)^{-1}.$$

First, we generalize [49, Theorem 6.1 and Theorem 6.4] to dilated operators. As in [49], Theorem 8 is the starting point for the investigation of the non-relativistic limit.

Theorem 8. *a) Suppose that $\theta \in S_{\pi/4}$ and $c \ge 1$. Then for $z \notin \sigma(D_{c,0}(\theta)) \cup \sigma(D_{\infty,0}(\theta))$ the resolvent relation*

$$\begin{aligned}(D_{c,0}(\theta) \mp c^2 - z)^{-1} &= \left(\beta_\pm \pm \frac{1}{2c^2}(-ic\boldsymbol{\alpha}\cdot\nabla \pm z)\right) \\ &\times \left(\mathbf{1} \mp \frac{1}{2c^2} z^2 (\pm D_{\infty,0}(\theta) - z)^{-1}\right)^{-1} (\pm D_{\infty,0}(\theta) - z)^{-1} \quad (67)\end{aligned}$$

holds.

b) Suppose that $\theta \in S_{\min\{\pi/4,\vartheta\}}$, $\frac{2\gamma}{c} C(\operatorname{Im}\theta) < 1$ and that (H1) holds. Then for $z \in M \setminus \mathbb{R}$ the relations

$$\begin{aligned}(D_{c,\gamma}(\theta) - c^2 - z)^{-1} &= \left(\beta_+ + \frac{1}{2c^2}(-ice^{-\theta}\boldsymbol{\alpha}\cdot\nabla + z)\right) \\ &\times K_{c,\gamma}(\theta)\left(\mathbf{1} - \frac{\gamma}{2c^2} V(\theta)(-ice^{-\theta}\boldsymbol{\alpha}\cdot\nabla + z) K_{c,\gamma}(\theta)\right)^{-1} \quad (68)\end{aligned}$$

and

$$K_{c,\gamma}(\theta) = \left(\mathbf{1} - \frac{z^2}{2c^2}(D_{\infty,\gamma}(\theta) - z)^{-1}\right)^{-1} (D_{\infty,\gamma}(\theta) - z)^{-1} \quad (69)$$

hold.

Proof.
a) We follow the proof of [49, Theorem 6.1], noting that $z \in \mathbb{C}$ with $z(1+\frac{z}{2c^2}) \notin e^{-2\mathrm{i}\,\mathrm{Im}\,\theta}[0,\infty)$ is equivalent to $z + c^2 \notin \sigma(D_{c,0}(\theta))$. In order to show Equation (67), we define the operators $A_\pm(\theta) := D_{c,0}(\theta) \pm c^2 \pm z = -ic\boldsymbol{\alpha}\cdot\nabla \pm 2c^2\beta_\pm \pm z$ and note that $A_+(\theta)A_-(\theta) = A_-(\theta)A_+(\theta) = -c^2e^{-2\mathrm{i}\,\theta}\Delta - 2c^2z - z^2$. This yields

$$A_\pm(\theta)^{-1} = \frac{A_\mp(\theta)}{2c^2}\left(D_{\infty,0}(\theta) - z - \frac{z^2}{2c^2}\right)^{-1}, \tag{70}$$

which in turn implies the claim. Note that all operators are equivalent to multiplication operators.
b) We follow the proof of [49, Theorem 6.2]. Theorem 3 implies that $z + c^2 \notin \sigma(D_{c,\gamma}(\theta))$. It follows that $D_{c,\gamma}(\theta) - (c^2 + z) = A_-(\theta) - \gamma V(\theta) = (\mathbf{1} + \gamma V(\theta)A_-(\theta)^{-1})A_-(\theta)$. Since $D_{c,\gamma}(\theta) - (c^2 + z)$ and $A_-(\theta)$ have bounded inverses, the bounded operator $\mathbf{1} + \gamma V(\theta)A_-(\theta)^{-1}$ is bijective, and is thus in particular bounded invertible. From Equation (70) it follows that

$$(D_{c,\gamma}(\theta)-c^2-z)^{-1} = (A_-(\theta)-\gamma V(\theta))^{-1} = A_-(\theta)^{-1}(1-\gamma V(\theta)A_-(\theta)^{-1})^{-1} \tag{71}$$

$$= \left(\beta_+ + \frac{1}{2c^2}(-ice^{-\theta}\boldsymbol{\alpha}\cdot\nabla + z)\right)(D_{\infty,0}(\theta) - z - \frac{z^2}{2c^2})^{-1}$$
$$\times\left(1 - \gamma V(\theta)\beta_+K_{c,0}(\theta) - \frac{\gamma}{2c^2}V(\theta)(-ice^{-\theta}\boldsymbol{\alpha}\cdot\nabla + z)K_{c,0}(\theta)\right)^{-1}.$$

$z \in M \setminus \mathbb{R}$ implies $z + z^2/(2c) \in M \setminus \mathbb{R}$ and in particular $z(1 + \frac{z}{2c^2}) \notin \sigma(D_{\infty,\gamma}(\theta))$, which shows (69). Moreover $K_{c,\gamma}(\theta) = K_{c,0}(\theta) - \gamma V(\theta)\beta_+ = (\mathbf{1} - \gamma V(\theta)\beta_+K_{c,0}(\theta)^{-1})K_{c,0}(\theta)$ holds. To see this, note that $z + c^2 \notin \sigma(D_{c,\gamma}(\theta))$ implies $z + c^2 \notin \sigma(D_{c,0}(\theta))$, which in turn implies $z(1 + \frac{z}{2c^2}) \notin \sigma(D_{\infty,0}(\theta))$, i.e. $K_{c,0}(\theta)$ is bounded invertible. Thus, the bounded operator $\mathbf{1} - \gamma V(\theta)\beta_+K_{c,0}(\theta)^{-1}$ has a bounded inverse, and

$$K_{c,\gamma}(\theta)^{-1} = K_{c,0}(\theta)^{-1}(\mathbf{1} - \gamma V(\theta)\beta_+K_{c,0}(\theta)^{-1})^{-1} \tag{72}$$

as well as

$$\left(1 - \gamma V(\theta)\beta_+K_{c,0}(\theta) - \frac{\gamma}{2c^2}V(\theta)(-ice^{-\theta}\boldsymbol{\alpha}\cdot\nabla + z)K_{c,0}(\theta)\right)^{-1}$$
$$= (\mathbf{1} - \gamma V(\theta)K_{c,0}(\theta)^{-1}\beta_+)^{-1}$$
$$\times\left(\mathbf{1} - \frac{\gamma}{2c^2}V(\theta)(-ice^{-\theta}\boldsymbol{\alpha}\cdot\nabla + z)K_{c,0}(\theta)(\mathbf{1} - \gamma V(\theta)K_{c,0}(\theta)^{-1}\beta_+)^{-1}\right)^{-1} \tag{73}$$

hold. Using (72) and (73), (68) follows from (71). □

We denote the real eigenvalues of $D_{\infty,\gamma}(\theta)$ by $E_n(\infty,\gamma) = E_n(\gamma)$, ordered by size and not counting multiplicities. Note that by dilation analyticity these eigenvalues are the same as the eigenvalues of $D_{\infty,\gamma}(0)$ and that $|E_n(\infty,\gamma) - E_{n,l}(c,\gamma)| = \mathcal{O}(1/c^2)$ for all $l \in \{1,\dots,N_n\}$ by [49, Theorem 6.8]. We pick now η as in Section 7 and define for each $\tilde\epsilon > 0$ the set

$$M_{\eta,\tilde\epsilon} := \{z \in \mathbb{C} | 1 \le \operatorname{Re} z \le -(\eta+\tilde\epsilon),\ |\operatorname{Im} z| \le 1,\ \operatorname{dist}(z, \sigma(D_{c,\gamma}(\theta))) \ge \tilde\epsilon\}.$$

Moreover, we set $D(w,r) := \{z \in \mathbb{C} | |z-w| < r\}$ for $w \in \mathbb{C}$ and $r > 0$. Fix $\tilde\epsilon > 0$ so small that for all $n, n' \in \mathbb{N}$ with $n \neq n'$ and $1 \le n, n' \le \tilde n$ the sets $D(E_n(\infty,\gamma), 2\tilde\epsilon)$ and $D(E_{n'}(\infty,\gamma), 2\tilde\epsilon)$ are disjoint and contained in the set $\{z \in \mathbb{C} | 1 \le \operatorname{Re} z \le -(\eta+\tilde\epsilon),\ |\operatorname{Im} z| \le 1\}$. Now we pick for $\tilde\epsilon > 0$ a contour Γ with positive orientation such that Γ is contained $M_{\eta,\tilde\epsilon}$ and has only the eigenvalue $E_n(\gamma)$ in its interior, but no other eigenvalues of $\sigma(D_{\infty,\gamma}(\theta))$. Then we define

$$P_n(\infty,\gamma;\theta) := -\frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{d}z\, R_{\infty,\gamma;\theta}(z)\beta_+.$$

We set $P_{\rm disc}(\infty,\gamma;\theta) := \sum_{i=1}^{\tilde n} P_i(\infty,\gamma;\theta)$ and $\bar P_{\rm disc}(\infty,\gamma;\theta) := 1 - P_{\rm disc}(\infty,\gamma;\theta)$. Note that using the definitions in Appendix A and in slight abuse of notation $P_{\rm disc}(\infty,\gamma;\theta) = P_{\rm disc}(\gamma;\theta)\beta_+$ and $\bar P_{\rm disc}(\infty,\gamma;\theta) = \beta_- + \bar P_{\rm disc}(\gamma;\theta)\beta_+$.
Now we are in the position to generalize [49, Corollary 6.5] to dilated operators.

Corollary 2. *Suppose that $|\theta| < \theta_0$, where θ_0 is sufficiently small (see Appendix A), and $\theta \in S_{\min\{\pi/4,\Theta\}}$ as well as $\frac{2\gamma}{c}C(\operatorname{Im}\theta) < 1$. Suppose moreover that (H1) holds. Then the resolvent expansion*

$$\left[D_{c,\gamma}(\theta) - (c^2+z)\right]^{-1} = \sum_{n=0}^{\infty} \frac{1}{c^n} R_n(z). \tag{74}$$

holds for all $z \in M_{\eta,\tilde\epsilon}$ and all sufficiently large c. The series converges in norm, uniformly in θ and z. In particular,

$$[D_{c,\gamma}(\theta) - (c^2+z)]^{-1} \underset{c\to\infty}{\longrightarrow} [D_{\infty,\gamma}(\theta) - z]^{-1}\beta_+$$

uniformly in θ and z.

Proof. First, we need an estimate on the resolvent of $D_{\infty,\gamma}(\theta)$. We split the resolvent according to

$$\begin{aligned}[D_{\infty,\gamma}(\theta) - z]^{-1} &= [D_{\infty,\gamma}(\theta)|_{\operatorname{Ran}\bar P_{\rm disc}(\infty,\gamma;\theta)} - z]^{-1} \bar P_{\rm disc}(\infty,\gamma;\theta) \\ &\quad + \sum_{n=1}^{\tilde n} [D_{\infty,\gamma}(\theta)|_{\operatorname{Ran} P_n(\infty,\gamma;\theta)} - z]^{-1} P_n(\infty,\gamma;\theta).\end{aligned} \tag{75}$$

Theorem A.1 implies that the norm of the first summand in (75) is bounded by $2/\eta$. The norms of the other summands can be estimated according to $\left\| [D_{\infty,\gamma}(\theta)|_{\operatorname{Ran} P_n(\infty,\gamma;\theta)} - z]^{-1} P_n(\infty,\gamma;\theta) \right\| \le \frac{\|P_n(\infty,\gamma;\theta)\|}{\operatorname{dist}(z,E_n(\gamma))} \le \frac{C|\theta|}{\operatorname{dist}(z,E_n(\gamma))}$ using

Corollary A.1. Thus, we have for sufficiently small $1/c$ (dependent on $\tilde{\epsilon}$) and all $z \in M_{\eta,\tilde{\epsilon}}$ the expansion

$$(\mathbf{1} - \frac{z^2}{2c^2}(D_{\infty,\gamma}(\theta) - z)^{-1})^{-1} = (D_{\infty,\gamma}(\theta) - z)^{-1} \sum_{n=0}^{\infty} (\frac{z^2}{2c^2}(D_{\infty,\gamma}(\theta) - z)^{-1})^n.$$

Hardy's inequality implies for $f \in H^2(\mathbb{R}^3;\mathbb{C}^4)$ the estimates $\|Vf\| \leq 2\|\nabla f\| \leq a\|\Delta f\| + (1/a)\|f\|$ and $e^{-2\mathrm{Re}\,\theta}\|\Delta f\| \leq 1/(1-2a\gamma)\|D_{\infty,\gamma}(\theta)f\| + 2\gamma/[a(1-2a\gamma)]\|f\|$ with a sufficiently small $a > 0$. It follows that

$$\|\frac{\gamma}{2c^2}V(\theta)(-ice^{-\theta}\boldsymbol{\alpha}\cdot\nabla+z)(D_{\infty,\gamma}(\theta)-z)^{-1}\| \leq \frac{\gamma}{c}[C_1+C_2\|(D_{\infty,\gamma}(\theta)-z)^{-1}\|]$$

holds with $C_1, C_2 > 0$ (independent of γ, c and θ), which implies that the last factor in (68) has a norm convergent series expansion in $1/c$ for $1/c$ small enough. □

REMARK 4. *We find $R_0(z) := \beta_+ R_{\infty,\gamma;\theta}(z)$ as in [49]. As in [49, Remark after Corollary 6.5], the operators occurring with even powers of $1/c$ are even, and the operators occurring with odd powers of $1/c$ are odd .*

LEMMA 7. *Suppose that the assumptions of Corollary 2 hold. Then there is a constant $C_{\mathrm{P},n} > 0$ (independent of c and θ) such that for sufficiently large c the estimate*

$$\|P_n(c,\gamma;\theta) - P_n(\infty,\gamma;\theta)\| \leq \frac{C_{\mathrm{P},n}}{c}$$

holds.

Proof. This follows immediately from Corollary 2. □

The following two corollaries extend Theorem 7.

COROLLARY 3. *Suppose that the assumptions of Corollary 2 hold. Then there is a constant $C > 0$ (possibly dependent on θ) such that for all $z \in \mathbb{C}$ with $-1 \leq \mathrm{Re}\, z \leq -\eta$ and $|\mathrm{Im}\, z| \leq 1$ and all sufficiently large c the estimate*

$$\|[D_{c,\gamma}(\theta)|_{\mathrm{Ran}\,\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)} - (c^2+z)]^{-1}\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)\| \leq C$$

holds.

Proof. Corollary 2 implies that $[D_{c,\gamma}(\theta)|_{\mathrm{Ran}\,\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)} - (c^2+z)]^{-1}$ is uniformly bounded in $z \in M_{\eta,\tilde{\epsilon}}$ and c (for sufficiently large c). Lemma 7 and Lemma 4 yield the existence of an upper bound on

$$\|\bar{P}_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta)\| = \|\mathbf{1} - (\Lambda_{c,\gamma}^{(-)}(\theta) + P_{\mathrm{disc},\tilde{n}}(c,\gamma;\theta))\|,$$

which does not depend on c. Thus the claim holds for $z \in M_{\eta,\tilde{\epsilon}}$.

Let $z_0 \in D(E_n(\infty,\gamma),\tilde\epsilon)$. Then $\Gamma := \{z \in \mathbb{C} | |z - E_n(\infty,\gamma)| = 2\tilde\epsilon\} \subset M_{\eta,\tilde\epsilon}$ holds because of the definition of the set $M_{\eta,\tilde\epsilon}$. Since $[D_{c,\gamma}(\theta)|_{\mathrm{Ran}\,\bar P_{\mathrm{disc},\tilde n}(c,\gamma;\theta))} - (c^2 + z)]^{-1}$ in $z \in \{z \in \mathbb{C} | -1 \le \mathrm{Re}\, z \le -\eta,\ |\mathrm{Im}\, z| \le 1\}$ is holomorphic,

$$[D_{c,\gamma}(\theta)|_{\mathrm{Ran}\,\bar P_{\mathrm{disc},\tilde n}(c,\gamma;\theta)} - (c^2 + z_0)]^{-1} \bar P_{\mathrm{disc},\tilde n}(c,\gamma;\theta) = -\frac{1}{2\pi \mathrm{i}} \times \int_\Gamma [D_{c,\gamma}(\theta)|_{\mathrm{Ran}\,\bar P_{\mathrm{disc},\tilde n}(c,\gamma;\theta))} - (c^2 + z)]^{-1} \bar P_{\mathrm{disc},\tilde n}(c,\gamma;\theta) \frac{1}{z - z_0} \mathrm{d}z$$

holds, where the contour is oriented in the positive sense. This implies the claim for $z_0 \in D(E_n(\infty,\gamma),\tilde\epsilon)$. □

Corollary 4. *Suppose that the assumptions of Theorem 7 and Corollary 2 hold. Then there is a $C > 0$ (possibly dependent on θ) such that for all $z \in \mathbb{C}$ with $-1 < \mathrm{Re}\, z \le -\eta$ and $|\mathrm{Im}\, z| \le 1$ or with $-\infty < \mathrm{Re}\, z \le -1$ and all sufficiently large c the estimate*

$$\|[D_{c,\gamma}(\theta)|_{\mathrm{Ran}\,\bar P_{\mathrm{disc},\tilde n}(c,\gamma;\theta)} - (c^2 + z)]^{-1} \bar P_{\mathrm{disc},\tilde n}(c,\gamma;\theta)\| \le \frac{C}{-\eta - \mathrm{Re}\, z}$$

is true.

Proof. This follows immediately from Corollary 3 and Theorem 7 together with Lemma 7. □

Now, we define a transformation function $\mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta) : L^2(\mathbb{R}^3;\mathbb{C}^4) \to L^2(\mathbb{R}^3;\mathbb{C}^4)$ by

$$\mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta) := [P_n(c,\gamma;\theta)P_n(\infty,\gamma;\theta) + (\mathbf{1} - P_n(c,\gamma;\theta))(\mathbf{1} - P_n(\infty,\gamma;\theta))] \times [\mathbf{1} - (P_n(c,\gamma;\theta) - P_n(\infty,\gamma;\theta))^2]^{-1/2}.$$

Lemma 8. *Suppose that the assumptions of Corollary 2 and the inequality $C_{\mathrm{P},n}/c < q < 1$ hold for some $0 < q < 1$. Then the mapping $\mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta)$ is bounded with bounded inverse $\mathcal{V}_{\mathrm{NR}}(c,\gamma;\theta)$. The relations*

$$\mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta) P_l(\infty,\gamma;\theta) \mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta)^{-1} = P_l(c,\gamma;\theta) \tag{76}$$

and

$$\|\mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta) - \mathbf{1}\| \le \frac{C_{\mathrm{NRP}}}{c} \tag{77}$$

hold with a constant $C_{\mathrm{NRP}} > 0$ independent of c and θ. $\mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta)$ is a holomorphic function of θ.

Proof. Using Lemma 7 this can be proven in the same way as Theorem 6. For the holomorphicity in θ note that the power series (in $1/c$) for $R_{c,\gamma;\theta}(z)$, $P_n(c,\gamma;\theta)$ and $\mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta)$ converge uniformly in θ. □

Remark 5. *As in [49] we obtain by Remark 4 that in the series expansion of* $\mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta)$ *the operators occurring with even powers of* $1/c$ *are even and the operators occurring with odd powers of* $1/c$ *are odd. In particular,*

$$\mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta) = \mathcal{U}_{NR,g}(c,\gamma;\theta) + \frac{1}{c}\mathcal{U}_{NR,ug}(c,\gamma;\theta), \tag{78}$$

where $\mathcal{U}_{NR,g}(c,\gamma;\theta)$ *and* $\mathcal{U}_{NR,ug}(c,\gamma;\theta)$ *are even and odd operators holomorphic in* $1/c$.

The following theorem generalizes [49, Theorem 6.7] and shows that the lower component of an eigen-spinor of the Dirac operator converges to zero as $c \to \infty$.

Theorem 9. *Suppose that the assumptions of Lemma 8 hold. Then the normed eigenfunctions* $\phi_n(c,\gamma;\theta)$ *of* $D_{c,\gamma}(\theta)$ *with eigenvalue* $E_{n,l}(c,\gamma)$ *have the form*

$$\phi_{n,l}(c,\gamma;\theta) = \phi_{n,l,+}(c,\gamma;\theta) + \frac{1}{c}\phi_{n,l,-}(c,\gamma;\theta),$$
$$\phi_{n,l,\pm}(c,\gamma;\theta) \in \beta_\pm L^2(\mathbb{R}^3;\mathbb{C}^4), \tag{79}$$

where $\phi_{n,l,\pm}(c,\gamma;\theta)$ *are continuous functions of* $1/c$.

Proof. We have $P_n(c,\gamma;\theta)D_{c,\gamma}(\theta)P_n(c,\gamma;\theta) = -\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{z}{D_{c,\gamma}(\theta)-z}\mathrm{d}z$. Any eigenvector $\tilde{\phi}_n(c,\gamma;\theta)$ of $P_n(c,\gamma;\theta)D_{c,\gamma}(\theta)P_n(c,\gamma;\theta)$ and thus any eigenvector of $D_{c,\gamma}(\theta)$ with eigenvalue $E_{n,l}(c,\gamma)$ is given by $\tilde{\phi}_{n,l}(c,\gamma;\theta)$ $= \mathcal{U}_{\mathrm{NR}}(c,\gamma;\theta)\phi_{n,l}(\infty,\gamma;\theta)$ for a $\phi_{n,l}(\infty,\gamma;\theta) \in \beta_+L^2(\mathbb{R}^3;\mathbb{C}^4)$. Remark 5 and the analytic perturbation theory imply $\tilde{\phi}_{n,l}(c,\gamma;\theta) = \tilde{\phi}_{n,l,+}(c,\gamma;\theta) + \frac{1}{c}\phi_{n,l,-}(c,\gamma;\theta)$, where $\tilde{\phi}_{n,l}(c,\gamma;\theta)$ and $\tilde{\phi}_{n,l,\pm}(c,\gamma;\theta)$ are holomorphic functions of $1/c$. Since the projections $P_n(c,\gamma;\theta)$ are nor orthogonal, the normed eigenfunctions are in general not holomorphic functions of $1/c$. But nevertheless $\|\tilde{\phi}_{n,l}(c,\gamma;\theta)\| \geq 1 - C\frac{1}{c}$ holds for some $C > 0$ and thus (79) follows. ☐

We use these statements to prove that eigenfunctions are bounded in the norm of $H^1(\mathbb{R}^3;\mathbb{C}^4)$.

Theorem 10. *Suppose the assumptions of Lemma 8 hold. Then there is a constant* $C_{\mathrm{EF}} > 0$*, independent of* c*, such that the normed eigenfunctions* $\phi_n(c,\gamma;\theta)$ *of* $D_{c,\gamma}(\theta)$ *with eigenvalue* $E_{n,l}(c,\gamma)$ *fulfill the estimates*

$$\|\nabla\phi_{n,l,+}(c,\gamma;\theta)\| \leq C_{\mathrm{EF}} \tag{80}$$

and

$$\|\nabla\phi_{n,l,-}(c,\gamma;\theta)\| \leq \frac{C_{\mathrm{EF}}}{c} \tag{81}$$

for sufficiently large c.

Proof. We follow Esteban and Séré [10, Proof of Lemma 7 and Theorem 3], who considered the non-relativistic limit of self-adjoint Dirac-Fock operators. Since $D_{c,\gamma}(\theta)$ is not self-adjoint, there are some additional difficulties. To simplify the notation, we suppress the dependence of $\phi_{n,l}(c,\gamma;\theta)$ on c, γ and θ. We have

$$\begin{aligned}&E_{n,l}(c,\gamma)^2\|\phi_{n,l}\|^2 = \|D_{c,\gamma}(\theta)\phi_{n,l}\|^2\\ &\geq e^{-2\operatorname{Re}\theta}[c^2(1-2\sin\operatorname{Im}\theta-\gamma/4)-4c\gamma]\|\nabla\phi_{n,l}\|^2\\ &+[c^4(1-2\sin\operatorname{Im}\theta)-16\gamma c^2]\|\phi_{n,l}\|^2,\end{aligned}$$

where we used Hardy's inequality. Since $E_{n,l}(c,\gamma)^2 - c^4 \leq 0$, it follows that

$$\begin{aligned}\|\nabla\phi_{n,l}\|^2 \leq &\frac{E_{n,l}(c,\gamma)^2 - c^4 + 2\sin\operatorname{Im}\theta c^2 + 16c^2}{c^2(1-2\sin\operatorname{Im}\theta - 1/4) - 4c\gamma}\|\phi_{n,l}\|^2\\ &\leq C(\sin\operatorname{Im}\theta c^2 + 1)\|\phi_{n,l}\|^2 \qquad (82)\end{aligned}$$

for sufficiently large c, where $C > 0$ does not depend on c.
Note that the term proportional to c^2 in (82) does not occur for $\operatorname{Im}\theta = 0$, which implies immediately the boundedness of $\|\nabla\phi_{n,l}\|$ in this case. To circumvent this difficulty, we write the Dirac equation in its components, where (in abuse of notation) $\phi_{n,l,\pm}$ denotes the upper and, respectively, lower components of $\phi_{n,l}$:

$$ce^{-\theta}\boldsymbol{\sigma}\cdot\nabla\phi_{n,l,-} - \gamma V(\theta)\phi_{n,l,+} + c^2\phi_{n,l,+} = E_{n,l}(c,\gamma)\phi_{n,l,+} \qquad (83)$$

$$ce^{-\theta}\boldsymbol{\sigma}\cdot\nabla\phi_{n,l,+} - \gamma V(\theta)\phi_{n,l,-} - c^2\phi_{n,l,-} = E_{n,l}(c,\gamma)\phi_{n,l,-} \qquad (84)$$

Dividing (83) by c, using Hardy's inequality and the boundedness of $E_{n,l}(c,\gamma) - c^2$, Formula (82) implies

$$\|\nabla\phi_{n,l,-}\| \leq \frac{2}{c}\|\nabla\phi_{n,l,+}\| + \frac{|E_{n,l}(c,\gamma)-c^2|}{c|e^{-\operatorname{Re}\theta}|}\|\phi_{n,l,+}\| \leq C \qquad (85)$$

for some $C > 0$ independent of c, i.e. $\|\nabla\phi_{n,l,-}\|$ is bounded in c. Dividing (84) by c, we obtain

$$\|\nabla\phi_{n,l,+}\| \leq \frac{2}{c}\|\nabla\phi_{n,l,-}\| + \frac{|E_{n,l}(c,\gamma)+c^2|}{c|e^{-\operatorname{Re}\theta}|}\|\phi_{n,l,-}\| \leq C \qquad (86)$$

for some $C > 0$ independent of c, where we used Theorem 9 and Equation (85). This shows (80). Inserting (86) in (85), Equation (81) follows. □

Remark 6. *Their validity of Theorem 9 and Theorem 10 in the Coulomb case could be derived from the explicit form of the eigenfunctions (see the proof of Lemma 11).*

Moreover, we need a bound on the norm of the dilation operator $\mathcal{U}(\theta)$, restricted to the spaces $\operatorname{Ran} P_n(c,\gamma;\theta)$.

LEMMA 9. *Suppose that the assumptions of Lemma 8 hold. Then the family of operators* $\mathcal{U}(\theta)|_{\operatorname{Ran} P_n(c,\gamma;0)} : \operatorname{Ran} P_n(c,\gamma;0) \to \operatorname{Ran} P_n(c,\gamma;\theta)$ *is uniformly bounded in* c *and* θ.

Proof. Surely $\mathcal{U}(\theta)|_{\operatorname{Ran} P_n(\infty,\gamma;0)} : \operatorname{Ran} P_n(\infty,\gamma;0) \to \operatorname{Ran} P_n(\infty,\gamma;\theta)$ is well defined for all $\theta \in \mathbb{C}$ with $|\theta| \leq \min\{\pi/4, \Theta\}$ (see [2, 6]) and (as a mapping between finite-dimensional vector spaces) bounded. Since the operator is a holomorphic function of θ for $|\theta| \leq \min\{\pi/4, \Theta\}$, there is a bound $C' > 0$ (independent of θ) on its norm.
Let $f \in \operatorname{Ran} P_n(c,\gamma;0)$. Then there is a $\tilde{f} \in \operatorname{Ran} P_n(\infty,\gamma;0)$ with $f = U_{\mathrm{NR}}(c,\gamma;0)\tilde{f}$, and for real θ $f(\theta) := \mathcal{U}_{\mathrm{el}}(\theta)f = \mathcal{U}(\theta)U_{\mathrm{NR}}(c,\gamma;0)\mathcal{U}(\theta)^{-1}\tilde{f}(\theta) = U_{\mathrm{NR}}(c,\gamma;\theta)\tilde{f}(\theta)$ holds, where $\tilde{f}(\theta) := \mathcal{U}_{\mathrm{el}}(\theta)\tilde{f}$. By holomorphic continuation we obtain for complex θ the equality $f(\theta) = U_{\mathrm{NR}}(c,\gamma;\theta)\tilde{f}(\theta)$. Thus Lemma 8 implies $\|f(\theta)\| \leq \|U_{\mathrm{NR}}(c,\gamma;\theta)\|\|\tilde{f}(\theta)\| \leq (1 + C_{\mathrm{NRP}}/c)C'\|\tilde{f}\| \leq (1 + C_{\mathrm{NRP}}/c)C'\|f\|$ for some $C' > 0$ independent of c and θ. □

The following corollary shows that also the projections on the fine structure components are bounded uniformly in c. This follows from the fact the dilated projections are similar to the corresponding orthogonal projections belonging to the corresponding self-adjoint Dirac operators because of Lemma 9. Note that in general such projections are not uniformly bounded in the perturbation parameter (see [33, Chapter II-1.5]).

COROLLARY 5. *Let* $1 \leq n \leq \tilde{n}$ *and suppose that the assumptions of Lemma 9 hold. Then* $\|P_{n,l}(c,\gamma;\theta)\| \leq C$ *for some* $C > 0$ *independent of* n, l, c *and* θ.

Proof. This follows from Lemma 5, since the projections $P_{n,l}(c,\gamma;0) = = \mathcal{U}(\theta)^{-1}P_{n,l}(c,\gamma;0)\mathcal{U}(\theta)$ are orthogonal. □

8.2 APPLICATION TO EXPECTATION VALUES OF DIRAC MATRICES

We are now in the position to investigate expectation values of the matrices $\boldsymbol{\alpha}$. Since these matrices are odd, such expectation values involve scalar products of the upper component of one spinor with the lower component of the other spinor. Therefore, one expects that such expectation values converge to zero like $1/c$ as $c \to \infty$ uniformly in a set of suitable spinors. We show in the following that this is true, if one of the spinors is in the set of eigenstates (in the positive part of the gap) and the other state is an arbitrary state from the positive spectral subspace. Note that this is not true, if both states are arbitrary states from the positive spectral subspace. At least for the free spectral subspaces this can be seen from the explicit form of the projections (see Section 5). We will apply this result in [30].

Lemma 10. *Suppose that the assumptions of Lemma 9 hold and let* $\tilde n$ *as in Section 7. Then there is a constant* $C > 0$*, independent of* c *and* θ*, such that for all* $1 \le n, n' \le \tilde n$*,* $1 \le l \le n$*,* $1 \le l' \le n'$ *and* $k_1, k_2 \in \mathbb{R}^3$

$$\|P_{n,l}(c,\gamma;\theta) k_1 \cdot \boldsymbol{\alpha} e^{ik_2\cdot x} P_{n',l'}(c,\gamma;\theta)\| \le \frac{C|k_1|}{c}.$$

Proof. This follows from Theorem 9 and Corollary 5, since $\boldsymbol{\alpha}$ is an odd operator. □

Lemma 11. *Suppose that* $V = V_C$*. Let* $c \ge 1$ *and* $\gamma/c < \sqrt{3}/2$*. Then there is a constant* $C > 0$*, independent of* c*, such that*

$$\||x| P_{n,l}(c,\gamma;0)\| \le C$$

holds, where x *denotes the operator of multiplication with the space variable.*

Proof. We define the unitary dilations $U_c\ f_c(x) := (U_c f)(x) := c^{-3/2} f(c^{-1}x)$ and note that $U_c D_{c,\gamma} U_c^{-1} = c^2 D_{1,\gamma/c}$. Thus, if $f \in H^1(\mathbb{R}^3;\mathbb{C}^4)$ is a normed eigenfunction of $D_{c,\gamma}$ with eigenvalue $E_{n,l}$, then f_c is a normed eigenfunction of $D_{1,\gamma/c}$ with eigenvalue $E_{n,l}/c^2$. The radial parts $f_{c,\pm}(r)$ of the upper and lower component, respectively, of f_c are (see [35, Abschnitt 36])

$$\begin{aligned} f_{c,\pm}(r) := \frac{\pm(2\lambda)^{3/2}}{\Gamma(2\tilde\gamma+1)} \frac{(1 \pm E_{n,l}/c^2)\Gamma(2\tilde\gamma + n_r + 1)}{4\frac{\gamma}{c\lambda}(\frac{\gamma}{c\lambda} - \kappa) n_r!} (2\lambda r)^{\tilde\gamma - 1} e^{-\lambda r} \\ \times \{(\frac{\gamma}{c\lambda} - \kappa) F(-n_r, 2\tilde\gamma + 1, 2\lambda r) \mp n_r F(1 - n_r, 2\tilde\gamma + 1, 2\lambda r)\}. \end{aligned}$$

Here the radial quantum number fulfills $n_r \in \mathbb{N}_0$ if $\kappa < 0$ and $n_r \in \mathbb{N}$ if $\kappa > 0$, and $\kappa \in \pm\mathbb{N}$ is the eigenvalue of the spin-orbit operator (see [49, Chapter 4.6]). F denotes the confluent hypergeometric function, which reduces to a polynomial in $2\lambda r$ here (see [35, Abschnitt 36] and [34, Abschnitt d]). Moreover, $\tilde\gamma := \sqrt{\kappa^2 - \gamma^2/c^2}$ and $\lambda := \sqrt{1 - E_{n,l}^2/c^4}$. Thus, the radial parts $f_\pm(r)$ of the upper respectively lower components of f are

$$\begin{aligned} f_{\pm}(r) := \frac{\pm(2c\lambda)^{3/2}}{\Gamma(2\tilde\gamma+1)} \frac{(1 \pm E_{n,l}/c^2)\Gamma(2\tilde\gamma + n_r + 1)}{4\frac{\gamma}{c\lambda}(\frac{\gamma}{c\lambda} - \kappa) n_r!} (2\lambda r)^{\tilde\gamma - 1} e^{-c\lambda r} \\ \times \{(\frac{\gamma}{c\lambda} - \kappa) F(-n_r, 2\tilde\gamma + 1, 2c\lambda r) \mp n_r F(1 - n_r, 2\tilde\gamma + 1, 2c\lambda r)\}. \end{aligned}$$

Using the explicit formula (see [35]) for the eigenvalues, we see that $c\lambda$ is a function bounded in c with $c\lambda \longrightarrow \gamma/n$ for $c \to \infty$. Moreover, obviously $\tilde\gamma \to |\kappa|$ holds. This shows the claim. □

Remark 7. *At this point we make use of the explicit from of the eigenfuntions of the Coulomb Dirac operator. There do not seem any results to be available in the literature about exponential decay of eifenfunctions of the Dirac operator uniformly in the velocity of light.*

Lemma 12. *Suppose that the assumptions of Lemma 9 are fulfilled and let $\tilde{n}$ as in Section 7. Let moreover $f : \mathbb{C} \to \mathbb{C}$ with $|f(z)| \leq |z|$. Then there is a constant $C > 0$, independent of c, such that for all $1 \leq n \leq \tilde{n}$, $1 \leq l \leq n$ and $k_1, k_2 \in \mathbb{R}^3$*

$$\|P_{n,l}(c,\gamma;0)k_1 \cdot \boldsymbol{\alpha} f(k_2 \cdot x) P_{n',l'}(c,\gamma;0)\| \leq \frac{C|k_1||k_2|}{c}.$$

Proof. Lemma 11 implies that $\|xP_{n,l}(c,\gamma;0)\|$ is uniformly bounded in c, in particular (using the notation of Theorem 9) $x\phi_{n,l,+}(c,\gamma;0)$. Now the claim follows exactly as in Lemma 10. □

The following theorem generalizes Lemma 10. Note that the statement of Lemma 10 is not completely obvious, since not even the lower component of the free positive spectral projection converges to zero in norm as $c \to \infty$. This is, however, compensated by the fact that the H^1-norm of the upper component of bound states is bounded uniformly in c (Theorem 10).

Theorem 11. *Suppose the assumptions of Lemma 9 hold and let $\tilde{n}$ as in Section 7. Then there is a constant $C > 0$, independent of c and θ such that for all $1 \leq n \leq \tilde{n}$, $1 \leq l \leq n$ and $k_1, k_2 \in \mathbb{R}^3$*

$$\|P_{n,l}(c,\gamma;\theta)k_1 \cdot \boldsymbol{\alpha} e^{ik_2 \cdot x} \Lambda^{(\pm)}_{c,\gamma}(\theta)\| \leq \frac{C|k_1|(1+|k_2|)}{c}.$$

Proof. Corollary 1 and Corollary 5 imply $\|P_{n,l}(c,\gamma;\theta)k_1 \cdot \boldsymbol{\alpha} e^{ik_2 \cdot x} \Lambda^{(\pm)}_{c,\gamma}(\theta)\| \leq \|P_{n,l}(c,\gamma;\theta)k_1 \cdot \boldsymbol{\alpha} e^{ik_2 \cdot x} \Lambda^{(\pm)}_{c,0}(\theta)\| + C_{\mathrm{NR}}|k_1|\frac{\gamma}{c}$ with some $C > 0$ independent of θ and c. Thus, it suffices to show $\|P_{n,l}(c,\gamma;\theta)k \cdot \boldsymbol{\alpha} e^{ik_2 \cdot x} \Lambda^{(\pm)}_{c,0}(\theta)\| \leq \frac{C|k_1|(1+|k_2|)}{c}$ for some $C > 0$. In a first step, we pick $f \in \operatorname{Ran} P_{n,l}(c,\gamma;\bar{\theta})$ and $g \in \operatorname{Ran} \Lambda^{(\pm)}_{c,0}(\theta)$ normed, but arbitrary otherwise. We have $g = V_{\mathrm{FW}}(c;\theta)(\tilde{g},0)^T$ for some $\tilde{g} \in L^2(\mathbb{R}^3;\mathbb{C}^2)$. It follows that $g = \mathcal{F}^{-1}(\frac{c^2+E_c(p;\theta)}{N_c(p;\theta)}\mathcal{F}\tilde{g}, \frac{ce^{-\theta}\boldsymbol{\sigma}\cdot p}{N_c(p;\theta)}\mathcal{F}\tilde{g})^T$, where $\mathcal{F}$ denotes both the Fourier transform on $L^2(\mathbb{R}^3;\mathbb{C}^2)$ and on $L^2(\mathbb{R}^3;\mathbb{C}^4)$. We decompose $f = (f_+, f_-)^T$ with $f_\pm \in L^2(\mathbb{R}^3;\mathbb{C}^2)$. It follows that

$$\begin{aligned}
&|(f, P_{n,l}(c,\gamma;\theta)k_1 \boldsymbol{\alpha} e^{ik_2 \cdot x} \Lambda^{(\pm)}_{c,0}(\theta) g)| \\
\leq &|(f_+, k_1 \cdot \boldsymbol{\sigma} \mathcal{F}^{-1} \frac{-ce^{-\theta}\boldsymbol{\sigma}\cdot p}{N_c(p;\theta)} \mathcal{F}\tilde{g})| + |k_1|\|f_-\|\|\tilde{g}\| \sup_{p\in\mathbb{R}^3} |\frac{c^2+E_c(p;\theta)}{N_c(p;\theta)}|.
\end{aligned}$$

Similarly to the proof of Theorem 1 we see that the supremum $\sup_{p\in\mathbb{R}^3} |\frac{c^2+E_c(p;\theta)}{N_c(p;\theta)}|$ is bounded independently of c and θ. Thus, Theorem 9 implies the claim for the second summand.
For the first summand, observe that $\sup_{p\in\mathbb{R}^3} |\frac{ce^{-\theta}}{N_c(p;\theta)}| \leq e^{\pi/4}/c$. Thus

$$\begin{aligned}
&|(f_+, k_1 \cdot \boldsymbol{\sigma} e^{ik_2 \cdot x} \mathcal{F} \frac{-ce^{-\theta}\boldsymbol{\sigma}\cdot p}{N_c(p;\theta)} \mathcal{F}^{-1}\tilde{g})| \\
= &|(\boldsymbol{\sigma}\cdot(-\mathrm{i}\,\nabla)k_1 \cdot \boldsymbol{\sigma} e^{-ik_2 \cdot x} f_+, \mathcal{F} \frac{-ce^{-\theta}}{N_c(p;\theta)} \mathcal{F}^{-1}\tilde{g})| \leq \frac{|k_1|e^{\pi/4}}{c} \|\nabla e^{ik_2 \cdot x} f_+\|\|\tilde{g}\|.
\end{aligned}$$

Theorem 1 yields $\|\tilde{g}\| \leq \sqrt{1 + C_{\mathrm{FW}}|\mathrm{Im}\,\theta|}\|g\|$, which shows

$$|(f, P_{n,l}(c,\gamma;\theta)k_1\boldsymbol{\alpha}e^{ik_2\cdot x}\Lambda_{c,0}^{(\pm)}(\theta)g)| \leq C\|f\|\|g\| \tag{87}$$

for some $C > 0$, if one takes Theorem 9 and Theorem 10 into account .
Now, pick $f, g \in L^2(\mathbb{R}^3;\mathbb{C}^4)$ arbitrarily and apply (87) to the functions $P_{n,l}(c,\gamma;\bar{\theta})f$ and $\Lambda_{c,0}^{(\pm)}(\theta)g$. This implies the claim together with Corollary 5 and Lemma 4. □

A Some Estimates taken from Bach, Fröhlich, and Sigal [5]

In this appendix we quote some results from [5] which we need for the investigation of the non-relativistic limit in Section 8. We quote the result only in the generality which we need here and would like to mention that it also holds for suitable multi-particle Schrödinger operators.
We define

$$H_\gamma(\theta) := -\frac{e^{-2\theta}}{2}\Delta - \gamma V(\theta) \tag{88}$$

as operator on $L^2(\mathbb{R}^3;\mathbb{C}^2)$ and pick some eigenvalue $E_{\tilde{n}}$. We define (with $r > 0$ small enough) $P_{\mathrm{el},n'}(\gamma;\theta) := -(2\pi\mathrm{i}\,)^{-1}\int_{|E_{n'}-z|=r}(H_\gamma(\theta) - z)^{-1}\mathrm{d}z$ as projection onto the eigenspace of $H_\gamma(\theta)$ with eigenvalue $E_{n'}$. We abbreviate $\overline{P}_{n'}(\gamma;\theta) := 1 - P_{n'}(\theta)$. For $\eta > 0$ with $E_{\tilde{n}} < -\eta < E_{\tilde{n}+1}$ we define $P_{\mathrm{disc}}(\gamma;\theta) := \sum_{i:E_i\leq-\eta} P_i(\gamma;\theta)$ and $\overline{P}_{\mathrm{disc}}(\gamma;\theta) := 1 - P_{\mathrm{disc}}(\gamma;\theta)$.
In the following, we pick a sufficiently small $\theta_0 > 0$.

Lemma A.1 ([5], Corollary 1.4.). *There is a constant $C > 0$ such that for all $|\theta| < \theta_0$ the estimate $\|[H_\gamma(\theta) - H_\gamma(0)](H_\gamma(0) \pm \mathrm{i}\,)^{-1}\| \leq C|\theta|$ holds.*

Lemma (A.1) implies

Corollary A.1 ([5], Equation (3.79)). *There is a $C > 0$ such that for all $|\theta| < \theta_0$ the estimate $\|P_n(\gamma;\theta) - P_n(\gamma;0)\| \leq C|\theta|$ holds. The same estimate is true if one replaces P_n with P_{disc}.*

Using Lemma A.1 and Corollary A.1 as well as a resolvent expansion one shows

Theorem A.1 ([5], Lemma 3.8.). *Let $z \in \mathbb{C}$ with $\mathrm{Re}\,z < \Sigma - \eta$. Then the operator $H_\gamma(\theta) - z$ is invertible on $\mathrm{Ran}\,\bar{P}_{\mathrm{disc}}(\gamma;\theta)$ for sufficiently small $|\theta|(1 + (-\eta - \mathrm{Re}\,z)^{-1})$ and the estimate*

$$\big\|(H_\gamma(\theta)|_{\bar{P}_{\mathrm{disc}}(\gamma;\theta)} - z)^{-1}\bar{P}_{\mathrm{disc}}(\gamma;\theta)\big\| \leq 2(-\eta - \mathrm{Re}\,z)^{-1}$$

holds.

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Mathias Huber
Fakultät für Mathematik
und Informatik
FernUniversität in Hagen
Lützowstr. 125
58095 Hagen
Germany
Matthias.Huber@gmx.de