Hasse Principle for G-Quadratic Forms

Eva Bayer–Fluckiger, Nivedita Bhaskhar,
and Raman Parimala

Received: August 19, 2012
Revised: February 14, 2013

Communicated by Ulf Rehmann

Abstract.

Introduction.

Let k be a global field of characteristic $\neq 2$. The classical Hasse–Minkowski theorem states that if two quadratic forms become isomorphic over all the completions of k, then they are isomorphic over k as well. It is natural to ask whether this is true for *G–quadratic forms*, where G is a finite group. In the case of number fields the Hasse principle for G–quadratic forms does not hold in general, as shown by J. Morales [M 86]. The aim of the present paper is to study this question when k is a global field of positive characteristic. We give a sufficient criterion for the Hasse principle to hold (see th. 2.1.), and also give counter–examples. These counter–examples are of a different nature than those for number fields : indeed, if k is a global field of positive characteristic, then the Hasse principle does hold for G–quadratic forms on projective $k[G]$–modules (see cor. 2.3), and in particular if $k[G]$ is semi–simple, then the Hasse principle is true for G–quadratic forms, contrarily to what happens in the case of number fields. On the other hand, there are counter–examples in the non semi–simple case, as shown in §3. Note that the Hasse principle holds in all generality for G–trace forms (cf. [BPS 13]).

The third named author is partially supported by National Science Foundation grant DMS-1001872.

§1. Definitions, notation and basic facts

Let k be a field of characteristic $\neq 2$. All modules are supposed to be left modules.

G–quadratic spaces

Let G be a finite group, and let $k[G]$ be the associated group ring. A *G–quadratic space* is a pair (V, q), where V is a $k[G]$–module that is a finite dimensional k–vector space, and $q : V \times V \to k$ is a non–degenerate symmetric bilinear form such that

$$q(gx, gy) = q(x, y)$$

for all $x, y \in V$ and all $g \in G$.

Two G–quadratic spaces (V, q) and (V', q') are *isomorphic* if there exists an isomorphism of $k[G]$–modules $f : V \to V'$ such that $q'(f(x), f(y)) = q(x, y)$ for all $x, y \in V$. If this is the case, we write $(V, q) \simeq_G (V', q')$, or simply $q \simeq_G q'$.

Hermitian forms

Let R be a ring endowed with an involution $r \mapsto \overline{r}$. For any R–module M, we denote by M^* its dual $\mathrm{Hom}_R(M, R)$. Then M^* has an R–module structure given by $(rf)(x) = f(x)\overline{r}$ for all $r \in R$, $x \in M$ and $f \in M^*$. If M and N are two R–modules and if $f : M \to N$ is a homomorphism of R–modules, then f induces a homomorphism $f^* : N^* \to M^*$ defined by $f^*(g) = gf$ for all $g \in N^*$, called the *adjoint* of f.

A *hermitian form* is a pair (M, h) where M is an R–module and $h : M \times M \to R$ is biadditive, satisfying the following two conditions:
(1.1) $h(rx, sy) = rh(x, y)\overline{s}$ and $\overline{h(x, y)} = h(y, x)$ for all $x, y \in M$ and all $r, s \in R$.
(1.2) The homomorphism $h : M \to M^*$ given by $y \mapsto h(\ , y)$ is an isomorphism.

Note that the existence of h implies that M is *self-dual*, i.e. isomorphic to its dual.

If G is a finite group, then the group algebra $R = k[G]$ has a natural k–linear involution, characterized by the formula $\overline{g} = g^{-1}$ for every $g \in G$. We have the following dictionary (see for instance [BPS 13, 2.1, Example]
a) R–module $M \iff k$-module M with a k–linear action of G;
b) R–dual $M^* \iff k$–dual of M, with the contragredient (i.e. dual) action of G.
c) hermitian space $(M, h) \iff$ symmetric bilinear form on M, which is G–invariant and defines an isomorphism of M onto its k–dual.

Therefore a hermitian space over $k[G]$ corresponds to a G–quadratic space, as defined above.

Hermitian elements

Let E be a ring with an involution $\sigma : E \to E$ and put

$$E^0 = \{z \in E^\times \mid \sigma(z) = z\}.$$

If $z \in E^0$, the map $h_z : E\times E \to E$ defined by $h_z(x,y) = x.z.\sigma(y)$ is a hermitian space over E; conversely, every hermitian space over E with underlying module E is isomorphic to h_z for some $z \in E^0$.

Define an equivalence relation on E^0 by setting $z \equiv z'$ if there exists $e \in E^\times$ with $z' = \sigma(e)ze$; this is equivalent to $(E, h_z) \simeq (E, h_{z'})$. Let $H(E,\sigma)$ be the quotient of E^0 by this equivalence relation. If $z \in E^0$, we denote by $[z]$ its class in $H(E,\sigma)$.

Classifying hermitian spaces via hermitian elements

Let (M, h_0) be a hermitian space over R. Set $E_M = \mathrm{End}(M)$. Let $\tau : E_M \to E_M$ be the involution of E_M *induced* by h_0, i.e.

$$\tau(e) = h_0^{-1} e^* h_0, \quad \text{for} \quad e \in E_M,$$

where e^* is the adjoint of e. If (M,h) is a hermitian space (with the same underlying module M), we have $\tau(h_0^{-1}h) = h_0^{-1}(h_0^{-1}h)^* h_0 = h_0^{-1}h^*(h_0^{-1})^* h_0 = h_0^{-1}h$. Hence $h_0^{-1}h$ is a hermitian element of (E_M,τ); let $[h_0^{-1}h]$ be its class in $H(E_M,\tau)$.

LEMMA 1.1. (see for instance [BPS 13, lemma 3.8.1]) *Sending a hermitian space (M,h) to the element $[h_0^{-1}h]$ of $H(E_M,\tau)$ induces a bijection between the set of isomorphism classes of hermitian spaces (M,h) and the set $H(E_M,\tau)$.*

Components of algebras with involution

Let A be a finite dimensional k–algebra, and let $\iota : A \to A$ be a k–linear involution. Let R_A be the radical of A. Then A/R_A is a semi–simple k–algebra, hence we have a decomposition $A/R_A = \prod_{i=1,\dots,r} M_{n_i}(D_i)$, where $D_1,\dots,D_r$ are division algebras. Let us denote by K_i the center of D_i, and let D_i^{op} be the opposite algebra of D_i.

Note that $\iota(R_A) = R_A$, hence ι induces an involution $\iota : A/R_A \to A/R_A$. Therefore A/R_A decomposes into a product of involution invariant factors. These can be of two types : either an involution invariant matrix algebra $M_{n_i}(D_i)$, or a product $M_{n_i}(D_i) \times M_{n_i}(D_i^{op})$, with $M_{n_i}(D_i)$ and $M_{n_i}(D_i^{op})$ exchanged by the involution. We say that a factor is *unitary* if the restriction of the involution to its center is not the identity : in other words, either an involution invariant $M_{n_i}(D_i)$ with $\iota|K_i$ not the identity, or a product $M_{n_i}(D_i)\times M_{n_i}(D_i^{op})$. Otherwise, the factor is said to be of the first kind. In this case, the component is of the form $M_{n_i}(D_i)$ and the restriction of ι to K_i is the identity. We say that the component is *orthogonal* if after base change to a separable closure ι is given by the transposition, and *symplectic* otherwise. A component $M_{n_i}(D_i)$ is said to be *split* if D_i is a commutative field.

Completions

If k is a global field and if v is a place of k, we denote by k_v the completion of k at v. For any k–algebra E, set $E_v = E \otimes_k k_v$. If K/k is a field extension of finite degree and if w is a place of K above v, then we use the notation $w|v$.

§2. Hasse principle

In this section, k will be a global field of characteristic $\neq 2$. Let us denote by Σ_k the set of all places of k. The aim of this section is to give a sufficient criterion for the Hasse principle for G–quadratic forms to hold. All modules are left modules, and finite dimensional k–vector spaces.

Theorem 2.1. *Let V be a $k[G]$–module , and let $E = \mathrm{End}(V)$. Let R_E be the radical of E, and set $\overline{E} = E/R_E$. Suppose that all the orthogonal components of $\overline{E}$ are split, and let (V,q), (V,q') be two G–forms. Then $q \simeq_G q'$ over k if and only if $q \simeq_G q'$ over all the completions of k.*

This is announced in [BP 13], and replaces th. 3.5 of [BP 11]. The proof of th. 2.1 relies on the following proposition

Proposition 2.2. *Let E be a finite dimensional k–algebra endowed with a k–linear involution $\sigma : E \to E$. Let R_E be the radical of E, and set $\overline{E} = E/R_E$. Suppose that all the orthogonal components of $\overline{E}$ are split. Then the canonical map $H(E,\sigma) \to \prod_{v\in\Sigma_k} H(E_v,\sigma_v)$ is injective.*

Proof. *The case of a simple algebra.* Suppose first that E is a simple k–algebra. Let K be the center of E, and let F be the fixed field of σ in K. Let Σ_F denote the set of all places of F. For all $v \in \Sigma_k$, set $E_v = E \otimes_k k_v$, and note that $E_v = \prod_{w|v} E_w$, therefore $\prod_{v\in\Sigma_k} H(E_v,\sigma_v) = \prod_{w\in\Sigma_F} H(E_w,\sigma_w)$. By definition, $H(E,\sigma)$ is the set of isomorphism classes of one dimensional hermitian forms over E. Moreover, if σ is orthogonal, then the hypothesis implies that E is split, in other words we have $E \simeq M_n(F)$. Therefore the conditions of [R 11, th. 3.3.1] are fulfilled, hence the Hasse principle holds for hermitian forms over E with respect to σ. This implies that the canonical map $H(E,\sigma) \to \prod_{v\in\Sigma_k} H(E_v,\sigma_v)$ is injective.

The case of a semi–simple algebra. Suppose now that E is semi–simple. Then

$$E \simeq E_1 \times \ldots \times E_r \times A \times A^{\mathrm{op}},$$

where $E_1,\ldots,E_r$ are simple algebras which are stable under the involution σ, and where the restriction of σ to $A \times A^{\mathrm{op}}$ exchanges the two factors. Applying [BPS 13, lemmas 3.7.1 and 3.7.2] we are reduced to the case where E is a simple algebra, and we already know that the result is true in this case.

General case. We have $\overline{E} = E/R_E$. Then $\overline{E}$ is semi–simple, and σ induces a k–linear involution $\overline{\sigma} : \overline{E} \to \overline{E}$. We have the following commutative diagram

$$\begin{array}{ccc} H(E,\sigma) & \xrightarrow{f} & \prod_{v\in\Sigma_k} H(E_v,\sigma) \\ \downarrow & & \downarrow \\ H(\overline{E},\overline{\sigma}) & \xrightarrow{\overline{f}} & \prod_{v\in\Sigma_k} H(\overline{E}_v,\overline{\sigma}), \end{array}$$

where the vertical maps are induced by the projection $E \to \overline{E}$. By [BPS 13, lemma 3.7.3], these maps are bijective. As $\overline{E}$ is semi–simple, the map $\overline{f}$ is injective, hence f is also injective. This concludes the proof.

Proof of th. 2.1. It is clear that if $q \simeq_G q'$ over k, then $q \simeq_G q'$ over all the completions of k. Let us prove the converse. Let (V, h) be the $k[G]$-hermitian space corresponding to (V, q), and let $\sigma : E \to E$ be the involution induced by (V, h) as in §1. Let (V, h') be the $k[G]$–hermitian space corresponding to (V, q'), and set $u = h^{-1}h'$. Then $u \in E^0$, and by lemma 1.1. the element $[u] \in H(E, \sigma)$ determines the isomorphism class of (V, q'); in other words, we have $q \simeq_G q'$ if and only if $[u] = [1]$ in $H(E, \sigma)$. Hence the theorem is a consequence of proposition 2.2.

Corollary 2.3 *Suppose that* $\mathrm{char}(k) = p > 0$*, and let* V *be a projective* $k[G]$*–module.. Let* (V, q)*,* (V, q') *be two* G*–forms. Then* $q \simeq_G q'$ *over* k *if and only if* $q \simeq_G q'$ *over all the completions of* k*.*

Proof. Since V is projective, there exists a $k[G]$–module W and $n \in N$ such that $V \oplus W \simeq k[G]^n$. The endomorphism ring of $k[G]^n$ is $M_n(k[G])$, and as $\mathrm{char}(k) = p > 0$, we have $k[G] = F_p[G] \otimes_{F_p} k$. Hence $M_n(k[G])$ is isomorphic to $M_n(F_p[G]) \otimes_{F_p} k$. Let $E = \mathrm{End}(V)$, let R_E be the radical of E, and let $\overline{E} = E/R_E$. Let us show that all the components of $\overline{E}$ are split. Let e be the idempotent endomorphism of $V \oplus W$ which is the identity of V. Set $\Lambda = \mathrm{End}(V \oplus W)$ and let R_Λ be the radical of Λ. Then $e\Lambda e = E$ and $eR_\Lambda e = R_E$. Set $\overline{\Lambda} = \Lambda/R_\Lambda$, and and let $\overline{e}$ be the image of e in $\overline{\Lambda}$. Set $\overline{k[G]} = k[G]/\mathrm{rad}(k[G])$. Then we have $\overline{E} \simeq \overline{e}\overline{\Lambda}\overline{e} \simeq \overline{e}M_n(\overline{k[G]})\overline{e}$. This implies that $\overline{E}$ is a component of the semi–simple algebra $M_n(\overline{k[G]})$. Let us show that all the components of $M_n(\overline{k[G]})$ are split. As F_p is a finite field, $F_p[G]/(\mathrm{rad}(F_p[G])$ is a product of matrix algebras over finite fields. Moreover, for any finite field F of characteristic p, the tensor product $F \otimes_{F_p} k$ is a product of fields. This shows that $(F_p[G]/(\mathrm{rad}(F_p[G])) \otimes_{F_p} k$ is a product of matrix algebras over finite extensions of k; in particular, it is semi–simple. The natural isomorphism $F_p[G] \otimes_{F_p} k \to k[G]$ induces an isomorphism $[F_p[G]/(\mathrm{rad}(F_p[G]))] \otimes_{F_p} k \to k[G]/(\mathrm{rad}(F_p[G]).k[G])$. Therefore $\mathrm{rad}(F_p[G].k[G])$ is the radical of $k[G]$, and we have an isomorphism $[F_p[G]/(\mathrm{rad}(F_p[G]))] \otimes_{F_p} k \to k[G]/(\mathrm{rad}(k[G]))$. Hence all the components of $k[G]/(\mathrm{rad}(k[G]))$ are split. This implies that all the components of $\overline{E}$ are split as well. Therefore the corollary follows from th. 2.1.

The following corollary is well–known (see for instance [R 11, 3.3.1 (b)]).

Corollary 2.4 *Suppose that* $\mathrm{char}(k) = p > 0$*, and that the order of* G *is prime to* p*. Then two* G*–quadratic forms are isomorphic over* k *if and only if they become isomorphic over all the completions of* k*.*

Proof. This follows immediately from cor. 2.3.

§3. Counter–examples to the Hasse principle

Let k be a field of characteristic $p > 0$, let C_p be the cyclic group of order p, and let $G = C_p \times C_p \times C_p$. In this section we give counter–examples to the Hasse principle for $G \times G$–quadratic forms over k in the case where k is a global field. We start with some constructions that are valid for any field of positive characteristic.

3.1 A construction

Let D be a division algebra over k. It is well–known that there exist indecomposable $k[G]$–modules such that their endomorphism ring modulo the radical is isomorphic to D. We recall here such a constuction, brought to our attention by R. Guralnick, in order to use it in 3.2 in the case of quaternion algebras.

The algebra D can be generated by two elements (see for instance [J 64, Chapter VII, §12, th. 3, p. 182]). Let us choose $i, j \in D$ be two such elements. Let us denote by D^{op} the opposite algebra of D, and let d be the degree of D. Then we have $D \otimes_k D^{\mathrm{op}} \simeq M_{d^2}(k)$. Let us choose an isomorphism $f : D \otimes_k D^{\mathrm{op}} \simeq M_{d^2}(k)$, and set $a_1 = f(1 \otimes 1) = 1$, $a_2 = f(i \otimes 1)$ and $a_3 = f(j \otimes 1)$.

Let $g_1, g_2, g_3 \in G$ be three elements of order p such that the set $\{g_1, g_2, g_3\}$ generates G.and let us define a representation $G \to \mathrm{GL}_{2d^2}(k)$ by sending g_m to the matrix

$$\begin{pmatrix} I & a_m \\ 0 & I \end{pmatrix}$$

for all $m = 1, 2, 3$. Note that this is well–defined because $\mathrm{char}(k) = p$. This endowes k^{2d^2} with a structure of $k[G]$–module. Let us denote by N this $k[G]$–module, and let E_N be its endomorphism ring. Then

$$E_N = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x \in D^{\mathrm{op}} \subset M_{d^2}(k),\ y \in M_{d^2}(k) \right\},$$

and its radical is

$$R_N = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in M_{d^2}(k) \right\},$$

hence $E_N / R_N \simeq D^{\mathrm{op}}$.

3.2. The case of a quaternion algebra

Let H be a quaternion algebra over k. Then by 3.1, we get a $k[G]$–module $N = N_H$ with endomorphism ring E_N such that $E_N/R_N \simeq H^{\mathrm{op}}$, where R_N is the radical of E_N. We now construct a G–quadratic form q over N in such a way that the involution it induces on $E_N/R_N \simeq H^{\mathrm{op}}$ is the canonical involution.

Let $i, j \in H$ such that $i^2, j^2 \in k^\times$ and that $ij = -ji$. Let $\tau : H \to H$ be the orthogonal involution of H obtained by composing the canonical involution of H with $\mathrm{Int}(ij)$. Let $\sigma : H^{\mathrm{op}} \to H^{\mathrm{op}}$ be the canonical involution of H^{op}. Let us consider the tensor product of algebras with involution

$$(H, \tau) \otimes (H^{\mathrm{op}}, \sigma) = (M_4(k), \rho).$$

Then ρ is a symplectic involution of $M_4(k)$ satisfying $\rho(a_m) = a_m$ for all $m = 1, 2, 3$, since $\tau(i) = (ij)(-i)(ij)^{-1} = i$, $\tau(j) = (ij)(-j)(ij)^{-1} = j$. Let $\alpha \in M_4(k)$ be a skew–symmetric matrix such that for all $x \in M_4(k)$, we have $\rho(x) = \alpha^{-1} x^T \alpha$, where x^T denotes the transpose of x. Set $A = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$. Then $A^T = A$. Let $q : N \times N \to k$ be the symmetric bilinear form defined by A :

$$q(v, w) = v^T A w$$

for all $v, w \in N$. Let $\gamma : M_8(k) \to M_8(k)$ be the involution adjoint to q, that is

$$\gamma(X) = A^{-1} X^T A$$

for all $X \in M_8(k)$, i.e. $q(fv, w) = q(v, \gamma(f)w)$ for all $f \in M_8(k)$ and all $v, w \in N$. The involution γ restricts to an involution of E_N, as for all $x, y \in M_4(k)$, we have

$$\gamma \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = \begin{pmatrix} \alpha^{-1} x^T \alpha & -\alpha^{-1} y^T \alpha \\ 0 & \alpha^{-1} x^T \alpha \end{pmatrix}.$$

It also sends R_N to itself, and induces an involution $\overline{\gamma}$ on $H^{\mathrm{op}} \simeq E_N/R_N$ that coincides with the canonical involution of H^{op}.

We claim that $q : N \times N \to k$ is a G–quadratic form. To check this, it suffices to show that $q(g_m v, g_m w) = q(v, w)$ for all $v, w \in N$ and for all $m = 1, 2, 3$. Since $\rho(a_m) = a_m$ for all $m = 1, 2, 3$, we have

$$\gamma \begin{pmatrix} I & a_m \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & a_m \\ 0 & I \end{pmatrix}^{-1}$$

and hence

$$q(g_m v, g_m w) = q(v, \gamma(g_m) g_m w) = q(v, w)$$

for all $m = 1, 2, 3$ and all $v, w \in N$. Thus q is a G–quadratic form, and by construction, the involution of E_N induced by q is the restriction of γ to E_N.

3.3. Two quaternion algebras

Let H_1 and H_2 be two quaternion algebras over k. By the construction of 3.2, we obtain two indecomposable $k[G]$–modules N_1 and N_2. Set $E_1 = E_{N_1}$ and $E_2 = E_{N_2}$. Let R_i be the radical of E_i for $i = 1, 2$, and set $\overline{E}_i = E_i/R_i$. We also obtain G–quadratic spaces $q_i : N_i \times N_i \to k$ inducing involutions $\gamma_i : E_i \to E_i$ such that the involutions $\overline{\gamma}_i : \overline{E}_i \to \overline{E}_i$ coincide with the canonical involution of H_i^{op}, for all $i = 1, 2$.

Let us consider the tensor product $(N, q) = (N_1, q_1) \otimes_k (N_2, q_2)$. Then (N, q) is a $G \times G$–quadratic space. Set $E = \mathrm{End}_{k[G\times G]}(N_1 \otimes N_2)$. Then $E \simeq E_1 \otimes E_2$. Let I be the ideal of E generated by R_1 and R_2. Then there is a natural isomorphism $f : E_1 \otimes E_2 \to E$ with $f(I) = R_E$, where R_E is the radical of E. Set $\overline{E} = E/R_E$. Then $\overline{E} \simeq \overline{E}_1 \otimes \overline{E}_2 \simeq H_1^{\mathrm{op}} \otimes H_2^{\mathrm{op}}$.

Set $\gamma = \gamma_1 \otimes \gamma_2$. Then $\gamma : E \to E$ is the involution induced by the $G \times G$–quadratic space (N, q). We obtain an involution $\overline{\gamma} : \overline{E} \to \overline{E}$, and $\overline{\gamma} = \overline{\gamma}_1 \otimes \overline{\gamma}_2$. Let us recall that $\overline{E}_i = H_i^{\mathrm{op}}$ for $i = 1, 2$, and that $\overline{\gamma}_i$ is the canonical involution of H_i^{op}. Hence $\overline{\gamma} : \overline{E} \to \overline{E}$ is an orthogonal involution.

3.4. A counter–example to the Hasse principle

Suppose now that k is a global field of characteristic p, with $p > 2$, and suppose that H_i is ramified at exactly two places v_i, v_i' of k, such that v_1, v_1', v_2, v_2' are all distinct. We have $H_1^{\mathrm{op}} \otimes H_2^{\mathrm{op}} \simeq M_2(Q)$ where Q is a quaternion division algebra over k, and Q is ramified exactly at the places v_1, v_1', v_2, v_2' of k. Recall that the involution $\overline{\gamma} : M_2(Q) \to M_2(Q)$ is the tensor product of the canonical involutions of H_i^{op}. In particular, $\overline{\gamma}$ is of orthogonal type. Note that at all $v \in \Sigma_k$, one of the algebras H_1^{op} or H_2^{op} is split. This implies that at all $v \in \Sigma_k$, the involution $\overline{\gamma}$ is hyperbolic.

Let $\delta : Q \to Q$ be an orthogonal involution of the division algebra Q. Then $\overline{\gamma}$ is induced by some hermitian space $h : Q^2 \times Q^2 \to Q$ with respect to the involution δ. As for all $v \in \Sigma_k$, the involution $\overline{\gamma}$ is hyperbolic at v, the hermitian form h is also hyperbolic at v. By lemma 1.1 the set of isomorphism classes of hermitian spaces on Q^2 is in bijection with the set $H(\overline{E}, \overline{\gamma})$, the hermitian space (Q^2, h) corresponding to the element $[1] \in H(\overline{E}, \overline{\gamma})$.

Let (Q^2, h') be a hermitian space which becomes isomorphic to (Q^2, h) over Q_v for all $v \in \Sigma_k$, but is not isomorphic to (Q^2, h) over Q (this is possible by [Sch 85, 10.4.6]). Let $u \in \overline{E}^0$ such that $[u] \in H(\overline{E}, \overline{\gamma})$ corresponds to (Q^2, h') by the bijection of lemma 1.1. Then $[u] \neq [1] \in H(\overline{E}, \overline{\gamma})$, and the images of $[u]$ and $[1]$ coincide in $\prod_{v\in\Sigma_k} H(\overline{E}_v, \overline{\gamma})$.

Recall that $H(E, \gamma)$ is in bijection with the isomorphism classes of $(G \times G)$–quadratic forms over N, the element $[1] \in H(E, \gamma)$ corresponding to the isomorphism class of (N, q). Let $\pi : E \to \overline{E}$ be the projection, and let $\tilde{u} \in E^0$ be such that $\pi(\tilde{u}) = u$ (cf. lemma 1.1). Let (N, q') be a $(G \times G)$–quadratic form corresponding to $\tilde{u}$. The diagram

$$\begin{array}{ccc} H(E,\gamma) & \xrightarrow{f} & \prod_{v\in\Sigma_k} H(E_v,\gamma) \\ \downarrow & & \downarrow \\ H(\overline{E},\overline{\gamma}) & \xrightarrow{\overline{f}} & \prod_{v\in\Sigma_k} H(\overline{E}_v,\overline{\gamma}), \end{array}$$

is commutative, and the vertical maps are bijective by [BPS 13, lemma 3.7.3]. Hence (N, q) and (N, q') are become isomorphic over all the completions of k, but are not isomorphic over k.

Bibliography

[BP 11] E. Bayer–Fluckiger, R. Parimala, Galois algebras, Hasse principle and induction–restriction methods, *Documenta Math.* 16 (2011), 677–707.

[BP 13] E. Bayer–Fluckiger, R. Parimala, Correction to : Galois algebras, Hasse principle and induction–restriction methods, *Documenta Math.* 16 (2011), 677–707 (to appear).

[BPS 13] E. Bayer–Fluckiger, R. Parimala and J-P. Serre, Hasse principle for G–trace forms, *Izvestia Math.* (2013), to appear.

[J 64] N. Jacobson, *Structure of rings*, AMS Colloquium Series XXXVII (1964).

[KMRT 98] M. Knus, A. Merkurjev, M. Rost & J–P. Tignol, *The Book of Involutions*, AMS Colloquium Publications 44, 1998.

[M 86] J. Morales, Integral bilinear forms with a group action, *J. Algebra* 98 (1986), 470 - 484.

[R 11] C.R. Riehm, *Introduction to Orthogonal, Symplectic and Unitary Representations of Finite Groups*, Fields Institute Monographies 28, AMS, 2011.

[Sch 85] W. Scharlau, *Quadratic and Hermitian Forms*, Grundlehren der Math. Wiss, Springer–Verlag (1985).

Eva Bayer–Fluckiger
École Polytechnique Fédérale
de Lausanne
EPFL/FSB/MATHGEOM/CSAG
Station 8
1015 Lausanne, Switzerland
eva.bayer@epfl.ch

Nivedita Bhaskhar
Department of Mathematics
& Computer Science
Emory University
Atlanta, GA 30322, USA.
nbhaskh@emory.edu

Raman Parimala
Department of Mathematics
& Computer Science
Emory University
Atlanta, GA 30322, USA.
parimala@mathcs.emory.edu