On Additive Higher Chow Groups of Affine Schemes

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Abstract. We show that the multivariate additive higher Chow groups of a smooth affine k-scheme $\mathrm{Spec}\,(R)$ essentially of finite type over a perfect field k of characteristic $\neq 2$ form a differential graded module over the big de Rham-Witt complex $\mathbb{W}_m\Omega^\bullet_R$. In the univariate case, we show that additive higher Chow groups of $\mathrm{Spec}\,(R)$ form a Witt-complex over R. We use these structures to prove an étale descent for multivariate additive higher Chow groups.

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1. Introduction

The additive higher Chow groups $\mathrm{TCH}^q(X, n; m)$ emerged originally in [5] in part as an attempt to understand certain relative higher algebraic K-groups of schemes in terms of algebraic cycles. Since then, several papers [16], [17], [18], [19], [26], [27], [28] have studied various aspects of these groups. But lack of a suitable moving lemma for smooth *affine* varieties has been a hindrance in studies of their local behaviors. Its projective sibling was known by [17]. During the period of stagnation, the subject has evolved into the notion of 'cycles with modulus' $\mathrm{CH}^q(X|D, n)$ by Binda-Kerz-Saito in [1], [15] associated to pairs (X, D) of schemes and effective Cartier divisors D, setting a more flexible ground, while this desired moving lemma for the affine case was obtained by W. Kai [14] (See Theorem 4.1).

The above developments now propel the authors to continue their program of realizing the relative K-theory $K_n(X \times \mathrm{Spec}\, k[t]/(t^{m+1}), (t))$ in terms of additive higher Chow groups. More specifically, one of the aims in the program considered in this paper is to understand via additive higher Chow groups, the part of the above relative K-groups which was proven in [2] to give the crystalline cohomology. This part turned out to be isomorphic to the de Rham-Witt complexes as seen in [12]. This article is the first of the authors' papers that relate the additive higher Chow groups to the big de Rham-Witt complexes $\mathbb{W}_m\Omega^\bullet_R$ of [8] and to the crystalline cohomology theory. This gives a motivic description of the latter two objects.

While the general notion of cycles with modulus for (X, D) provides a wider picture, the additive higher Chow groups still have a non-trivial operation not shared by the general case. One such is an analogue of the Pontryagin product on homology groups of Lie groups, which turns the additive higher Chow groups into a differential graded algebra (DGA). This product is induced by the structure of algebraic groups on $\mathbb{A}^1$ and $\mathbb{G}_m$ and their action on $X \times \mathbb{A}^r =: X[r]$ for $r \geq 1$.

The usefulness of such a product was already observed in the earliest papers on additive 0-cycles by Bloch-Esnault [5] and Rülling [28]. This product on higher dimensional additive higher Chow cycles was given in [19] for smooth projective varieties. In §5 of this paper, we extend this product structure in two directions: (1) toward multivariate additive higher Chow groups and (2) on smooth affine varieties. In doing so, we generalize some of the necessary tools, such as the following normalization theorem, proven as Theorem 3.2. Necessary definitions are recalled in §2.

Theorem 1.1. *Let X be a smooth scheme which is either quasi-projective or essentially of finite type over a field k. Let D be an effective Cartier divisor on X. Then each cycle class in $\mathrm{CH}^q(X|D, n)$ has a representative, all of whose codimension 1 faces are trivial.*

The above theorem for ordinary higher Chow groups was proven by Bloch and has been a useful tool in dealing with algebraic cycles. In this paper, we use the above theorem to construct the following structure of differential graded algebra and differential graded modules on the multivariate additive higher Chow groups, where Theorem 1.2 is proven in Theorems 7.1, 7.10, and 7.11, while Theorem 1.3 is proven in Theorem 6.13.

Theorem 1.2. *Let X be a smooth scheme which is either affine essentially of finite type or projective over a perfect field k of characteristic $\neq 2$*

1. *The additive higher Chow groups $\{\mathrm{TCH}^q(X, n; m)\}_{q,n,m\in\mathbb{N}}$ has a functorial structure of a restricted Witt-complex over k.*
2. *If $X = \mathrm{Spec}\,(R)$ is affine, then $\{\mathrm{TCH}^q(X, n; m)\}_{q,n,m\in\mathbb{N}}$ has a structure of a restricted Witt-complex over R.*
3. *For X as in (2), there is a natural map of restricted Witt-complexes $\tau^R_{n,m} : \mathbb{W}_m\Omega^{n-1}_R \to \mathrm{TCH}^n(R, n; m)$.*

Theorem 1.3. *Let $r \geq 1$. For a smooth scheme X which is either affine essentially of finite type or projective over a perfect field k of characteristic $\neq 2$, the multivariate additive higher Chow groups $\{\mathrm{CH}^q(X[r]|D_{\underline{m}}, n)\}_{q,n\geq 0}$ with modulus $\underline{m} = (m_1, \cdots, m_r)$, where $m_i \geq 1$, form a differential graded module over* *the DGA* $\{\mathrm{TCH}^q(X, n; |\underline{m}| - 1)\}_{q,n\geq 1}$*, where* $|\underline{m}| = \sum_{i=1}^{r} m_i$*. In particular, each* $\mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$ *is a* $\mathbb{W}_{(|\underline{m}|-1)}(R)$*-module, when* $X = \mathrm{Spec}\,(R)$ *is affine.*

The above structures on the univariate and multivariate additive higher Chow groups suggest an expectation that these groups may describe the algebraic K-theory relative to nilpotent thickenings of the coordinate axes in an affine space over a smooth scheme. The calculations of such relative K-theory by Hesselholt in [9] and [10] show that any potential motivic cohomology which describes the above relative K-theory may have such a structure.

As part of our program of connecting the additive higher Chow groups with the relative K-theory, we show in [22] that the above map $\tau^R_{n,m}$ is an isomorphism when X is semi-local in addition, and we show how one deduces crystalline cohomology from additive higher Chow groups. The results of this paper form a crucial part in the process.

Recall that the higher Chow groups of Bloch and algebraic K-theory do not satisfy étale descent with integral coefficients. As an application of Theorem 1.3, we show that the étale descent is actually true for the multivariate additive higher Chow groups in the following setting:

Theorem 1.4. *Let* $r \geq 1$ *and let* X *be a smooth scheme which is either affine essentially of finite type or projective over a perfect field* k *of characteristic* $\neq 2$*. Let* G *be a finite group of order prime to* $\mathrm{char}(k)$*, acting freely on* X *with the quotient* $f : X \to X/G$*. Then for all* $q, n \geq 0$ *and and* $\underline{m} = (m_1, \cdots, m_r)$ *with* $m_i \geq 1$ *for* $1 \leq i \leq r$*, the pull-back map* f^* *induces an isomorphism*

$$\mathrm{CH}^q(X/G[r]|D_{\underline{m}}, n) \xrightarrow{\simeq} \mathrm{H}^0(G, \mathrm{CH}^q(X[r]|D_{\underline{m}}, n)).$$

Note that the quotient X/G exists under the hypothesis on X. Since the corresponding descent is not yet known for the relative K-theory of nilpotent thickenings of the coordinate axes in an affine space over a smooth scheme, the above theorem suggests that this descent could be indeed true for the relative K-theory.

Conventions. In this paper, k will denote the base field which will be assumed to be perfect after §4. A k-scheme is a separated scheme of finite type over k. A k-variety is a reduced k-scheme. The product $X \times Y$ means usually $X \times_k Y$, unless said otherwise. We let $\mathbf{Sch}_k$ be the category of k-schemes, $\mathbf{Sm}_k$ of smooth k-schemes, and $\mathbf{SmAff}_k$ of smooth affine k-schemes. A scheme essentially of finite type is a scheme obtained by localizing at a finite subset (including $\emptyset$) of a finite type k-scheme. For $\mathcal{C} = \mathbf{Sch}_k, \mathbf{Sm}_k, \mathbf{SmAff}_k$, we let $\mathcal{C}^{\mathrm{ess}}$ be the extension of the category $\mathcal{C}$ obtained by localizing at a finite subset (including $\emptyset$) of objects in $\mathcal{C}$. We let $\mathbf{SmLoc}_k$ be the category of smooth semi-local k-schemes essentially of finite type over k. So, $\mathbf{SmAff}^{\mathrm{ess}}_k = \mathbf{SmAff}_k \cup \mathbf{SmLoc}_k$ for the objects. When we say a semi-local k-scheme, we always mean one that is essentially of finite type over k. Let $\mathbf{SmProj}_k$ be the category of smooth projective k-schemes.

2. Recollection of basic definitions

For $\mathbb{P}^1 = \mathrm{Proj}_k(k[s_0, s_1])$, we let $y = s_1/s_0$ its coordinate. Let $\square := \mathbb{P}^1 \setminus \{1\}$. For $n \geq 1$, let $(y_1, \cdots, y_n) \in \square^n$ be the coordinates. A face $F \subset \square^n$ means a closed subscheme defined by the set of equations of the form $\{y_{i_1} = \epsilon_1, \cdots, y_{i_s} = \epsilon_s\}$ for an increasing sequence $\{i_j | 1 \leq j \leq s\} \subset \{1, \cdots, n\}$ and $\epsilon_j \in \{0, \infty\}$. We allow $s = 0$, in which case $F = \square^n$. Let $\overline{\square} := \mathbb{P}^1$. A face of $\overline{\square}^n$ is the closure of a face in $\square^n$. For $1 \leq i \leq n$, let $F^1_{n,i} \subset \overline{\square}^n$ be the closed subscheme given by $\{y_i = 1\}$. Let $F^1_n := \sum_{i=1}^n F^1_{n,i}$, which is the cycle associated to the closed subscheme $\overline{\square}^n \setminus \square^n$. Let $\square^0 = \overline{\square}^0 := \mathrm{Spec}\,(k)$. Let $\iota_{n,i,\epsilon} : \square^{n-1} \hookrightarrow \square^n$ be the inclusion $(y_1, \cdots, y_{n-1}) \mapsto (y_1, \cdots, y_{i-1}, \epsilon, y_i, \cdots, y_{n-1})$.

2.1. Cycles with modulus. Let $X \in \mathbf{Sch}^{\mathrm{ess}}_k$. Recall ([21, §2]) that for effective Cartier divisors D_1 and D_2 on X, we say $D_1 \leq D_2$ if $D_1 + D = D_2$ for some effective Cartier divisor D on X. A *scheme with an effective divisor* (sed) is a pair (X, D), where $X \in \mathbf{Sch}^{\mathrm{ess}}_k$ and D an effective Cartier divisor. A morphism $f : (Y, E) \to (X, D)$ of seds is a morphism $f : Y \to X$ in $\mathbf{Sch}^{\mathrm{ess}}_k$ such that $f^*(D)$ is defined as a Cartier divisor on Y and $f^*(D) \leq E$. In particular, $f^{-1}(D) \subset E$. If $f : Y \to X$ is a morphism of k-schemes, and (X, D) is a sed such that $f^{-1}(D) = \emptyset$, then $f : (Y, \emptyset) \to (X, D)$ is a morphism of seds.

Definition 2.1 ([1], [15]). Let (X, D) and $(\overline{Y}, E)$ be schemes with effective divisors. Let $Y = \overline{Y} \setminus E$. Let $V \subset X \times Y$ be an integral closed subscheme with closure $\overline{V} \subset X \times \overline{Y}$. We say V *has modulus D (relative to E)* if $\nu_V^*(D \times \overline{Y}) \leq \nu_V^*(X \times E)$ on $\overline{V}^N$, where $\nu_V : \overline{V}^N \to \overline{V} \hookrightarrow X \times \overline{Y}$ is the normalization followed by the closed immersion.

Recall the following containment lemma from [21, Proposition 2.4] (see also [1, Lemma 2.1] and [17, Proposition 2.4]):

Proposition 2.2. *Let (X, D) and $(\overline{Y}, E)$ be schemes with effective divisors and $Y = \overline{Y} \setminus E$. If $V \subset X \times Y$ is a closed subscheme with modulus D relative to E, then any closed subscheme $W \subset V$ also has modulus D relative to E.*

Definition 2.3 ([1], [15]). Let (X, D) be a scheme with an effective divisor. For $s \in \mathbb{Z}$ and $n \geq 0$, let $\underline{z}_s(X|D, n)$ be the free abelian group on integral closed subschemes $V \subset X \times \square^n$ of dimension $s + n$ satisfying the following conditions:

(1) (Face condition) for each face $F \subset \square^n$, V intersects $X \times F$ properly.
(2) (Modulus condition) V has modulus D relative to F^1_n on $X \times \square^n$.

We usually drop the phrase "relative to F^1_n" for simplicity. A cycle in $\underline{z}_s(X|D, n)$ is called an *admissible cycle with modulus D*. One checks that $(n \mapsto \underline{z}_s(X|D, n))$ is a cubical abelian group. In particular, the groups $\underline{z}_s(X|D, n)$ form a complex with the boundary map $\partial = \sum_{i=1}^n (-1)^i (\partial_i^\infty - \partial_i^0)$, where $\partial_i^\epsilon = \iota^*_{n,i,\epsilon}$.

Definition 2.4 ([1], [15]). The complex $(z_s(X|D, \bullet), \partial)$ is the nondegenerate complex associated to $(n \mapsto \underline{z}_s(X|D, n))$, i.e., $z_s(X|D, n) :=$

$\underline{z}_s(X|D,n)/\underline{z}_s(X|D,n)_{\rm degn}$. The homology $\mathrm{CH}_s(X|D,n) := \mathrm{H}_n(z_s(X|D,\bullet))$ for $n \geq 0$ is called *higher Chow group of X with modulus D*. If X is equidimensional of dimension d, for $q \geq 0$, we write $\mathrm{CH}^q(X|D,n) = \mathrm{CH}_{d-q}(X|D,n)$.

Here is a special case from [21]:

Definition 2.5. Let $X \in \mathbf{Sch}_k^{\rm ess}$. For $r \geq 1$, let $X[r] := X \times \mathbb{A}^r$. When $(t_1, \cdots, t_r) \in \mathbb{A}^r$ are the coordinates, and $m_1, \cdots, m_r \geq 1$ are integers, let $D_{\underline{m}}$ be the divisor on $X[r]$ given by the equation $\{t_1^{m_1} \cdots t_r^{m_r} = 0\}$. The groups $\mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$ are called *multivariate additive higher Chow groups* of X. For simplicity, we often say "a cycle with modulus $\underline{m}$" for "a cycle with modulus $D_{\underline{m}}$." For an r-tuple of integers $\underline{m} = (m_1, \cdots, m_r)$, we write $|\underline{m}| = \sum_{i=1}^r m_i$. We shall say that $\underline{m} \geq p$ if $m_i \geq p$ for each i.
When $r = 1$, we obtain additive higher Chow groups, and as in [19], we often use the older notations $\mathrm{Tz}^q(X, n+1; m-1)$ for $z^q(X[1]|D_m, n)$ and $\mathrm{TCH}^q(X, n+1; m-1)$ for $\mathrm{CH}^q(X[1]|D_m, n)$. In such cases, note that the modulus m is shifted by 1 from the above sense.

Definition 2.6. Let $\mathcal{W}$ be a finite set of locally closed subsets of X and let $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ be a set function. Let $\underline{z}^q_{\mathcal{W},e}(X|D,n)$ be the subgroup generated by integral cycles $Z \in \underline{z}^q(X|D,n)$ such that for each $W \in \mathcal{W}$ and each face $F \subset \square^n$, we have $\mathrm{codim}_{W \times F}(Z \cap (W \times F)) \geq q - e(W)$. They form a subcomplex $\underline{z}^q_{\mathcal{W},e}(X|D,\bullet)$ of $\underline{z}^q(X|D,\bullet)$. Modding out by degenerate cycles, we obtain the subcomplex $z^q_{\mathcal{W},e}(X|D,\bullet) \subset z^q(X|D,\bullet)$. We write $z^q_{\mathcal{W}}(X|D,\bullet) := z^q_{\mathcal{W},0}(X|D,\bullet)$. For additive higher Chow cycles, we write $\underline{\mathrm{Tz}}^q_{\mathcal{W}}(X,n;m)$ for $\underline{z}^q_{\mathcal{W}[1]}(X[1]|D_{m+1}, n-1)$, where $\mathcal{W}[1] = \{W[1] \mid W \in \mathcal{W}\}$.

Here are some basic lemmas used in the paper:

Lemma 2.7 ([21, Lemma 2.2]). *Let $f : Y \to X$ be a dominant map of normal integral k-schemes. Let D be a Cartier divisor on X such that the generic points of* $\mathrm{Supp}(D)$ *are contained in $f(Y)$. Suppose that $f^*(D) \geq 0$ on Y. Then $D \geq 0$ on X.*

Lemma 2.8 ([21, Lemma 2.9]). *Let $f : Y \to X$ be a proper morphism of quasi-projective k-varieties. Let $D \subset X$ be an effective Cartier divisor such that $f(Y) \not\subset D$. Let $Z \in z^q(Y|f^*(D), n)$ be an irreducible cycle. Let $W = f(Z)$ on $X \times \square^n$. Then $W \in z^s(X|D,n)$, where $s = \mathrm{codim}_{X \times \square^n}(W)$.*

Lemma 2.9. *Let X be a k-scheme, and let $\{U_i\}_{i \in I}$ be an open cover of X. Let $Z \in z^q(X \times \square^n)$ and let Z_{U_i} be the flat pull-back to $U_i \times \square^n$. Then $Z \in z^q(X|D,n)$ if and only if for each $i \in I$, we have $Z_{U_i} \in z^q(U_i|D_{U_i}, n)$, where D_{U_i} is the restriction of D on U_i.*

Proof. The direction ($\Rightarrow$) is obvious since flat pull-backs respect admissibility of cycles with modulus by [21, Proposition 2.12]. For the direction ($\Leftarrow$), we may assume Z is irreducible. In this case, it is easily checked that the face and the modulus conditions are both local on the base X. $\square$

2.2. de Rham-Witt complexes.

2.2.1. *Ring of big Witt-vectors.* Let R be a commutative ring with unit. We recall the definition of the ring of big Witt-vectors of R (see [11, §4] or [28, Appendix A]). A *truncation set* $S \subset \mathbb{N}$ is a non-empty subset such that if $s \in S$ and $t|s$, then $t \in S$. As a set, let $\mathbb{W}_S(R) := R^S$ and define the map $w : \mathbb{W}_S(R) \to R^S$ by sending $a = (a_s)_{s\in S}$ to $w(a) = (w(a)_s)_{s\in S}$, where $w(a)_s := \sum_{t|s} t a_t^{s/t}$. When R^S on the target of w is given the component-wise ring structure, it is known that there is a unique functorial ring structure on $\mathbb{W}_S(R)$ such that w is a ring homomorphism (see [11, Proposition 1.2]). When $S = \{1, \cdots, m\}$, we write $\mathbb{W}_m(R) := \mathbb{W}_S(R)$.
There is another description. Let $\mathbb{W}(R) := \mathbb{W}_{\mathbb{N}}(R)$. Consider the multiplicative group $(1 + tR[[t]])^\times$, where t is an indeterminate. Then there is a natural bijection $\mathbb{W}(R) \simeq (1+tR[[t]])^\times$, where the addition in $\mathbb{W}(R)$ corresponds to the multiplication of formal power series. For a truncation set S, we can describe $\mathbb{W}_S(R)$ as the quotient of $(1+tR[[t]])^\times$ by a suitable subgroup I_S. See [28, A.7] for details. In case $S = \{1, \cdots, m\}$, we can write $\mathbb{W}_m(R) = (1+tR[[t]])^\times/(1+t^{m+1}R[[t]])^\times$ as an additive group.
For $a \in R$, the Teichmüller lift $[a] \in \mathbb{W}_S(R)$ corresponds to the image of $1 - at \in (1 + tR[[t]])^\times$. This yields a multiplicative map $[-] : R \to \mathbb{W}_S(R)$. The additive identity element of $\mathbb{W}_m(R)$ corresponds to the unit polynomial 1 and the multiplicative identity element corresponds to the polynomial $1 - t$.

2.2.2. *de Rham-Witt complex.* Let p be an odd prime and R be a $\mathbb{Z}_{(p)}$-algebra.[1] For each truncation set S, there is a differential graded algebra $\mathbb{W}_S\Omega^\bullet_R$ called the big de Rham-Witt complex over R. This defines a contravariant functor on the category of truncation sets. This is an initial object in the category of V-complexes and in the category of Witt-complexes over R. For details, see [8] and [28, §1]. When S is a finite truncation set, we have $\mathbb{W}_S\Omega^\bullet_R = \Omega^\bullet_{\mathbb{W}_S(R)/\mathbb{Z}}/N^\bullet_S$, where $N^\bullet_S$ is the differential graded ideal given by some generators ([28, Proposition 1.2]). In case $S = \{1, 2, \cdots, m\}$, we write $\mathbb{W}_m\Omega^\bullet_R$ for this object.
Here is another relevant object for this paper from [8, Definition 1.1.1]; *a restricted Witt-complex over R* is a pro-system of differential graded $\mathbb{Z}$-algebras $((E_m)_{m\in\mathbb{N}}, \mathfrak{R} : E_{m+1} \to E_m)$, with homomorphisms of graded rings $(F_r : E_{rm+r-1} \to E_m)_{m,r\in\mathbb{N}}$ called the *Frobenius* maps, and homomorphisms of graded groups $(V_r : E_m \to E_{rm+r-1})_{m,r\in\mathbb{N}}$ called the *Verschiebung* maps, satisfying the following relations for all $n, r, s \in \mathbb{N}$:

(i) $\mathfrak{R}F_r = F_r\mathfrak{R}^r, \mathfrak{R}^rV_r = V_r\mathfrak{R}, F_1 = V_1 = \mathrm{Id}, F_rF_s = F_{rs}, V_rV_s = V_{rs}$;
(ii) $F_rV_r = r$. When $(r, s) = 1$, $F_rV_s = V_sF_r$ on E_{rm+r-1};
(iii) $V_r(F_r(x)y) = xV_r(y)$ for all $x \in E_{rm+r-1}$ and $y \in E_m$; (projection formula)
(iv) $F_r dV_r = d$, where d is the differential of the DGAs.

Furthermore, we require that there is a homomorphism of pro-rings $(\lambda : \mathbb{W}_m(R) \to E^0_m)_{m\in\mathbb{N}}$ that commutes with F_r and V_r, satisfying

[1] A definition of Witt-complex over a more general ring R can be found in [11, Definition 4.1].

(v) $F_r d\lambda([a]) = \lambda([a]^{r-1})d\lambda([a])$ for all $a \in R$ and $r \in \mathbb{N}$.

The pro-system $\{\mathbb{W}_m\Omega^\bullet_R\}_{m\geq 1}$ is the initial object in the category of restricted Witt-complexes over R (See [28, Proposition 1.15]).

3. Normalization theorem

Let k be any field. The aim of this section is to prove Theorem 3.2. Such results were known when $D = \emptyset$, or when X is replaced by $X \times \mathbb{A}^1$ with $D = \{t^{m+1} = 0\}$ for $t \in \mathbb{A}^1$. We generalize it to higher Chow groups with modulus.

Definition 3.1. Let (X, D) be a scheme with an effective divisor. Let $z^q_N(X|D, n)$ be the subgroup of cycles $\alpha \in z^q(X|D, n)$ such that $\partial^0_i(\alpha) = 0$ for all $1 \leq i \leq n$ and $\partial^\infty_i(\alpha) = 0$ for $2 \leq i \leq n$. One checks that $\partial^\infty_1 \circ \partial^\infty_1 = 0$. Writing ∂^∞_1 as ∂^N, we obtain a subcomplex $\iota : (z^q_N(X|D, \bullet), \partial^N) \hookrightarrow (z^q(X|D, \bullet), \partial)$.

Theorem 3.2. *Let $X \in \mathbf{Sm}^{\mathrm{ess}}_k$ and let $D \subset X$ be an effective Cartier divisor. Then $\iota : z^q_N(X|D, \bullet) \to z^q(X|D, \bullet)$ is a quasi-isomorphism. In particular, every cycle class in $\mathrm{CH}^q(X|D, n)$ can be represented by a cycle α such that $\partial^\epsilon_i(\alpha) = 0$ for all $1 \leq i \leq n$ and $\epsilon = 0, \infty$.*

Let Cube be the standard category of cubes (see [24, §1]) so that a cubical abelian group is a functor $\text{Cube}^{\mathrm{op}} \to (\mathbf{Ab})$. Recall also from *loc.cit.* that an extended cubical abelian is a functor $\text{ECube}^{\mathrm{op}} \to (\mathbf{Ab})$, where ECube is the smallest symmetric monoidal subcategory of Sets containing Cube and the morphism $\mu : \underline{2} \to \underline{1}$. The essential point of the proof of Theorem 3.2 is

Theorem 3.3. *Let $X \in \mathbf{Sm}^{\mathrm{ess}}_k$ and $D \subset X$ be an effective Cartier divisor. Then $(\underline{n} \mapsto z^q(X|D; n))$ is an extended cubical abelian group.*

If Theorem 3.3 holds, then [24, Lemma 1.6] implies Theorem 3.2. We suppose (X, D) is as in Theorem 3.2 in what follows. The idea is similar to that of [19, Appendix].

Let $q_1 : \square^2 \to \square$ be the morphism $(y_1, y_2) \mapsto y_1 + y_2 - y_1 y_2$ if $y_1, y_2 \neq \infty$, and $(y_1, y_2) \mapsto \infty$ if y_1 or $y_2 = \infty$. Under the identification $\psi : \square \simeq \mathbb{A}^1$ given by $y \mapsto 1/(1-y)$ (which sends $\{\infty, 0\}$ to $\{0, 1\}$), this map q_1 is equivalent to $q_{1,\psi} : \mathbb{A}^2 \to \mathbb{A}^1$ given by $(y_1, y_2) \mapsto y_1 y_2$. For our convenience, we use this $\square_\psi := (\mathbb{A}^1, \{0, 1\})$ and cycles on $X \times \square^n_\psi$. The boundary operator is $\partial = \sum_{i=1}^n (-1)^i (\partial^0_i - \partial^1_i)$, and we replace $F^1_{n,i}$ by $F^\infty_{n,i} = \{y_i = \infty\}$. We write $F^\infty_n = \sum_{i=1}^n F^\infty_{n,i}$. We write $\overline{\square}_\psi = (\mathbb{P}^1, \{0, 1\})$. The group of admissible cycles is $\underline{z}^q_\psi(X|D, n)$. Consider $q_{n,\psi} : X \times \square^{n+1}_\psi \to X \times \square^n_\psi$ given by $(x, y_1, \cdots, y_{n+1}) \mapsto (x, y_1, \cdots, y_{n-1}, y_n y_{n+1})$.

Proposition 3.4. *For $Z \in z^q_\psi(X|D, n)$, we have $q^*_{n,\psi}(Z) \in z^q_\psi(X|D, n+1)$.*

The delicacy of its proof lies in that the product map $q_{1,\psi} : \mathbb{A}^2 \to \mathbb{A}^1$ *does not* extend to a morphism $(\mathbb{P}^1)^2 \to \mathbb{P}^1$ of varieties so that checking the modulus condition becomes nontrivial. We use a correspondence instead. For $n \geq 1$, let

$i_n : W_n \hookrightarrow X \times \square_\psi^{n+1} \times \overline{\square}_\psi^1$ be the closed subscheme defined by the equation $u_0 y_n y_{n+1} = u_1$, where $(y_1, \cdots, y_{n+1}) \in \square_\psi^{n+1}$ and $(u_0; u_1) \in \overline{\square}_\psi^1$ are the coordinates. Let $y := u_1/u_0$. Its Zariski closure $\overline{W}_n \hookrightarrow X \times \overline{\square}_\psi^{n+1} \times \overline{\square}_\psi^1$ is given by the equation $u_0 u_{n,1} u_{n+1,1} = u_1 u_{n,0} u_{n+1,0}$, where $(u_{1,0}, u_{1,1}), \cdots, (u_{n+1,0}, u_{n+1,1})$ are the homogeneous coordinates of $\overline{\square}_\psi^{n+1}$ with $y_i = u_{i,1}/u_{i,0}$.

Consider $\theta_n : X \times \square_\psi^{n+1} \times \overline{\square}_\psi^1 \to X \times \square_\psi^n$ given by $(x, y_1, \cdots, y_{n+1}, (u_0; u_1)) \mapsto (x, y_1, \cdots, y_{n-1}, y_n y_{n+1})$, and let $\pi_n := \theta_n|_{W_n}$. To extend this π_n to a morphism $\overline{\pi}_n$ on $\overline{W}_n$, we use the projection $\overline{\theta}_n : X \times \overline{\square}_\psi^{n+1} \times \overline{\square}_\psi^1 \to X \times \overline{\square}_\psi^{n-1} \times \overline{\square}_\psi^1$, that drops the coordinates $(u_{n,0}; u_{n,1})$ and $(u_{n+1,0}; u_{n+1,1})$, and the projection $p_n : X \times \square_\psi^{n+1} \times \overline{\square}_\psi^1 \to X \times \square_\psi^{n+1}$, that drops the last coordinate $(u_0; u_1)$.

Lemma 3.5. (1) *$W_n \cap \{u_0 = 0\} = \emptyset$, so that $W_n \subset X \times \square_\psi^{n+1} \times \square_\psi^1$.* (2) *$\overline{\theta}_n|_{W_n} = \pi_n$. Thus, we define $\overline{\pi}_n := \overline{\theta}_n|_{\overline{W}_n}$, which extends π_n.* (3) *The varieties W_n and $\overline{W}_n$ are smooth.* (4) *Both π_n and $\overline{\pi}_n$ are surjective flat morphisms of relative dimension* 1.

Proof. Its proof is almost identical to that of [19, Lemma A.5]. Part (1) follows from the defining equation of W_n, and (2) holds by definition. Let $\rho_n := p_n|_{W_n} : W_n \to X \times \square_\psi^{n+1}$. Since X is smooth, using Jacobian criterion we check that W_n is smooth. Furthermore, ρ_n is an isomorphism with the obvious inverse. Under this identification, the morphism π_n can also be regarded as the projection $(x, y_1, \cdots, y_n, y) \mapsto (x, y_1, \cdots, y_{n-1}, y)$ that drops y_n. In particular, π_n is a smooth and surjective of relative dimension 1. To check that $\overline{W}_n$ is smooth, one can do it locally on each open set where each of $u_{n,i}, u_{n+1,i}, u_i$ is nonzero for $i = 0, 1$. In each such open set, the equation for $\overline{W}_n$ takes the same form as for W_n, so that it is smooth again by Jacobian criterion. Similarly as for π_n, one sees $\overline{\pi}_n$ is of relative dimension 1. Since $\overline{\theta}_n$ is projective and π_n is surjective, the morphism $\overline{\pi}_n$ is projective and surjective. So, since $\overline{W}_n$ is smooth, the map $\overline{\pi}_n$ is flat by [7, Exercise III-10.9, p.276]. Thus, we have (3) and (4). $\square$

Lemma 3.6. *Let $n \geq 1$ and let $Z \subset X \times \square_\psi^n$ be a closed subscheme with modulus D. Then $Z' := (i_n)_*(\pi_n^*(Z))$ also has modulus D.*

Proof. Let $\overline{Z}$ and $\overline{Z}'$ be the Zariski closures of Z and Z' in $X \times \overline{\square}_\psi^n$ and $X \times \overline{\square}_\psi^{n+1}$, respectively. By Lemma 3.5 and the projectivity of $\overline{\theta}_n$, we see that $\overline{\theta}_n(\overline{Z}') = \overline{Z}$. Consider the commutative diagram

(3.1)
$$\begin{tikzcd}
\overline{Z}'^N \arrow[r, "g"] \arrow[rr, bend left, "\nu_{Z'}"] \arrow[d, "f"] & \overline{W}_n \arrow[r, hook, "\overline{i}_n"] \arrow[dr, "\overline{\pi}_n"] & X \times \overline{\square}_\psi^{n+1} \times \overline{\square}_\psi^1 \arrow[d, "\overline{\theta}_n"] \\
\overline{Z}^N \arrow[rr, "\nu_Z"] & & X \times \overline{\square}_\psi^n,
\end{tikzcd}$$

where f is induced by the surjection $\overline{\theta}_n|_{\overline{Z}'} : \overline{Z}' \to \overline{Z}$, the maps g and ν_Z are normalizations of $\overline{Z}'$ and $\overline{Z}$ composed with the closed immersions, and $\nu_{Z'} := \overline{i}_n \circ g$. By the definition of $\overline{\theta}_n$, we have $\overline{\theta}_n^*(D \times \overline{\square}_\psi^n) = D \times \overline{\square}_\psi^{n+2}$, $\overline{\theta}_n^*(F_{n,n}^\infty) = F_{n+2,n+2}^\infty$, while $\overline{\theta}_n^*(F_{n,i}^\infty) = F_{n+2,i}^\infty$ for $1 \leq i \leq n-1$. By the defining equation of $\overline{W}_n$, we have $\overline{\pi}_n^* F_{n,n}^\infty = \overline{i}_n^* F_{n+2,n+2}^\infty = \overline{i}_n^*\{u_0 = 0\} \leq \overline{i}_n^*(\{u_{n,0} = 0\} + \{u_{n+1,0} = 0\}) = \overline{i}_n^*(F_{n+2,n}^\infty + F_{n+2,n+1}^\infty)$.
Thus, $\nu_{Z'}^* \overline{\theta}_n^* \sum_{i=1}^n F_{n,i}^\infty = \sum_{i=1}^{n-1} \nu_{Z'}^* F_{n+2,i}^\infty + g^* \overline{\pi}_n^* F_{n,n}^\infty \leq \sum_{i=1}^{n-1} \nu_{Z'}^* F_{n+2,i}^\infty + g^* \overline{i}_n^*(F_{n+2,n}^\infty + F_{n+2,n+1}^\infty) = \sum_{i=1}^{n+1} \nu_{Z'}^* F_{n+2,i}^\infty \leq \sum_{i=1}^{n+2} \nu_{Z'}^* F_{n+2,i}^\infty$. (In case $n=1$, we just ignore the terms with $\sum_{i=1}^{n-1}$ in the above.)
That Z has modulus D means $\nu_Z^*(D \times \overline{\square}_\psi^n) \leq \sum_{i=1}^n \nu_Z^* F_{n,i}^\infty$. Applying f^* and using (3.1), we have $\nu_{Z'}^*(D \times \overline{\square}_\psi^{n+2}) = \nu_{Z'}^* \overline{\theta}_n^*(D \times \overline{\square}_\psi^n) \leq \nu_{Z'}^* \overline{\theta}_n^* \sum_{i=1}^n F_{n,i}^\infty$, which is bounded by $\sum_{i=1}^{n+2} \nu_{Z'}^* F_{n+2,i}^\infty$ as we saw above. This means Z' has modulus D. □

Definition 3.7. For any closed subscheme $Z \subset X \times \square_\psi^n$, we define $W_n(Z) := p_{n*} i_{n*} \pi_n^*(Z)$, which is closed in $X \times \square_\psi^{n+1}$.

Lemma 3.8. *Let $n \geq 1$. If a closed subscheme $Z \subset X \times \square_\psi^n$ intersects all faces properly, then $W_n(Z)$ intersects all faces of $X \times \square_\psi^{n+1}$ properly.*

Proof. Our W_n is equal to $\tau^* \tau_n^* \tau_{n+1}^* W_n^X$, where W_n^X is that of [23, Lemma 4.1], and τ, τ_n, τ_{n+1} are the involutions $(x \mapsto 1-x)$ for y, y_n, y_{n+1}, respectively. So, the lemma is a special case of *loc.cit.* □

Proof of Proposition 3.4. Consider the commutative diagram

$$\begin{tikzcd} W_n \arrow[r, hook, "i_n"] \arrow[d, "\pi_n"'] \arrow[dr, "\rho_n = p_n|_{W_n}"] & X \times \square_\psi^{n+1} \times \overline{\square}_\psi \arrow[d, "p_n"] \\ X \times \square_\psi^n & X \times \square_\psi^{n+1}. \arrow[l, "q_{n,\psi}"'] \end{tikzcd}$$

By Lemma 3.5, ρ_n is an isomorphism so that $\rho_{n*} i_n^* p_n^* = \mathrm{Id}$. Hence, $q_{n,\psi}^*(Z) = \rho_{n*} i_n^* p_n^* q_{n,\psi}^*(Z) =^\dagger \rho_{n*} \pi_n^*(Z) =^\ddagger p_{n*} i_{n*} \pi_n^*(Z) = W_n(Z)$, where †, ‡ are due to commutativity. So, we have reduced to showing that $W_n(Z) \in z_\psi^q(X|D, n+1)$. But, by Lemmas 3.6 and 3.8, we have $i_{n*}\pi_n^*(Z) \in z_\psi^{q+1}(X \times \mathbb{P}^1 | D \times \mathbb{P}^1, n+1)$. Now, for the projection p_n, by Lemma 2.8, we have $W_n(Z) = p_{n*} i_{n*} \pi_n^*(Z) \in z_\psi^q(X|D, n+1)$. This proves Proposition 3.4. □

Proof of Theorem 3.3. Since we know that $(\underline{n} \mapsto z^q(X|D; n))$ is a cubical abelian group, every morphism $h : \underline{r} \to \underline{s}$ in Cube induces a morphism $h : \square^r \to \square^s$ which gives a homomorphism $h^* : z^q(X|D, s) \to z^q(X|D, r)$. Furthermore, the morphism $\mu : \underline{2} \to \underline{1}$ induces the morphism $q_1 : \square^2 \to \square^1$ of varieties, and for each $Z \in z^q(X|D, 1)$, we have $q_1^*(Z) \in z^q(X|D, 2)$. Indeed, under the isomorphism $\psi : \square \simeq \mathbb{A}^1, y \mapsto 1/(1-y)$, this is equivalent to

show that $q_{1,\psi}^*$ sends admissible cycles to admissible cycles, which we know by Proposition 3.4.
So, it only remains to show the following "stability under products": if $h_i : \underline{r_i} \to \underline{s_i}$, $i = 1, 2$, are morphisms in ECube such that the corresponding morphisms $h_i : \square^{r_i} \to \square^{s_i}$ induce homomorphisms $h_i^* : z^q(X|D, s_i) \to z^q(X|D, r_i)$, for $i = 1, 2$ and all $q \geq 0$, then $h := h_1 \times h_2 : \square^{r_1+r_2} \to \square^{s_1+s_2}$ induces a homomorphism $h^* : z^q(X|D, s) \to z^q(X|D, r)$ for all $q \geq 0$, where $r = r_1 + r_2$ and $s = s_1 + s_2$.
Since $h = h_1 \times h_2 = (\mathrm{Id}_{r_1} \times h_2) \circ (h_1 \times \mathrm{Id}_{r_2})$, we reduce to prove it when h is either $\mathrm{Id}_{r_1} \times h_2$ or $h_1 \times \mathrm{Id}_{r_2}$. But the statement obviously holds for these cases. □

4. On moving lemmas

Let k be any field. In this section, we discuss some of moving lemmas on algebraic cycles with modulus conditions. By a 'moving lemma', we ask whether the inclusion $z^q_{\mathcal{W}}(Y|D, \bullet) \subset z^q(Y|D, \bullet)$ in Definition 2.6 is a quasi-isomorphism. It is known when Y is smooth quasi-projective and $D = 0$ (by [4]), and when $Y = X \times \mathbb{A}^1$, with X smooth projective, $D = X \times \{t^{m+1} = 0\}$, and $\mathcal{W}$ consists of $W \times \mathbb{A}^1$ for finitely many locally closed subsets $W \subset X$ (by [17]). Recently, W. Kai [14] proved it when Y is smooth affine with a suitable condition. Kai's cases include the above case of $Y = X \times \mathbb{A}^1$, where X is this time smooth affine. His proof applies to more general cases, possibly after Nisnevich sheafifications. In §4.1, we sketch the argument of Kai in the case of multivariate additive higher Chow groups of smooth affine k-variety. In §4.2, we generalize the moving lemma of [17] in the case of pairs $(X \times S, X \times D)$ where X is smooth projective. In §4.3 and 4.4, we discuss the standard pull-back property and its consequences. In §4.5, we discuss a moving lemma for additive higher Chow groups of smooth semi-local k-schemes essentially of finite type.

4.1. Kai's affine method for multivariate additive higher Chow groups. The moving lemma of W. Kai [14] is the first moving result that applies to cycle groups with a *non-zero* modulus over a smooth *affine* scheme. Since the work *loc. cit.* is at present not yet refereed, we give a detailed sketch the proof of the following special case on multivariate additive higher Chow groups. But, we emphasize that the most crucial part is due to Kai. Following Definition 2.5, we write $X[r] := X \times \mathbb{A}^r$.

Theorem 4.1 (W. Kai). *Let X be a smooth affine variety over any field k. Let $\mathcal{W}$ be a finite set of locally closed subsets of X. Let $\mathcal{W}[r] := \{W[r] \mid W \in \mathcal{W}\}$. Let $\underline{m} = (m_1, \cdots, m_r) \geq 1$. Then the inclusion $z^q_{\mathcal{W}[r]}(X[r]|D_{\underline{m}}, \bullet) \hookrightarrow z^q(X[r]|D_{\underline{m}}, \bullet)$ is a quasi-isomorphism.*

First recall some preparatory results:

Lemma 4.2 ([17, Lemma 4.5]). *Let $f : X \to Y$ be a dominant morphism of normal varieties. Suppose that Y is integral with the generic point $\eta \in Y$, and let X_η be the fiber over η, with the inclusion $j_\eta : X_\eta \hookrightarrow X$. Let D be a Weil* *divisor on X such that $j_\eta^*(D) \geq 0$. Then there exists a non-empty open subset $U \subset Y$ such that $j_U^*(D) \geq 0$, where $j_U : f^{-1}(U) \hookrightarrow X$ is the inclusion.*

The following generalizes [17, Proposition 4.7]:

Proposition 4.3 (Spreading lemma). *Let $k \subset K$ be a purely transcendental extension. Let (X, D) be a smooth quasi-projective k-scheme with an effective Cartier divisor, and let $\mathcal{W}$ be a finite collection of locally closed subsets of X. Let (X_K, D_K) and $\mathcal{W}_K$ be the base changes via* $\operatorname{Spec}(K) \to \operatorname{Spec}(k)$. *Let $p_{K/k} : X_K \to X_k$ be the base change map. Then the pull-back map*

$$p_{K/k}^* : \frac{z^q(X|D, \bullet)}{z^q_{\mathcal{W}}(X|D, \bullet)} \to \frac{z^q(X_K|D_K, \bullet)}{z^q_{\mathcal{W}_K}(X_K|D_K, \bullet)}$$

is injective on homology.

Proof. It is similar to [17, Proposition 4.7]. We sketch its proof for the reader's convenience. If k is finite, then we can use the standard pro-ℓ-extension argument to reduce the proof to the case when k is infinite, which we assume from now. We may also assume that $\mathrm{tr.deg}_k K < \infty$ and furthermore that $\mathrm{tr.deg}_k K = 1$, by induction. So, we have $K = k(\mathbb{A}^1_k)$.

Suppose $Z \in z^q(X|D, n)$ is a cycle that satisfies $\partial Z \in z^q_{\mathcal{W}}(X|D, n-1)$, and $Z_K = \partial(B_K) + V_K$ for some $B_K \in z^q(X_K|D_K, n+1)$ and $V_K \in z^q_{\mathcal{W}_K}(X_K|D_K, n)$. Consider the inclusion $z^q(X_K|D_K, \bullet) \hookrightarrow z^q(X_K, \bullet)$. Then there is a non-empty open $U' \subset \mathbb{A}^1_k$ such that $B_K = B_{U'}|_\eta, V_K = V_{U'}|_\eta$, $Z \times U' = \partial(B_{U'}) + V_{U'}$ for some $B_{U'} \in z^q(X \times U', n+1)$, $V_{U'} \in z^q_{\mathcal{W}\times U'}(X \times U', n)$, where η is the generic point of U'. Let $j_\eta : X \times \eta \to X \times U'$ be the inclusion, which is flat.

Since B_K, V_K satisfy the modulus condition, we have $j_\eta^*(X \times U' \times F^1_{n+1} - D \times U' \times \overline{\square}^{n+1}) \geq 0$ on $\overline{B}^N_K$ and similarly for $\overline{V}^N_K$. Furthermore, $\overline{B}^N_{U'} \to U', \overline{V}^N_{U'} \to U'$ are dominant. Thus by Lemma 4.2, there is a non-empty open $U \subset U'$ such that $j_U^*(X \times U' \times F^1_{n+1} - D \times U' \times \overline{\square}^{n+1}) \geq 0$ on $\overline{B}^N_U$ and similarly for $\overline{V}^N_U$, for $j_U : X \times U \hookrightarrow X \times U'$. This proves that B_U and V_U have modulus $D \times U$. Hence, $B_U \in z^q(X \times U|D \times U, n+1)$ and $V_U \in z^q_{\mathcal{W}\times U}(X \times U|D \times U, n)$ with $Z \times U = \partial(B_U) + V_U$.

Since k is infinite, the set $U(k) \hookrightarrow U$ is dense. We claim the following:

Claim: *There is a point $u \in U(k)$ such that the pull-backs of B_U and V_U under the inclusion $i_u : X \times \{u\} \hookrightarrow X \times U$ are both defined in $z^q(X, n+1)$ and $z^q_{\mathcal{W}}(X, n)$, respectively.*

Its proof requires the following elementary fact:

Lemma: *Let Y be any k-scheme. Let $B \in z^q(Y \times U)$ be a cycle. Then there exists a nonempty open subset $U'' \subset U$ such that for each $u \in U''(k)$, the closed subscheme $Y \times \{u\}$ intersects B properly on $Y \times U$, thus it defines a cycle $i_u^*(B) \in z^q(Y)$, where Y is identified with $Y \times \{u\}$.*

Note that for each $u \in U(k)$, the subscheme $Y \times \{u\} \subset Y \times U$ is an effective divisor, so its proper intersection with B is equivalent to that $Y \times \{u\}$ does not contain any irreducible component of B. If there exists a point $u_i \in U(k)$

such that $Y \times \{u_i\}$ contains an irreducible component B_i of B, then for any other $u \in U(k) \setminus \{u_i\}$, we have $(Y \times \{u\}) \cap B_i = \emptyset$. So, for every irreducible component B_i of B, there exists at most one $u_i \in U(k)$ such that $Y \times \{u_i\}$ contains B_i. Let S be the union of such points u_i, if they exist. There are only finitely many irreducible components of B, so $|S| < \infty$. Taking $U'' := U \setminus S$, we have Lemma.

We now prove Claim. Let $F \subset \square^{n+1}$ be any face, including the case $F = \square^{n+1}$. Since $B_U \in z^q(X \times U, n+1)$, by definition $X \times U \times F$ and B_U intersect properly on $X \times U \times \square^{n+1}$, so their intersection gives a cycle $B_{U,F} \in z^q(X \times U \times F)$. By Lemma with $Y = X \times F$, there exists a nonempty open subset $U_F \subset U$ such that $B_{U,F}$ defines a cycle in $z^q(X \times \{u\} \times F)$ for every $u \in U_F(k)$. Let $\mathcal{U}_1 := \bigcap_F U_F$, where the intersection is taken over all faces F of $\square^{n+1}$. This is a nonempty open subset of U. Similarly, let $F \subset \square^n$ be any face, including the case $F = \square^n$. Here, $V_U \in z^q_{\mathcal{W} \times U}(X \times U, n)$, and repeating the above argument involving Lemma with $Y = W \times F$ for $W \in \mathcal{W}$, we get a nonempty open subset $U_{W,F} \subset U$ such that we have an induced cycle in $z^q(W \times \{u\} \times F)$ for every $u \in U_{W,F}(k)$. Let $\mathcal{U}_2 := \bigcap_{W,F} U_{W,F}$, where the intersection is taken over all pairs (W, F), with $W \in \mathcal{W}$ and a face $F \subset \square^n$. Taking $\mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2$, which is a nonempty open subset of U, we now obtain Claim for every $u \in \mathcal{U}(k)$.

Finally, for such a point u as in Claim, by the containment lemma (Proposition 2.2), $i_u^*(B_U)$ and $i_u^*(V_U)$ have modulus D. Hence, $i_u^*(B_U) \in z^q(X|D, n+1)$ and $i_u^*(V_U) \in z^q_{\mathcal{W}}(X|D, n)$. This finishes the proof. $\square$

Sketch of the proof of Theorem 4.1. Step 1. We first show it when $X = \mathbb{A}^d_k$. Let $K = k(\mathbb{A}^d_k)$ and let $\eta \in X$ be the generic point. To facilitate the proof, as we did previously in §3, using the automorphism $y \mapsto 1/(1-y)$ of $\mathbb{P}^1$ we replace $(\square, \{\infty, 0\})$ by $(\mathbb{A}^1, \{0, 1\})$, and write $\square = \mathbb{A}^1$. We use the homogeneous coordinates $(u_{i,0}; u_{i,1}) \in \overline{\square}^1 = \mathbb{P}^1$, where $y_i = u_{i,1}/u_{i,0}$, then the divisor $F^1_{n,i}$ in the modulus condition is replaced by $F^\infty_{n,i} = \{y_i = \infty\}$ and $F^\infty_n = \sum_{i=1}^n F^\infty_{n,i}$.

For any $g \in \mathbb{A}^d$ and an integer $s > 0$, define $\phi_{g,s} : \mathbb{A}^d_{k(g)}[r] \times_{k(g)} \square^1_{k(g)} \to \mathbb{A}^d_{k(g)}[r]$ by $\phi_{g,s}(\underline{x}, \underline{t}, y) := (\underline{x} + y(t_1^{m_1} \cdots t_r^{m_r})^s g, \underline{t})$, where $k(g)$ is the residue field of g. (N.B. In terms of W. Kai's homotopy, our $g \in \mathbb{A}^d$ corresponds to his $v = (g, 0, \cdots, 0) \in \mathbb{A}^d[r] = \mathbb{A}^{d+r}$.) For any cycle $V \in z^q(X[r]|D_{\underline{m}}, n)$, define $H^*_{g,s}(V) := (\phi_{g,s} \times \mathrm{Id}_{\square^n})^* p^*_{k(g)/k}(V)$, where $p_{k(g)/k} : \mathbb{A}^d_{k(g)}[r] \times \square^n \to \mathbb{A}^d_k[r] \times \square^n$ is the base change.

Using [3, Lemma 1.2], one checks that $H^*_{g,s}(V)$ preserves the face condition for V. Moreover, if $V \in z^q_{\mathcal{W}}(X[r], n)$, then so does $H^*_{g,s}(V)$. When $g = \eta$, another application of [3, Lemma 1.2] shows that $H^*_{g,s}(V)$ intersects with all $W[r] \times F$ properly, where $W \in \mathcal{W}$ and a $F \subset \square^n$ is a face. The argument for proving these face conditions follows the same steps as that of the proof of [17, Lemma 5.5, Case 2] though the present case is slightly different so that we use [3, Lemma 1.2] instead of [3, Lemma 1.1] (see [14, Lemma 3.5] for more detail). On the other hand, we have the following crucial and central assertion due to W. Kai (*cf.* [14, Proposition 3.3]):

Claim: *For each irreducible $V \in z^q(\mathbb{A}^d_k[r]|D_{\underline{m}}, n)$, there is $s(V) \in \mathbb{Z}_{\geq 0}$ such that for any $s > s(V)$ and for any $g \in \mathbb{A}^d$, the cycle $H^*_{g,s}(V)$ has modulus $D_{\underline{m}}$.* Once it is proven, call the smallest such integer $s(V)$, *the threshold of* V, for simplicity. Here, instead of translations by $g \in \mathbb{A}^d$ used in usual higher Chow groups of $\mathbb{A}^d$ (which correspond to $s = 0$), Kai uses *adjusted* translations as in the definition of $\phi_{g,s}$, so that near the divisors $\{t_i = 0\}$, the effect of adjusted translation is also small, while away from the divisors $\{t_i = 0\}$, the effect of adjusted translation gets larger, so that for a sufficiently large s, this imbues the desired modulus condition into cycles. Note the following elementary fact (*cf.* [14, Lemma 3.2]), which amounts to rewriting the definitions: *Let A be a commutative ring with unity, $\mathfrak{p} \subset A$ a prime ideal, $\zeta \in A$, and $u \in A \setminus \mathfrak{p}$. Then the element ζ/u of $\kappa(\mathfrak{p})$ is integral over $A/\mathfrak{p}$ if and only if there is a homogeneous polynomial $E(a, b) \in A[a, b]$, which is monic in the variable a, with $E(\zeta, b) \in \mathfrak{p}$ in A.*

For each $I \subset \{1, \cdots, n\}$, consider the open subset $U_I \subset \mathbb{A}^d_k \times \overline{\square}^n$ given by the conditions $u_{i,0} \neq 0$ for $i \in I$ and $u_{i,1} \neq 0$ for $i \notin I$. For $i \notin I$, we let $\overline{y}_i = u_{i,0}/u_{i,1} = y_i^{-1}$. Hence, $U_I = \mathrm{Spec}\,(R_I)$, where $R_I := k[\underline{x}, \underline{t}, \{y_i\}_{i \in I}, \{\bar{y}_i\}_{i \notin I}]$, where $\underline{x} = (x_1, \cdots, x_d)$ and $\underline{t} = (t_1, \cdots, t_r)$. On U_I, the divisor F^∞_n used in the definition of the modulus condition is given by the polynomial $\prod_{i \notin I} \overline{y}_i$.

For an irreducible $V \in z^q(\mathbb{A}^d_k[r]|D_{\underline{m}}, n)$, let $\overline{V}$ be its Zariski closure in $\mathbb{A}^d_k[r] \times \overline{\square}^n$. For a given I, the restriction $\overline{V} \cap (\mathbb{A}^d_k[r] \times U_I)$ is given by an ideal of R_I, say, generated by a finite set of polynomials $f^I_\lambda(\underline{x}, \underline{t}, \{y_i\}_{i \in I}, \{\overline{y}_i\}_{i \notin I}) \in R_I$ for $\lambda \in \Lambda_I$.

By the above fact and the assumption that V has the modulus condition, there is a polynomial $E_I(a, b) = E_I(\underline{x}, \underline{t}, \{y_i\}_{i \in I}, \{\overline{y}_i\}_{i \notin I}, a, b) \in R_I[a, b]$, homogeneous in a, b and monic in a, satisfying the condition inside the ring R_I:

$$E_I(\prod_{i \notin I} \overline{y}_i, t^{\underline{m}}) \in \sum_{\lambda \in \Lambda_I} (f^I_\lambda), \text{where } t^{\underline{m}} = t_1^{m_1} \cdots t_r^{m_r}. \tag{4.1}$$

If necessary, by multiplying a power of a to E_I, we may assume $\deg E_I \geq \deg_{\underline{x}} f^I_\lambda$, where deg is the homogeneous degree of E_I in the variables a, b and $\deg_{\underline{x}}$ is the total degree with respect to $\underline{x}$. In doing so, we may further assume that $\deg E_I$ is the same for all subset $I \subset \{1, \cdots, n\}$. For this choice of degrees, we let $s(V) = \deg E_I$. If V is not irreducible, then take the maximum of $s(V_i)$ over all irreducible components V_i of V to define $s(V)$. The heart of the proof is to show that this number satisfies the assertions of Claim, which we do now.

We may assume V is irreducible. For any fixed $s > s(V)$ and $g \in \mathbb{A}^d$, let V' be an irreducible component of $H^*_{g,s}(V)$ and let $\overline{V}'$ be its Zariski closure in $\mathbb{A}^d_\kappa[r] \times \overline{\square}^{n+1}$, where $\kappa = k(g)$. We use the coordinates $(y, y_1, \cdots, y_n) \in \overline{\square}^{n+1}$, and for the first $\overline{\square} = \mathbb{P}^1$, use the homogeneous coordinate $(u_0; u_1)$ so that $y = u_1/u_0$ and $\overline{y} := u_0/u_1 = y^{-1}$. Let $\nu : \overline{V}'^N \to \overline{V}$ be the normalization. Note that whether a divisor is effective or not on $\overline{V}'^N$ is a Zariski local question on $\overline{V}'^N$ (thus on $\overline{V}'$), so we may check the modulus condition Zariski locally

near any point $P \in \overline{V}'$. Fix a point P. Let $I \subset \{1, \cdots, n\}$ be the set points i such that P does not map to $\infty \in \mathbb{P}^1_\kappa$ of the $(i+1)$-th projection $\overline{V}' \hookrightarrow \mathbb{A}^d_\kappa[r] \times \overline{\square}^{n+1} \to \overline{\square}_\kappa = \mathbb{P}^1_\kappa$.
There are two possibilities. In the first case $P \in \mathbb{A}^d_\kappa[r] \times \mathbb{A}^1 \times \overline{\square}^n$, i.e. P does not map to $\infty \in \mathbb{P}^1$ for the first projection to $\overline{\square}_\kappa$, the morphism $p_{\kappa/k} \circ (\phi_{g,s} \times \mathrm{Id}^n_\square) : \mathbb{A}^d_\kappa[r] \times \mathbb{A}^1 \times \square^n \to \mathbb{A}^d_k[r] \times \square^n$ extends uniquely to $\mathbb{A}^d_\kappa[r] \times \mathbb{A}^1 \times \overline{\square}^n \to \mathbb{A}^d_k[r] \times \overline{\square}^n$. Thus, by pulling-back the relation (4.1), we obtain in the ring $R_I[y]$,

$$E_I(\underline{x} + y(t^{\underline{m}})^s g, \underline{t}, \{y_i\}_{i\in I}, \{\overline{y}_i\}_{i\notin I}, \prod_{i\notin I} \overline{y}_i, t^{\underline{m}}) \in \in \sum_{\lambda \in \Lambda_I} (f^I_\lambda(\underline{x} + y(t^{\underline{m}})^s g, \underline{t}, \{y_i\}_{i\in I}, \{\overline{y}_i\}_{i\notin I})). \tag{4.2}$$

Here, the polynomials $f^I_\lambda(\underline{x} + y(t^{\underline{m}})^s g, \{y_i\}_{i\in I}, \{\overline{y}_i\}_{i\notin I})$ over $\lambda \in \Lambda_I$ define the underlying closed subscheme of the Zariski closure of $H^*_{g,s}(V)$ restricted on the region $\mathrm{Spec}\,(R_I[y])$. Due to the choice of the degrees of E_I and f^I_λ, the relation (4.2) implies that the rational function $\prod_{i\notin I} \overline{y}_i / t^{\underline{m}}$ is integral using fact. In particular, V' satisfies the modulus condition in a neighborhood of P.
In the remaining case $P \notin \mathbb{A}^d_\kappa[r] \times \mathbb{A}^1 \times \overline{\square}^n$, i.e. P *does* map to $\infty \in \mathbb{P}^1$ for the first projection to $\overline{\square}_\kappa$, we use the affine open chart $\mathrm{Spec}\,(R_I[\overline{y}])$ where $u_1 \neq 0$. The defining ideal of $\overline{V}' \cap \mathrm{Spec}\,(R_I[\overline{y}])$ in the ring $R_I[\overline{y}]$ contains the polynomials $\phi^I_\lambda(\underline{x}, \underline{t}, \overline{y}, \{y_i\}_{i\in I}, \{\overline{y}_i\}_{i\notin I}) := f^I_\lambda(\underline{x} + \frac{1}{\overline{y}}(t^{\underline{m}})^s g, \underline{t}, \{y_i\}_{i\in I}, \{\overline{y}_i\}_{i\notin I}) \cdot \overline{y}^{\deg_{\underline{x}}(f^I_\lambda)}$, where $\lambda \in \Lambda_I$. By expanding the definition of ϕ^I_λ, we see that it is of the form

$$\phi^I_\lambda = \overline{y}^{\deg_{\underline{x}}(f^I_\lambda)} f^I_\lambda(\underline{x}, \underline{t}, \{y_i\}_{i\in I}, \{\overline{y}_i\}_{i\notin I}) + (t^{\underline{m}})^s h, \quad h \in R_I[\overline{y}]. \tag{4.3}$$

Express (4.1) as $E_I(\prod_{i\notin I} \overline{y}_i, t^{\underline{m}}) = \sum_{\lambda \in \Lambda_I} b_\lambda f^I_\lambda$ for some $b_\lambda \in R_I$. Let $c_\lambda := \overline{y}^{s(V) - \deg_{\underline{x}}(f^I_\lambda)} \cdot b_\lambda$ (which is in R_I because $s(V) \geq \deg_{\underline{x}}(f^I_\lambda)$). Then from (4.3),

$$\sum_{\lambda \in \Lambda_I} c_\lambda \phi^I_\lambda = \overline{y}^{s(V)} \cdot E_I(\prod_{i\notin I} \overline{y}_i, t^{\underline{m}}) + (t^{\underline{m}})^s g, \tag{4.4}$$

where (keep in mind that $s \geq s(V)$) the right hand side becomes $(\overline{y} \prod_{i\notin I} \overline{y}_i)^{s(V)} + e_1 \overline{y} (\overline{y} \prod_{i\notin I} \overline{y}_i)^{s(V)-1} t^{\underline{m}} + \cdots + (e_{s(v)} \overline{y}^{s(V)} + (t^{\underline{m}})^{s - s(V)} h) \cdot (t^{\underline{m}})^{s(V)}$, which we write as $E'(\overline{y} \prod_{i\notin I} \overline{y}_i, t^{\underline{m}})$ for a polynomial $E'(a,b) \in R_I[\overline{y}][a,b]$, homogeneous in a, b and monic in a. Thus (4.4) is $\sum_{\lambda\in\Lambda_I} c_\lambda \phi^I_\lambda = E'(\overline{y} \prod_{i\notin I} \overline{y}_i, t^{\underline{m}})$, which implies that the rational function $\overline{y} \prod_{i\notin I} \overline{y}_i / t^{\underline{m}}$ is integral on $\overline{V}' \cap \mathrm{Spec}\,(R_I[\overline{y}])$ using fact. Thus V' also satisfies the modulus condition near P. Combining these two cases, we have now proven Claim.
Now consider the subgroup $z^q_{\mathcal{W}[r],e}(X[r]|D_{\underline{m}}, n)^{\leq s} \subset z^q_{\mathcal{W}[r],e}(X[r]|D_{\underline{m}}, n)$ for $s > 0$, consisting of cycles V with its threshold $s(V) \leq s$ (*cf.* [14, §3.4]). We

deduce

$$\frac{z^q_{\mathcal{W}[r],e}(X[r]|D_{\underline{m}}, n)}{z^q_{\mathcal{W}[r]}(X[r]|D_{\underline{m}}, n)} = \lim_{\to s} \frac{z^q_{\mathcal{W}[r],e}(X[r]|D_{\underline{m}}, n)^{\leq s}}{z^q_{\mathcal{W}[r]}(X[r]|D_{\underline{m}}, n)^{\leq s}}.$$

Then one has the induced map from $H^*_{\eta,s}$,

$$H^*_{\eta,s} : \frac{z^q_{\mathcal{W}[r],e}(X[r]|D_{\underline{m}}, n)^{\leq s}}{z^q_{\mathcal{W}[r]}(X[r]|D_{\underline{m}}, n)^{\leq s}} \to \frac{z^q_{\mathcal{W}[r],e}(X_K[r]|D_{\underline{m}}, n+1)}{z^q_{\mathcal{W}[r]}(X_K[r]|D_{\underline{m}}, n+1)},$$

which gives a homotopy between the base change $p^*_{K/k}$ and $H^*_{\eta,s}|_{y=1}$. However, $H^*_{\eta,s}|_{y=1}$ is zero on the quotient, while $p^*_{K/k}$ is injective on homology by Proposition 4.3, after taking $s \to \infty$, so that the map $p^*_{K/k}$ is in fact zero on homology. This means, the quotient $z^q_{\mathcal{W}[r],e}(X[r]|D_{\underline{m}}, n)/z^q_{\mathcal{W}[r]}(X[r]|D_{\underline{m}}, n)$ is acyclic, proving the moving lemma for $X = \mathbb{A}^d_k$.

Step 2. If X is a general smooth affine k-variety of dimension d, we use the standard generic linear projection trick. We choose a closed immersion $X \hookrightarrow \mathbb{A}^N$ for some $N \gg d$ and run the steps of §6 of [17] (with $\mathbb{P}^n$ replaced by $\mathbb{A}^N$ everywhere) *mutatis mutandis* to conclude the proof of the moving lemma for X from that of affine spaces. We leave the details for the reader. $\square$

4.2. Projective method for multivariate additive higher Chow groups. The following theorem generalizes the moving lemma for additive higher Chow groups of smooth projective schemes [17, Theorem 4.1] to a general setting which includes the multivariate additive higher Chow groups.

Theorem 4.4. *Let (S, D) be a smooth quasi-projective k-variety with an effective Cartier divisor. Let X be a smooth projective k-variety. Let $\mathcal{W}$ be a finite collection of locally closed subsets of X. We let $\mathcal{W} \times S := \{W \times S | W \in \mathcal{W}\}$. Then the inclusion $z^q_{\mathcal{W}\times S}(X \times S|X \times D, \bullet) \hookrightarrow z^q(X \times S|X \times D, \bullet)$ is a quasi-isomorphism. In particular, when $\underline{m} = (m_1, \cdots, m_r) \geq 1$, and $(S, D) = (\mathbb{A}^r, D_{\underline{m}})$, the moving lemma holds for multivariate additive higher Chow groups of smooth projective varieties over k.*

Proof. Most arguments of [17, Theorem 4.1] work with minor changes, so we sketch the proof.

Step 1. We first prove the theorem when $X = \mathbb{P}^d_k$. The algebraic group $SL_{d+1,k}$ acts on $\mathbb{P}^d$. Let $K = k(SL_{d+1,k})$. Then there is a K-morphism $\phi : \square^1_K \to SL_{d+1,K}$ such that $\phi(0) = \mathrm{Id}$, and $\phi(\infty) = \eta$, where η is the generic point of $SL_{d+1,k}$. See [17, Lemma 5.4]. For such ϕ, consider the composition H_n of morphisms

$$\mathbb{P}^d \times S \times \square^{n+1}_K \overset{\mu_\phi}{\to} \mathbb{P}^d \times S \times \square^{n+1}_K \overset{\mathrm{pr}'_K}{\to} \mathbb{P}^d \times S \times \square^n_K \overset{p_{K/k}}{\to} \mathbb{P}^d \times S \times \square^n_k,$$

where $\mu_\phi(\underline{x}, s, y_1, \cdots, y_{n+1}) = (\phi(y_1)\underline{x}, s, y_1, \cdots, y_{n+1})$, pr'_K is the projection dropping y_1, and $p_{K/k}$ is the base change. We claim that H^*_n carries $z^q_{\mathcal{W}\times S}(\mathbb{P}^d \times S|\mathbb{P}^d \times D, n)$ to $z^q_{\mathcal{W}\times S}(\mathbb{P}^d_K \times S|\mathbb{P}^d_K \times D, n+1)$, i.e., for an irreducible cycle $Z \in z^q_{\mathcal{W}\times S}(\mathbb{P}^d \times S|\mathbb{P}^d \times S, n)$, we show that $Z' := H^*_n(Z) \in z^q_{\mathcal{W}\times S}(\mathbb{P}^d_K \times S|\mathbb{P}^d_K \times D, n+1)$.

To do so, we first claim that Z' intersects with $W \times S \times F_K$ properly for each $W \in \mathcal{W}$ and each face $F \subset \square^{n+1}$.
(1) In case $F = \{0\} \times F'$ for some face $F' \subset \square^n$, because $\phi(0) = \mathrm{Id}$, we have $Z' \cap (W \times S \times F_K) \simeq Z_K \cap (W \times S \times F'_K)$. Note that $\dim(W \times S \times F_K) = \dim(W \times S \times F'_K)$. Hence, $\mathrm{codim}_{W\times S\times F_K}(Z' \cap (W \times S \times F_K)) = \dim(W \times S \times F_K) - \dim(Z' \cap (W \times S \times F_K)) = \dim(W \times S \times F'_K) - \dim(Z_K \cap (W \times S \times F'_K)) = \dim(W \times S \times F') - \dim(Z \cap (W \times S \times F')) = \mathrm{codim}_{W\times S\times F'}(Z \cap (W \times S \times F')) \geq q$, because $Z \in z^q_{\mathcal{W}\times S}(\mathbb{P}^d \times S|\mathbb{P}^d \times D, n)$.
(2) In case $F = \{\infty\} \times F'$ for some face $F' \subset \square^n$, $\dim(W \times S \times F_K) = \dim(W \times S \times F'_K)$ and $Z' \cap (W \times S \times F_K) \simeq \eta\cdot(Z_K) \cap (W \times S \times F'_K)$, where $SL_{d+1,k}$ acts on $\mathbb{P}^d \times S \times F'$, naturally on $\mathbb{P}^d$ and trivially on $S \times F'$. Let $A := W \times S \times F'$ and $B := Z \cap (\mathbb{P}^d \times S \times F')$. Thus, $\mathrm{codim}_{W\times S\times F_K}(Z' \cap (W \times S \times F_K)) = \dim(W \times S \times F_K) - \dim(Z' \cap (W \times S \times F_K)) = \dim(W \times S \times F'_K) - \dim(\eta\cdot(Z_K) \cap (W \times S \times F'_K)) =^\dagger \dim(A_K) - \dim(\eta \cdot B_K \cap A_K) = \mathrm{codim}_{A_K}(\eta \cdot B_K \cap A_K)$, where $\dagger$ holds because $Z \cap A = B \cap A$. By applying [3, Lemma 1.1] to $G = SL_{d+1,k}$, and the above A, B on $\mathcal{X} := \mathbb{P}^d \times S \times F'$, there is a non-empty open subset $U \subset G$ such that for all $g \in U$, the intersection $(g \cdot A) \cap B$ is proper on $\mathcal{X}$. By shrinking U, we may assume U is invariant under inverse map, so $g = \eta^{-1} \in U$. Thus, $\mathrm{codim}_{A_K}((\eta \cdot B_K) \cap A_K) \geq \mathrm{codim}_{\mathcal{X}_K}(\eta \cdot B_K)$. Since $\mathrm{codim}_{\mathcal{X}_K}(\eta \cdot B_K) = \mathrm{codim}_{\mathcal{X}_K} B_K$ and $\mathrm{codim}_{\mathcal{X}_K} B_K = q$, we get $\mathrm{codim}_{W\times S\times F_K}(Z' \cap (W \times S \times F_K)) = \mathrm{codim}_{A_K}((\eta \cdot B_K) \cap A_K) \geq \mathrm{codim}_{\mathcal{X}_K} B_K = q$.
(3) In case $F = \square \times F'$ for some face $F' \subset \square^n$, the projection $Z' \cap (W \times S \times \square \times F'_K) \to \square_K$ is flat, being a dominant map to a curve, so $\dim(Z' \cap (W \times S \times \square \times F'_K)) = \dim(Z' \cap (W \times S \times \{\infty\} \times F'_K)) + 1$. We also have $\dim(W \times S \times \square \times F'_K) = \dim(W \times S \times \{\infty\} \times F'_K) + 1$. Hence, we deduce $\mathrm{codim}_{W\times S\times F_K}(Z' \cap (W \times S \times F_K)) = \dim(W \times S \times \square \times F_K) - \dim(Z' \cap (W \times S \times \square \times F'_K)) = \mathrm{codim}_{W\times S\times\{\infty\}\times F'_K}(Z' \cap (W \times S \times \{\infty\} \times F'_K)) \geq^\dagger q$, where $\dagger$ follows from case (2). This shows Z' intersects all faces properly.
Now we show that Z' has modulus $\mathbb{P}^d \times D$. We drop all exchange of the factors, for simplicity. For $p : \mathbb{P}^d \to \mathrm{Spec}\,(k)$, we take $V = p(Z)$ on $S \times \square^n$. Because $Z \subset p^{-1}(p(Z)) = \mathbb{P}^r \times V$, we have $Z' = \mu^*_\phi(\square^1_K \times Z) \subset \mu^*_\phi(\mathbb{P}^d \times \square^1_K \times V) = \mathbb{P}^d \times \square^1_K \times V := Z_1$. By Lemma 2.8, V is admissible on $S \times \square^n$. So, $p^*[V] = \mathbb{P}^d \times V$ is admissible on $\mathbb{P}^d \times S \times \square^n$. In particular, $\mathbb{P}^d \times V$ has modulus $\mathbb{P}^d \times D$. Hence, $Z_1 = \mathbb{P}^d \times \square^1_K \times V$ also has modulus $\mathbb{P}^d_K \times D$. Now, $Z' \subset Z_1$ shows that Z' has modulus $\mathbb{P}^d_K \times D$ by Proposition 2.2. Thus, we proved $Z' \in z^q_{\mathcal{W}\times S}(\mathbb{P}^d_K \times S|\mathbb{P}^d_K \times D, n + 1)$.
Going back to the proof, one checks that $H^*_\bullet : z^q(\mathbb{P}^d \times S|\mathbb{P}^d \times D, \bullet) \to z^q(\mathbb{P}^d_K \times S|\mathbb{P}^d \times D, \bullet + 1)$ is a chain homotopy satisfying $\partial H^*(Z) + H^*\partial(Z) = Z_K - \eta \cdot (Z_K)$, and the same holds for $z_{\mathcal{W}\times S}$ by a straightforward computation (see [17, Lemma 5.6]). Furthermore, for each admissible Z, we have $\eta \cdot Z_K \in z^q_{\mathcal{W}_K\times S}(\mathbb{P}^d_K \times S|\mathbb{P}^d_K \times D, n)$, by the above proof of proper intersection of Z' with $W \times S \times F_K$, where $F = \{\infty\} \times F'$ for a face $F' \subset \square^n$. Hence, the base change $p^*_{K/k} : z^q(\mathbb{P}^d_k \times S|\mathbb{P}^d_k \times D, \bullet)/z^q_{\mathcal{W}\times S}(\mathbb{P}^d_k \times S|\mathbb{P}^d_k \times D, \bullet) \to z^q(\mathbb{P}^d_K \times S|\mathbb{P}^d_K \times D, \bullet)/z^q_{\mathcal{W}_K\times S}(\mathbb{P}^d_K \times S|\mathbb{P}^d_K \times D, \bullet)$ is homotopic to $\eta \cdot p^*_{K/k}$, which is

zero on the quotient. That is, $p^*_{K/k}$ on the above quotient complex is zero on homology. However, by the spreading argument (Proposition 4.3), $p^*_{K/k}$ is injective on homology. (N.B. We used here an elementary fact that $k(SL_{d+1,k})$ is purely transcendental over k. To check this fact, first note that by definition $k[SL_{d+1,k}] \simeq k[\{T_{i,j} | 1 \leq i,j \leq d+1\}]/(\det(M)-1)$ for the $(d+1,d+1)$-matrix $M = [T_{ij}]$ consisting of indeterminates $T_{i,j}$ for $1 \leq i,j \leq d+1$. Here by Cramer's rule we can write $\det(M) - 1 = \alpha T_{d+1,d+1} - \beta - 1$, where $\alpha = \det(M_{d+1,d+1})$, $\beta = \sum_{1 \leq j \leq d} (-1)^{d+1+j} \det(M_{d+1,j})$ and M_{ij} is the (i,j)-minor of M. Here both α and β do not have $T_{d+1,d+1}$. Hence $k[SL_{d+1,k}] \simeq k[\{T_{ij} | 1 \leq i,j \leq d+1, (i,j) \neq (d+1,d+1)\}, \frac{\beta+1}{\alpha}]$. Thus, $k(SL_{d+1,k}) \simeq k(\{T_{ij} | 1 \leq i,j \leq d+1, (i,j) \neq (d+1,d+1)\})$, which is purely transcendental over k.) Hence, the quotient complex $z^q(\mathbb{P}^d \times S | \mathbb{P}^d \times D, \bullet)/z^q_{\mathcal{W} \times S}(\mathbb{P}^d \times S | \mathbb{P}^d \times D, \bullet)$ is acyclic, i.e., the moving lemma holds for $(\mathbb{P}^d \times S, \mathbb{P}^d \times D)$, finishing Step 1.

Step 2. Now let X be a general smooth projective variety of dimension d. In this case, we choose a closed immersion $X \hookrightarrow \mathbb{P}^N$ for some $N \gg d$. We now run the linear projection argument of [17, §6] again without any extra argument to deduce the proof of the moving lemma for X from that of the projective spaces. We leave out the details. □

4.3. Contravariant functoriality. The following contravariant functoriality of multivariate additive higher Chow groups is an immediate application of the moving lemma and the proof is identical to that of [17, Theorem 7.1].

Theorem 4.5. *Let $f : X \to Y$ be a morphism of k-varieties, with Y smooth affine or smooth projective. Let $r \geq 1$ and $\underline{m} = (m_1, \cdots, m_r) \geq 1$. Then there exists a pull-back $f^* : \mathrm{CH}^q(Y[r]|D_{\underline{m}}, n) \to \mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$.*
If $g : Y \to Z$ is another morphism with Z smooth affine or smooth projective, then we have $(g \circ f)^* = f^* \circ g^*$.

Remark 4.6. As a special case, when $r = 1$, we have the pull-back map $f^* : \mathrm{TCH}^q(Y, n; m) \to \mathrm{TCH}^q(X, n; m)$.

4.4. The presheaf $\mathcal{TCH}$. For the rest of the section, we concentrate on additive higher Chow groups. Let $m \geq 0$. By Theorem 4.5, we see that $T^q_{n,m} := \mathrm{TCH}^q(-, n; m)$ is a presheaf of abelian groups on the category $\mathbf{SmAff}_k$, but we do not know if it is a presheaf on the categories $\mathbf{Sm}_k$ or $\mathbf{Sch}_k$. However, we can exploit Theorem 4.5 further to define a new presheaf on $\mathbf{Sm}_k$ and $\mathbf{Sch}_k$. The idea of this detour occurred to the authors while working on [20]. We do it for somewhat more general circumstances.

Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a full subcategory. Let F be a presheaf of abelian groups on $\mathcal{D}$, i.e. $F : \mathcal{D}^{\mathrm{op}} \to (\mathrm{Ab})$ is a functor to the category of abelian groups. For each object $X \in \mathcal{C}$, let $(X \downarrow \mathcal{D})$ be the category whose objects are the morphisms $X \to A$ in $\mathcal{C}$, with $A \in \mathcal{D}$, and a morphism from $h_1 : X \to A$ to $h_2 : X \to B$, with $A, B \in \mathcal{D}$, is given by a morphism $g : A \to B$ in $\mathcal{C}$ such that $g \circ h_1 = h_2$. The functor $F : \mathcal{D}^{\mathrm{op}} \to (\mathrm{Ab})$ induces the functor $(X \downarrow \mathcal{D})^{\mathrm{op}} \to (\mathrm{Ab})$ given by $(X \xrightarrow{h} A) \mapsto F(A)$, also denoted by F.

Definition 4.7. Suppose that for each $X \in \mathcal{C}$, the category $(X \downarrow \mathcal{D})$ is cofiltered. Then define $\mathcal{F}(X) := \underset{(X\downarrow\mathcal{D})^{\mathrm{op}}}{\mathrm{colim}} F$.

In particular, when $\mathcal{C} = \mathbf{Sch}_k$ and $\mathcal{D} = \mathbf{SmAff}_k$, one checks that $(X \downarrow \mathbf{SmAff}_k)$ is cofiltered, and for $X \in \mathbf{Sch}_k$, we define $\mathcal{TCH}^q(X, n; m) := \underset{(X\downarrow\mathbf{SmAff}_k)^{\mathrm{op}}}{\mathrm{colim}} T^q_{n,m}$.

Proposition 4.8. *Let $\mathcal{C}$ be a category and $\mathcal{D}$ be a full subcategory such that for each $X \in \mathcal{C}$, the category $(X \downarrow \mathcal{D})$ is cofiltered. Let F be a presheaf of abelian groups on $\mathcal{D}$ and let $\mathcal{F}$ be as in Definition 4.7.*
Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then for $X \in \mathcal{C}$, the association $X \mapsto \mathcal{F}(X)$ satisfies the following properties:

1. *There is a canonical homomorphism $\alpha_X : \mathcal{F}(X) \to F(X)$.*
2. *If $X \in \mathcal{D}$, then α_X is an isomorphism, and $\alpha : \mathcal{F} \to F$ defines an isomorphism of presheaves on $\mathcal{D}$.*
3. *There is a canonical pull-back $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$. If $g : Y \to Z$ is another morphism in $\mathcal{C}$, then we have $(g\circ f)^* = f^*\circ g^*$. So, $\mathcal{F}$ is a presheaf of abelian groups on $\mathcal{C}$. In particular, $\mathcal{TCH}^q(-, n; m)$ is a presheaf of abelian groups on $\mathbf{Sch}_k$, which is isomorphic to* $\mathrm{TCH}^q(-, n; m)$ *on $\mathbf{SmAff}_k$.*

Proof. (1) Let $(X \xrightarrow{h} A) \in (X \downarrow \mathcal{D})^{\mathrm{op}}$. By the given assumption, we have the pull-back $h^* : F(A) \to F(X)$. Regarding $F(X)$ as a constant functor on $(X \downarrow \mathcal{D})^{\mathrm{op}}$, this gives a morphism of functors $F \to F(X)$. Taking the colimits over all h, we obtain $\mathcal{F}(X) \to F(X)$, where $\alpha_X = \mathrm{colim}_h h^*$.
(2) When $X \in \mathcal{D}$, the category $(X \downarrow \mathcal{D})^{\mathrm{op}}$ has the terminal object $\mathrm{Id}_X : X \to X$. Hence, the colimit $\mathcal{F}(X)$ is just $F(X)$.
(3) A morphism $f : X \to Y$ in $\mathcal{C}$ defines a functor $f^\sharp : (Y \downarrow \mathcal{D})^{\mathrm{op}} \to (X \downarrow \mathcal{D})^{\mathrm{op}}$ given by $(Y \xrightarrow{h} A) \mapsto (X \xrightarrow{f} Y \xrightarrow{h} A)$. Thus, taking the colimits of the functors induced by F, we obtain $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$. For another morphism $g : Y \to Z$, that $(g \circ f)^* = f^* \circ g^*$ can be checked easily using the universal property of the colimits.
In the special case when $\mathcal{C} = \mathbf{Sch}_k$ and $\mathcal{D} = \mathbf{SmAff}_k$ with $F = \mathrm{TCH}^q(-, n; m)$, by Theorem 4.5 we know that F is a presheaf on $\mathbf{SmAff}_k$. So, the above general discussion holds. $\square$

Remark 4.9. Since additive higher Chow groups have pull-backs for flat maps (see [16, Lemma 4.7]), it follows that for $X \in \mathbf{Sm}_k$, $\alpha_{(-)}$ defines a map of presheaves $\mathcal{TCH}^q(-, n; m) \to \mathrm{TCH}^q(-, n; m)$ on the small Zariski site X_{Zar} of X. Proposition 4.8(2) says that this map is an isomorphism for affine open subsets of X. Thus, this map of presheaves on X_{Zar} induces an isomorphism of their Zariski sheafifications.

4.5. Moving lemma for smooth semi-local schemes. One remaining objective in Section 4 is to prove the following semi-local variation of Theorem 4.1:

Theorem 4.10. *Let $Y \in \mathbf{SmLoc}_k$. Let $\mathcal{W}$ be a finite set of locally closed subsets of Y. Then the inclusion* $\mathrm{Tz}^q_{\mathcal{W}}(Y, \bullet; m) \hookrightarrow \mathrm{Tz}^q(Y, \bullet; m)$ *is a quasi-isomorphism.*

We begin with some basic results related to cycles over semi-local schemes. Recall that when A is a ring and $\Sigma = \{p_1, \cdots, p_N\}$ is a finite subset of $\mathrm{Spec}\,(A)$, the localization at Σ is the localization $A \to S^{-1}A$, where $S = \bigcap_{i=1}^N (A \setminus p_i)$. For a quasi-projective k-scheme X and a finite subset Σ of (not necessarily closed) points of X, the localization X_Σ is defined by reducing it to the case when X is affine by the following elementary fact (see [25, Proposition 3.3.36]) that we use often.

Lemma 4.11. *Let X be a quasi-projective k-scheme. Given any finite subset $\Sigma \subset X$ and an open subset $U \subset X$ containing Σ, there exists an affine open subset $V \subset U$ containing Σ.*

For $X \in \mathbf{Sch}_k$ and a point $x \in X$, the open neighborhoods of x form a cofiltered category and we have functorial flat pull-back maps $(j^V_U)^* : \underline{\mathrm{Tz}}^q(V, n; m) \to \underline{\mathrm{Tz}}^q(U, n; m)$ for $j^V_U : U \hookrightarrow V$ in this category.

Lemma 4.12. *Let $X \in \mathbf{Sch}_k$ and let $x \in X$ be a scheme point. Let $Y = \mathrm{Spec}\,(\mathcal{O}_{X,x})$. Then we have* $\mathrm{colim}_{x \in U}\, \underline{\mathrm{Tz}}^q(U, n; m) \xrightarrow{\simeq} \underline{\mathrm{Tz}}^q(Y, n; m)$*, where the colimit is taken over all open neighborhoods U of x.*

Proof. Replacing x by an affine open neighborhood of $x \in X$, we may assume that X is affine and write $X = \mathrm{Spec}\,(A)$. Let $\mathfrak{p}_x \subset A$ be the prime ideal that corresponds to the point x and let $S := A \setminus \mathfrak{p}_x$, so that $Y = \mathrm{Spec}\,(S^{-1}A)$. To facilitate our proof, using the automorphism $y \mapsto 1/(1-y)$ of $\mathbb{P}^1$, we identify $\square$ with $\mathbb{A}^1$ and take $\{0, 1\} \subset \mathbb{A}^1$ as the faces. So, $X \times B_n = X \times \mathbb{A}^1 \times \mathbb{A}^{n-1} = \mathrm{Spec}\,(A[t, y_1, \cdots, y_{n-1}])$.

Let $\alpha \in \underline{\mathrm{Tz}}^q(Y, n; m)$. We need to find an open subset $U \subset X$ containing x such that the closure of α in $U \times \mathbb{A}^1 \times \mathbb{A}^{n-1}$ is admissible. For this, we may assume α is irreducible, i.e., it is a closed irreducible subscheme $Z \subset Y \times \mathbb{A}^1 \times \mathbb{A}^{n-1}$. Let $\overline{Z}$ be its Zariski closure in $X \times \mathbb{A}^1 \times \mathbb{A}^{n-1}$. Let $\mathfrak{p}$ be the prime ideal of $B := A[t, y_1, \cdots, y_{n-1}]$ such that $V(\mathfrak{p}) = \overline{Z}$.

For the proper intersection with faces, let $\mathfrak{q} \subset B$ be the prime ideal $(y_{i_1} - \epsilon_1, \cdots, y_{i_s} - \epsilon_s)$, where $1 \le i_1 < \cdots < i_s \le n-1$ and $\epsilon_j \in \{0, 1\}$. Let $\mathfrak{P}$ be a minimal prime of $\mathfrak{p} + \mathfrak{q}$. One checks immediately from the behavior of prime ideals under localizations that there is $a \in S$ such that either $\mathfrak{P}B[a^{-1}] = B[a^{-1}]$ or $\mathrm{ht}(\mathfrak{P}B[a^{-1}]) \ge q + s$. This means, over $U_{\mathfrak{q}} := \mathrm{Spec}\,(A[a^{-1}])$, either the intersection of $\overline{Z}_{U_{\mathfrak{q}}}$ with $V(\mathfrak{q})$ is empty, or has codimension $\ge q + s$. Applying this argument to all faces, we can take $U_1 := \bigcap_{\mathfrak{q}} U_{\mathfrak{q}}$. Then $\overline{Z}_{U_1}$ intersects all faces of $U_1 \times \mathbb{A}^1 \times \mathbb{A}^{n-1}$ properly.

For the modulus condition, let $\nu : \widehat{Z}^N \to \widehat{Z} \hookrightarrow X \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1}$ be the normalization composed with the closed immersion of the further Zariski closure $\widehat{Z}$ of $\overline{Z}$. Let $F^\infty_n = \sum_{i=1}^{n-1} \{y_i = \infty\}$ be the divisor at infinity. For an open set $j : U \hookrightarrow X$, the modulus condition of $\overline{Z}_U$ means $(m+1)[j^*\nu^*\{t = 0\}] \le$

$[j^*\nu^*(F_n^\infty)]$ on $\widehat{Z}_U^N$. Note that there exist only finitely many prime Weil divisors $P_1, \cdots, P_\ell$ on $\widehat{Z}^N$ such that $\mathrm{ord}_{P_i}(\nu^*(F_n^\infty) - (m+1)\nu^*\{t=0\}) < 0$. Their images Q_i under the normalization map $\widehat{Z}^N \to \widehat{Z}$ are still irreducible proper closed subsets of $\widehat{Z}$, thus of $X \times \mathbb{P}^1 \times (\mathbb{P}^1)^{n-1}$. Since $Z = \overline{Z}_Y$ has the modulus condition on $Y \times B_n$ by the given assumption, we have $(Y \times \widehat{B}_n) \cap Q_i = \emptyset$ for each $1 \le i \le \ell$. Thus, there is an affine open subset $U_2 \subset X$ containing x such that $(U_2 \times \widehat{B}_n) \cap Q_i = \emptyset$ for each $1 \le i \le \ell$. Now, by construction, $\overline{Z}_{U_2}$ on $U_2 \times B_n$ satisfies the modulus condition. So, taking an affine open subset $U \subset U_1 \cap U_2$ containing x, we have $\overline{Z}_U \in \underline{\mathrm{Tz}}^q(U, n; m)$. That $(\overline{Z}_U)_Y = Z$ is obvious. $\square$

We can extend this colimit description to semi-local schemes:

Lemma 4.13. *Let Y be a semi-local k-scheme obtained by localizing at a finite set Σ of scheme points of a quasi-projective k-variety X. For a cycle Z on $Y \times B_n$, let $\overline{Z}$ be its Zariski closure in $X \times B_n$.*
Then $Z \in \underline{\mathrm{Tz}}^q(Y, n; m)$ if and only if there exists an affine open subset $U \subset X$ containing Σ, such that $\overline{Z}_U \in \underline{\mathrm{Tz}}^q(U, n; m)$, where $\overline{Z}_U$ is the pull-back of $\overline{Z}$ via the open immersion $U \to X$.

Proof. The direction ($\Leftarrow$) is obvious by pulling back via the flat morphism $Y \hookrightarrow U$. For the direction ($\Rightarrow$), by Lemma 4.12, for each $x \in \Sigma$ we have an affine open neighborhood $U_x \subset X$ of x such that $\overline{Z}_{U_x} \in \underline{\mathrm{Tz}}^q(U_x, n; m)$. Take $W = \bigcup_{x \in \Sigma} U_x$. This is an open subset of X containing Σ. By Lemma 2.9, we have $\overline{Z}_W \in \underline{\mathrm{Tz}}^q(W, n; m)$. On the other hand, by Lemma 4.11, there exists an affine open subset $U \subset W$ containing Σ. By taking the flat pull-back via the open immersion $U \hookrightarrow W$, we get $\overline{Z}_U \in \underline{\mathrm{Tz}}^q(U, n; m)$. $\square$

Lemma 4.14. *Let Y be a semi-local integral k-scheme obtained by localizing at a finite set Σ of scheme points of an integral quasi-projective k-scheme X. Let $Z \in \mathrm{Tz}^q(Y, n; m)$, $W \in \mathrm{Tz}^q(Y, n+1; m)$, and let $\overline{Z}$, $\overline{W}$ be their Zariski closures in $X \times B_n$ and $X \times B_{n+1}$, respectively. For every open subset $U \subset X$, the subscript U means the pull-back to U. Then we have the following:*

1. *If $\partial Z = 0$, we can find an affine open subset $U \subset X$ containing Σ such that $\overline{Z}_U \in \mathrm{Tz}^q(U, n; m)$ and $\partial \overline{Z}_U = 0$.*
2. *If $Z = \partial W$, we can find an affine open subset $U \subset X$ containing Σ such that $\overline{Z}_U \in \mathrm{Tz}^q(U, n; m)$, $\overline{W}_U \in \mathrm{Tz}^q(U, n+1; m)$ and $\overline{Z}_U = \partial \overline{W}_U$.*

Proof. Note that (1) is a special case of (2), so we prove (2) only. Let $Z' := \overline{Z} - \partial \overline{W} \in z^q(X \times B_n)$. If Z' is 0 as a cycle, then take $U_0 = X$. If not, let $Z'_1, \cdots, Z'_s$ be the irreducible components of Z'. Since $Z = \partial W$, each component Z'_i has empty intersection with $Y \times B_n$. So, each $\pi((Z'_i)^c)$ is a non-empty open subset of X containing Σ, where $\pi : X \times B_n \to X$ is the projection, which is open. Take $U_0 = \bigcap_{i=1}^s \pi((Z'_i)^c)$.
On the other hand, Lemma 4.13 implies that there exist open sets $U_1, U_2 \subset X$ containing Σ such that $\overline{Z}_{U_1} \in \mathrm{Tz}^q(U_1, n; m)$ and $\overline{W}_{U_2} \in \mathrm{Tz}^q(U_2, n+1; m)$.

Choose an affine open subset $U \subset U_0 \cap U_1 \cap U_2$ containing Σ, using Lemma 4.11. Then part (2) holds over U by construction. $\square$

Proof of Theorem 4.10. We show that the chain map $\mathrm{Tz}^q_{\mathcal{W}}(Y, \bullet; m) \hookrightarrow \mathrm{Tz}^q(Y, \bullet; m)$ is a quasi-isomorphism. Let X be a smooth affine k-variety with a finite subset $\Sigma \subset X$ such that $Y = \mathrm{Spec}\,(\mathcal{O}_{X,\Sigma})$.

For surjectivity on homology, let $Z \in \underline{\mathrm{Tz}}^q(Y, n; m)$ be such that $\partial Z = 0$. Let $\overline{Z}$ be the Zariski closure of Z in $X \times B_n$. Here, $\partial \overline{Z}$ may not be zero, but by Lemma 4.14(1), there exists an affine open subset $U \subset X$ containing Σ such that we have $\partial \overline{Z}_U = 0$, where $\overline{Z}_U$ is the pull-back of $\overline{Z}$ to U. Let $\mathcal{W}_U = \{W_U | W \in \mathcal{W}\}$, where W_U is the Zariski closure of W in U. Then the quasi-isomorphism $\mathrm{Tz}^q_{\mathcal{W}_U}(U, \bullet; m) \hookrightarrow \mathrm{Tz}^q(U, \bullet; m)$ of Theorem 4.1 shows that there are some $C \in \mathrm{Tz}^q(U, n+1; m)$ and $Z'_U \in \mathrm{Tz}^q_{\mathcal{W}_U}(U, n; m)$ such that $\partial C = \overline{Z}_U - Z'_U$. Let $\iota : Y \hookrightarrow U$ be the inclusion. So, by applying the flat pull-back ι^* (which is equivariant with respect to taking faces), we obtain $\partial(\iota^* C) = Z - \iota^* Z'_U$, and here $\iota^* Z'_U \in \mathrm{Tz}^q_{\mathcal{W}}(Y, n; m)$, *i.e.*, Z is equivalent to a member in $\mathrm{Tz}^q_{\mathcal{W}}(Y, n; m)$.

For injectivity on homology, let $Z \in \mathrm{Tz}^q_{\mathcal{W}}(Y, n; m)$ be such that $Z = \partial Z'$ for some $Z' \in \mathrm{Tz}^q(Y, n+1; m)$. Let $\overline{Z}$ and $\overline{Z}'$ be the Zariski closures of Z and Z' on $X \times B_n$ and $X \times B_{n+1}$, respectively. Then by Lemma 4.14(2), there exists a nonempty open affine subset $U \subset X$ containing Σ such that $\overline{Z}_U = \partial \overline{Z}'_U$. Then the quasi-isomorphism $\mathrm{Tz}^q_{\mathcal{W}_U}(U, \bullet; m) \hookrightarrow \mathrm{Tz}^q(U, \bullet; m)$ of Theorem 4.1 shows that there exists $Z'' \in \mathrm{Tz}^q_{\mathcal{W}_U}(U, n+1; m)$ such that $\overline{Z}_U = \partial Z''$. Pulling back via $\iota : Y \hookrightarrow U$ then shows $Z = \partial(\iota^* Z'')$, with $\iota^* Z'' \in \mathrm{Tz}^q_{\mathcal{W}}(Y, n+1; m)$. $\square$

Using an argument identical to Theorem 4.5 (see [17, Theorem 7.1]), we get:

Corollary 4.15. *Let $f : Y_1 \to Y_2$ be a morphism in $\mathbf{Sch}^{\mathrm{ess}}_k$, where $Y_2 \in \mathbf{SmLoc}_k$. Then there is a natural pull-back $f^* : \mathrm{TCH}^q(Y_2, n; m) \to \mathrm{TCH}^q(Y_1, n; m)$.*

5. The Pontryagin product

Let R be a commutative ring and let (A, d_A) be a differential graded algebra over R. Recall that (*left*) differential graded module M over A is a left A-module M with a grading $M = \oplus_{n \in \mathbb{Z}} M_n$ and a differential d_M such that $A_m M_n \subset M_{m+n}$, $d_M(M_n) \subset M_{n+1}$ and $d_M(ax) = d_A(a)x + (-1)^n a d_M(x)$ for $a \in A_n$ and $x \in M$. A homomorphism of differential graded modules $f : M \to N$ over A is an A-module map which is compatible with gradings and differentials.

In this section, we show that the multivariate additive higher Chow groups have a product structure that resembles the Pontryagin product. We construct a differential operator on these groups in the next section and show that the product and the differential operator together turn multivariate additive higher Chow groups groups into a differential graded module over $\mathbb{W}_m \Omega^\bullet_R$ for suitable m, when $X = \mathrm{Spec}\,(R)$ is in $\mathbf{SmAff}^{\mathrm{ess}}_k$. This generalizes the DGA-structure on additive higher Chow groups of smooth projective varieties in [19]. The base field k is perfect in this section.

5.1. SOME CYCLE COMPUTATIONS. We generalize some of [19, §3.2.1, 3.2.2, 3.3]. Let (X, D) be a k-scheme with an effective divisor.
Recall that a permutation $\sigma \in \mathfrak{S}_n$ acts naturally on $\square^n$ via $\sigma(y_1, \cdots, y_n) := (y_{\sigma(1)}, \cdots, y_{\sigma(n)})$. This action extends to cycles on $X \times \square^n$ and $X \times \overline{\square}^n$.
Let $n, r \geq 1$ be given. Consider the finite morphism $\chi_{n,r} : X \times \square^n \to X \times \square^n$ given by $(x, y_1, \cdots, y_n) \mapsto (x, y_1^r, y_2, \cdots, y_n)$. Given an irreducible cycle $Z \subset X \times \square^n$, define $Z\{r\} := (\chi_{n,r})_*([Z]) = [k(Z) : k(\chi_{n,r}(Z))] \cdot [\chi_{n,r}(Z)]$. We extend it $\mathbb{Z}$-linearly.

LEMMA 5.1. *If Z is an admissible cycle with modulus D, then so is $Z\{r\}$.*

Proof. The proof is almost identical to that of [19, Lemma 3.11], except that the divisor $(m+1)\{t = 0\}$ there should be replaced by $D \times \overline{\square}^n$. We give its argument for the reader's convenience.
We may assume Z is irreducible. It is enough to show that $\chi_{n,r}(Z)$ is admissible with modulus D. We first check that it satisfies the face condition of Definition 2.3. When $n = 1$, the proper faces of $\square$ are of codimension 1, and for $\epsilon \in \{0, \infty\}$, we have $\partial_1^\epsilon(\chi_{n,r}(Z)) = r\partial_1^\epsilon(Z)$. When $n \geq 2$, for $\epsilon \in \{0, \infty\}$, we have $\partial_1^\epsilon(\chi_{n,r}(Z)) = r\partial_1^\epsilon(Z)$ and $\partial_i^\epsilon(\chi_{n,r}(Z)) = \chi_{n-1,r}(\partial_i^\epsilon(Z))$ if $i \geq 2$. For faces $F \subset \square^n$ of higher codimensions, we consequently have $F \cdot (\chi_{n,r}(Z)) = r(F \cdot Z)$ if F involves the equations $\{y_1 = \epsilon\}$, and $F \cdot (\chi_{n,r}(Z)) = \chi_{n-c,r}(F \cdot Z)$, otherwise, where c is the codimension of F. Since the intersection $F \cdot Z$ is proper, so is $\chi_{n-c,r}(F \cdot Z)$ by induction on the codimension of faces. This shows $\chi_{n,r}(Z)$ satisfies the face condition.
To show that $W := \chi_{n,r}(Z)$ has modulus D, consider the commutative diagram

$$\begin{tikzcd}
\overline{Z}^N \arrow[r, "\nu_Z"] \arrow[d, "\overline{\chi}^N_{n,r}"] & \overline{Z} \arrow[r, hook, "\iota_Z"] \arrow[d, "\overline{\chi}_{n,r}"] & X \times \overline{\square}^n \arrow[d, "\chi_{n,r}"] \\
\overline{W}^N \arrow[r, "\nu_W"] & \overline{W} \arrow[r, hook, "\iota_W"] & X \times \overline{\square}^n,
\end{tikzcd}$$

where $\overline{Z}, \overline{W}$ are the Zariski closures of Z and W in $X \times \overline{\square}^n$ and ν_Z, ν_W are the respective normalizations. The morphisms $\chi_{n,r}$, $\overline{\chi}_{n,r}$ are the natural induced maps, and $\overline{\chi}^N_{n,r}$ is induced by the universal property of normalization. Since Z has modulus D, we have the inequality

$$(5.1) \qquad [\nu_Z^* \iota_Z^* (D \times \overline{\square}^n)] \leq \sum_{i=1}^n [\nu_Z^* \iota_Z^* \{y_i = 1\}].$$

By the definition of $\chi_{n,r}$, we have $\chi_{n,r}^*(D \times \overline{\square}^n) = D \times \overline{\square}^n$, $\chi_{n,r}^*\{y_1 = 1\} \geq \{y_1 = 1\}$, and $\chi_{n,r}^*\{y_i = 1\} = \{y_i = 1\}$ for $i \geq 2$. Hence (5.1) implies that $[\nu_Z^* \iota_Z^* \chi_{n,r}^*(D \times \overline{\square}^n)] \leq \sum_{i=1}^n [\nu_Z^* \iota_Z^* \chi_{n,r}^* \{y_i = 1\}]$. By the commutativity of the diagram, this implies that ${\overline{\chi}^N_{n,r}}^* \left(\sum_{i=1}^n \nu_W^* \iota_W^* \{y_i = 1\} - \nu_W^* \iota_W^* (D \times \overline{\square}^n) \right) \geq 0$.

By Lemma 2.7, this implies $\sum_{i=1}^n \nu_W^* \iota_W^* \{y_i = 1\} - \nu_W^* \iota_W^* (D \times \overline{\square}^n) \geq 0$, which means W has modulus D. This completes the proof. $\square$

Let $n, i \geq 1$. Suppose X is smooth quasi-projective essentially of finite type over k. Let $(x, y_1, \cdots, y_n, y, \lambda)$ be the coordinates of $X \times \overline{\square}^{n+2}$. Consider the closed subschemes V_X^i on $X \times \square^{n+2}$ given by the equation $(1-y)(1-\lambda) = 1 - y_1$ if $i = 1$, and $(1-y)(1-\lambda) = (1-y_1)(1 + y_1 + \cdots + y_1^{i-1} - \lambda(1 + y_1 + \cdots + y_1^{i-2}))$ if $i \geq 2$.

Let $\widehat{V}_X^i$ be the Zariski closure of V_X^i in $X \times \overline{\square}^{n+2}$. Let $\pi_1 : X \times \overline{\square}^{n+2} \to X \times \overline{\square}^{n+1}$ be the projection that drops y_1, and let $\pi_1' := \pi_1|_{V_X^i}$. As in [19, Lemma 3.12], one sees that π_1' is proper surjective. For an irreducible cycle $Z \subset X \times \square^n$, define (see [19, Definition 3.13]) $\gamma_Z^i := \pi'_{1*}(V_X^i \cdot (Z \times \square^2))$ as an abstract algebraic cycle. One checks that it is also the Zariski closure of $\nu^i(Z \times \square)$, where $\nu^i : X \times \square^n \times \square \to X \times \square^{n+1}$ is the rational map given by $\nu^i(x, y_1, \cdots, y_n, y) = (x, y_2, y_3, \cdots, y_n, y, \frac{y - y_1^i}{y - y_1^{i-1}})$. We extend the definition of γ_Z^i $\mathbb{Z}$-linearly.

Lemma 5.2. *Let $Z \in z^q(X|D, n)$. Then $\gamma_Z^i \in z^q(X|D, n+1)$.*

Proof. Once we have Lemma 5.1, the proof of Lemma 5.2 is very similar to that of [19, Lemma 3.15], except we replace $(m+1)\{t = 0\}$ by $D \times \overline{\square}^{n+1}$. We give its argument for the reader's convenience.

We may assume Z is irreducible. To keep track of n, we write $\gamma_{Z,n}^i = \gamma_Z^i$. We first check that it satisfies the face condition of Definition 2.3. Let $\epsilon \in \{0, \infty\}$. Let $F \subset \square^{n+1}$ be a face. If F involves the equation $\{y_j = \epsilon\}$ for $j = n, n+1$, then by direction computations, we see that $\partial_n^0(\gamma_{Z,n}^i) = \sigma \cdot Z, \partial_{n+1}^0(\gamma_{Z,n}^i) = \sigma \cdot (Z\{i\})$ for the cyclic permutation $\sigma = (1, 2, \cdots, n)$, and $\partial_n^\infty(\gamma_{Z,n}^i) = 0$, $\partial_{n+1}^\infty(\gamma_{Z,n}^i) = \sigma \cdot (Z\{i-1\})$. Since Z is admissible with modulus D, so are $Z\{i\}$ and $Z\{i-1\}$ by Lemma 5.1. In particular, all of $\sigma \cdot Z, \sigma \cdot (Z\{i\})$, and $\sigma \cdot (Z\{i-1\})$ intersect all faces properly. Hence $\gamma_{Z,n}^i$ intersects F properly.

In case F does not involve the equations $\{y_j = \epsilon\}$ for $j = n, n+1$, we prove it by induction on $n \geq 1$. By direction calculations, for $j < n$, we have $\partial_j^\epsilon(\gamma_{Z,n}^i) = \gamma^i_{\partial_j^\epsilon Z, n-1}$ so that the dimension of $\partial_j^i(\gamma_{Z,n}^i)$ is at least one less by the induction hypothesis. Repeated applications of this argument for all other defining equations of F then give the result.

It remains to show that γ_Z^i has modulus D. Every irreducible component of γ_Z^i is of the form $W' = \pi_1'(Z')$, where Z' is an irreducible component of $V_X^i \cdot (Z \times \square^2)$. We prove W' has modulus D. Consider the following commutative diagram

$$\begin{tikzcd}
\overline{Z}'^N \arrow[r, "\nu_{Z'}"] \arrow[d, "\pi_1^N"] & \widehat{V}_X^i \arrow[r, hook, "\iota"] \arrow[d, "\overline{\pi}_1'"] \arrow[rd, "\pi_1'"] & X \times \overline{\square}^{n+2} \arrow[d, "\pi_1"] \\
\overline{W}'^N \arrow[r, "\nu"] & \overline{W}' \arrow[r, hook, "\iota_{W'}"] & X \times \overline{\square}^{n+1},
\end{tikzcd}$$

where $\nu_{Z'}$ is the normalization of the Zariski closure $\overline{Z}'$ of Z' in $\widehat{V}^i_X$, ν is the normalization of the Zariski closure $\overline{W}'$ of W' in $X \times \overline{\square}^{n+1}$, and π_1^N, $\overline{\pi}'_1$ are the induced morphisms. We use $(x, y_1, \cdots, y_n, y, \lambda) \in X \times \overline{\square}^{n+2}$ and $(x, y_2, \cdots, y_n, y, \lambda) \in X \times \overline{\square}^{n+1}$ as the coordinates. From the modulus D condition of Z, we deduce

$$\nu^*_{Z'} \iota^*(D \times \overline{\square}^{n+2}) \leq \sum_{j=1}^{n} \nu^*_{Z'} \iota^* \{y_j = 1\}. \tag{5.2}$$

Note that the above does not involve the divisors $\{y = 1\}$ and $\{\lambda = 1\}$. Since V^i_X is an effective divisor on $X \times \square^{n+2}$ defined by the equation $(1 - y_1)(*) = (1 - y)(1 - \lambda)$ for some polynomial $(*)$, we have $[\nu^*_{Z'} \iota^* \{y_1 = 1\}] \leq [\nu^*_{Z'} \iota^* \{y = 1\}] + [\nu^*_{Z'} \iota^* \{\lambda = 1\}]$.
Since the above diagram commutes, from (5.2) we deduce ${\pi_1^N}^* \nu^* \iota^*_{W'}(D \times \overline{\square}^{n+1}) \leq {\pi_1^N}^* \left(\sum_{j=2}^n \nu^* \iota^*_{W'} \{y_j = 1\} + \{y = 1\} + \{\lambda = 1\}\right)$. Hence by Lemma 2.7, we deduce $\nu^* \iota^*_{W'}(D \times \overline{\square}^{n+1}) \leq \sum_{j=2}^n \nu^* \iota^*_{W'} \{y_j = 1\} + \{y = 1\} + \{\lambda = 1\}$, which means W' has modulus D. This finishes the proof. $\square$

Lemma 5.3. *Let $n \geq 2$ and let $Z \in z^q(X|D, n)$ such that $\partial^\epsilon_i(Z) = 0$ for all $1 \leq i \leq n$ and $\epsilon \in \{0, \infty\}$. Let $\sigma \in \mathfrak{S}_n$. Then there exists $\gamma^\sigma_Z \in z^q(X|D, n+1)$ such that $Z = (\mathrm{sgn}(\sigma))(\sigma \cdot Z) + \partial(\gamma^\sigma_Z)$.*

Proof. Its proof is almost identical to that of [19, Lemma 3.16], except that we use Lemma 5.2 instead of [19, Lemma 3.15]. We give its argument for the reader's convenience.
First consider the case when σ is the transposition $\tau = (p, p+1)$ for $1 \leq p \leq n-1$. We do it for $p = 1$ only, i.e. $\tau = (1, 2)$. Other cases of τ are similar. Let ξ be the unique permutation such that $\xi \cdot (x, y_1, \cdots, y_{n+1}) = (x, y_n, y_1, y_{n+1}, y_2, \cdots, y_{n-1})$. Consider the cycle $\gamma^\tau_Z := \xi \cdot \gamma^1_Z$, where γ^1_Z is as in Lemma 5.2. Being a permutation of an admissible cycle, so is this cycle γ^ξ_Z. Furthermore, by direction calculations, we have $\partial^\infty_1(\gamma^\tau_Z) = 0$, $\partial^0_1(\gamma^\tau_Z) = \tau \cdot Z$, $\partial^\infty_3(\gamma^\tau_Z) = 0$ and $\partial^0_3(\gamma^\tau_Z) = Z$. On the other hand, for $\epsilon \in \{0, \infty\}$, $\partial^\epsilon_2(\gamma^\tau_Z)$ is a cycle obtained from $\gamma^1_{\partial^\epsilon_2(Z)}$ by a permutation action. So, it is 0 because $\partial^\epsilon_2(Z) = 0$ by the given assumptions. Similarly for $j \geq 4$, we have $\partial^\epsilon_j(\gamma^\tau_Z) = 0$. Hence $\partial(\gamma^\tau_Z) = Z + \tau \cdot Z$, as desired.
Now let $\sigma \in \mathfrak{S}_n$ be any. By a basic result from group theory, we can express $\sigma = \tau_r \tau_{r-1} \cdots \tau_2 \tau_1$, where each τ_i is a transposition of the form $(p, p+1)$ as considered before. Let $\sigma_0 := \mathrm{Id}$ and $\sigma_\ell := \tau_\ell \tau_{\ell-1} \cdots \tau_1$ for $1 \leq \ell \leq r$. For each such ℓ, by the previous case considered, we have $(-1)^{\ell-1} \sigma_{\ell-1} \cdot Z + (-1)^{\ell-1} \tau_\ell \cdot \sigma_{\ell-1} \cdot Z = \partial((-1)^{\ell-1} \gamma^{\tau_\ell}_{\sigma_{\ell-1} \cdot Z})$. Since $\tau_\ell \cdot \sigma_{\ell-1} = \sigma_\ell$, by taking the sum of the above equations over all $1 \leq \ell \leq r$, after cancellations, we obtain $Z + (-1)^{r-1} \sigma \cdot Z = \partial(\gamma^\sigma_Z)$, where $\gamma^\sigma_Z := \sum_{\ell=1}^r (-1)^{\ell-1} \gamma^{\tau_\ell}_{\sigma_{\ell-1} \cdot Z}$. Since $(-1)^r = \mathrm{sgn}(\sigma)$, we obtain the desired result. $\square$

5.2. Pontryagin product. Let $X \in \mathbf{Sch}_k^{\mathrm{ess}}$ be an equidimensional scheme. For $\underline{m} = (m_1, \cdots, m_r) \geq 1$, let $\mathrm{CH}(X[r]|D_{\underline{m}}) := \oplus_{q,n} \mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$. For $m \geq 1$, we let $\mathrm{TCH}(X; m) = \oplus_{q,n} \mathrm{TCH}^q(X, n; m) = \oplus_{q,n} \mathrm{CH}^q(X[1]|D_{m+1}, n-1)$. The objective of §5.2 is to prove the following result which generalizes [19, §3].

Theorem 5.4. *Let k be a perfect field. Let $m \geq 0$ and let $\underline{m} = (m_1, \cdots, m_r) \geq 1$. Let X, Y be both either in $\mathbf{SmAff}_k^{\mathrm{ess}}$ or in $\mathbf{SmProj}_k$. Then we have the following:*

1. $\mathrm{TCH}(X; m)$ *is a graded commutative algebra with respect to a product* $\wedge_X$.
2. $\mathrm{CH}(X[r]|D_{\underline{m}})$ *is a graded module over* $\mathrm{TCH}(X; |\underline{m}| - 1)$.
3. *For $f : Y \to X$ with $d = \dim Y - \dim X$, $f^* : \mathrm{CH}(X[r]|D_{\underline{m}}) \to \mathrm{CH}(Y[r]|D_{\underline{m}})$ and $f_* : \mathrm{CH}(Y[r]|D_{\underline{m}}) \to \mathrm{CH}(X[r]|D_{\underline{m}})[-d]$ (if f is proper in addition) are morphisms of graded* $\mathrm{TCH}(X; |\underline{m}| - 1)$*-modules.*

The proof requires a series of results and will be over after Lemma 5.13.

Lemma 5.5. *Let $X_1, X_2 \in \mathbf{Sch}_k^{\mathrm{ess}}$. For $i = 1, 2$ and $r_i \geq 1$, let V_i be a cycle on $X_i \times \mathbb{A}^{r_i} \times \square^{n_i}$ with modulus $\underline{m}_i = (m_{i1}, \cdots, m_{ir_i})$, respectively. Then $V_1 \times V_2$, regarded as a cycle on $X_1 \times X_2 \times \mathbb{A}^{r_1 + r_2} \times \square^{n_1 + n_2}$ after a suitable exchange of factors, has modulus $(\underline{m}_1, \underline{m}_2)$.*

Proof. We may assume that V_1 and V_2 are irreducible. It is enough to show that each irreducible component $W \subset V_1 \times V_2$ has modulus $(\underline{m}_1, \underline{m}_2)$. Let $\iota_i : \overline{V}_i \hookrightarrow X_i \times \mathbb{A}^{r_i} \times \overline{\square}^{n_i}$ be the Zariski closure of V_i, and let $\nu_{\overline{V}_i} : \overline{V}_i^N \to \overline{V}_i$ be the normalization for $i = 1, 2$. Since k is perfect, [16, Lemma 3.1] says that the morphism $\nu := \nu_{\overline{V}_1} \times \nu_{\overline{V}_2} : \overline{V}_1^N \times \overline{V}_2^N \to \overline{V}_1 \times \overline{V}_2 = \overline{V_1 \times V_2}$ is the normalization. Hence, the composite $\overline{W}^N \overset{\nu_W}{\to} \overline{W} \overset{\iota}{\hookrightarrow} \overline{V}_1 \times \overline{V}_2$, where $\overline{W}$ is the Zariski closure of W and ν_W is the normalization of $\overline{W}$, factors into $\overline{W}^N \overset{\iota^N}{\to} \overline{V}_1^N \times \overline{V}_2^N \overset{\nu}{\to} \overline{V}_1 \times \overline{V}_2$, where ι^N is the natural inclusion.
Let $(t_1, \cdots, t_{r_1}, t'_1, \cdots, t'_{r_2}, y_1, \cdots, y_{n_1 + n_2}) \in \mathbb{A}^{r_1 + r_2} \times \overline{\square}^{n_1 + n_2}$ be the coordinates. Consider two divisors $D^1 := \sum_{i=1}^{n_1} \{y_i = 1\} - \sum_{j=1}^{r_1} m_{1j} \{t_j = 0\}, D^2 := \sum_{i=n_1+1}^{n_1+n_2} \{y_i = 1\} - \sum_{j=1}^{r_2} m_{2j} \{t'_j = 0\}$. By the modulus conditions satisfied by V_1 and V_2, we have $((\iota_1 \times 1) \circ (\nu_{\overline{V}_1} \times 1))^* D^1 \geq 0$ and $((1 \times \iota_2) \circ (1 \times \nu_{\overline{V}_2}))^* D^2 \geq 0$. Thus, we have $\nu^* (\iota_1 \times \iota_2)^* (D^1 + D^2) \geq 0$ on $\overline{V}_1^N \times \overline{V}_2^N$ so that $(\iota^N)^* \nu^* (\iota_1 \times \iota_2)^* (D^1 + D^2) \geq 0$ on $\overline{W}^N$. Since $\iota \circ \nu_W = \nu \circ \iota^N$, this is equivalent to $\nu_W^* \iota^* (\iota_1 \times \iota_2)^* (D^1 + D^2) \geq 0$, which shows W has modulus $(\underline{m}_1, \underline{m}_2)$. $\square$

Definition 5.6. Let $r \geq 1$ be an integer and define $\mu : X_1 \times \mathbb{A}^1 \times \square^{n_1} \times X_2 \times \mathbb{A}^r \times \square^{n_2} \to X_1 \times X_2 \times \mathbb{A}^r \times \square^{n_1 + n_2}$ by $(x_1, t, \{y_j\}) \times (x_2, \{t_i\}, \{y'_j\}) \mapsto (x_1, x_2, \{tt_i\}, \{y_j\}, \{y'_j\})$.

The map μ is flat, but not proper. But, the following generalization of [19, Lemma 3.4] gives a way to take a push-forward:

Proposition 5.7. *Let $V_1 \subset X_1 \times \mathbb{A}^1 \times \square^{n_1}$ and $V_2 \subset X_2 \times \mathbb{A}^r \times \square^{n_2}$ be closed subschemes with moduli m and $\underline{m} \geq 1$, respectively. Then $\mu|_{V_1 \times V_2}$ is finite.*

Proof. Since μ is an affine morphism, the proposition is equivalent to show that $\mu|_{V_1 \times V_2}$ is projective.
Set $X = X_1 \times X_2 \times \square^{n_1+n_2}$. Let $\Gamma \hookrightarrow X_1 \times X_2 \times \mathbb{A}^1 \times \mathbb{A}^r \times \mathbb{A}^r \times \square^{n_1+n_2} = X \times \mathbb{A}^1 \times \mathbb{A}^r \times \mathbb{A}^r$ denote the graph of the morphism μ and let $\overline{\Gamma} \hookrightarrow X \times \mathbb{P}^1 \times (\mathbb{P}^1)^r \times (\mathbb{P}^1)^r = X \times P_1 \times P_2 \times P_3$ be its closure, where $P_1 = \mathbb{P}^1$ and $P_2 = P_3 = (\mathbb{P}^1)^r$. Let p_i be the projection of $X \times \mathbb{P}^1 \times (\mathbb{P}^1)^r \times (\mathbb{P}^1)^r$ to $X \times P_i$ for $1 \leq i \leq 3$. Set $\overline{\Gamma}^0 = p_3^{-1}(X \times \mathbb{A}^r)$. Then $p_3 : \overline{\Gamma}^0 \to X \times \mathbb{A}^r$ is projective.
Using the homogeneous coordinates of $P_1 \times P_2 \times P_3$, one checks easily that $Z := \overline{\Gamma}^0 \setminus \Gamma \subset E \cup (\bigcup_{i=1}^r E_i)$ (the union is taken inside $X \times P_1 \times P_2 \times P_3$), where $E = X \times \{\infty\} \times (\{0\})^r \times \mathbb{A}^r$ and $E_i = X \times \{0\} \times ((\mathbb{P}^1)^{i-1} \times \{\infty\} \times (\mathbb{P}^1)^{r-i}) \times \mathbb{A}^r$.
Let $V = V_1 \times V_2$. Let Γ_V be the graph Γ restricted to V and let $\overline{\Gamma}_V$ be its Zariski closure in $X \times P_1 \times P_2 \times P_3$. Since $p_3 : \overline{\Gamma}^0 \to X \times \mathbb{A}^r$ is projective, so is the map $\overline{\Gamma}^0_V := \overline{\Gamma}_V \cap \overline{\Gamma}^0 \to X \times \mathbb{A}^r$. So, if we show $\overline{\Gamma}^0_V \cap Z = \emptyset$, then $V \simeq \Gamma_V = \overline{\Gamma}^0_V$ is projective over $X \times \mathbb{A}^r$, which is the assertion of the proposition.
To show $\overline{\Gamma}^0_V \cap Z = \emptyset$, consider the projections $X \times P_1 \times P_2 \times P_3 \xrightarrow{p_1} X \times P_1 \xrightarrow{\pi_1} X_1 \times P_1 \times \square^{n_1}$. Since the closure $\overline{V}_1$ has modulus $m \geq 1$ on $X_1 \times P_1 \times \square^{n_1}$, we have $\overline{V}_1 \cap (X_1 \times \{0\} \times \square^{n_1}) = \emptyset$. In particular, $\overline{\Gamma}_V \cap E_i \hookrightarrow (\pi_1 \circ p_1)^{-1}(\overline{V}_1 \cap (X_1 \times \{0\} \times \square^{n_1})) = \emptyset$ for $1 \leq i \leq r$.
To show that $\overline{\Gamma}^0_V \cap E = \emptyset$, consider the projections $X \times P_1 \times P_2 \times P_3 \xrightarrow{p_2} X \times P_2 \xrightarrow{\pi_2} X_2 \times P_2 \times \square^{n_2}$. Since the closure $\overline{V}_2$ has modulus $\underline{m} \geq 1$ on $X_2 \times P_2 \times \square^{n_2}$, we have $\overline{V}_2 \cap (X_2 \times (\{0\})^r \times \square^{n_2}) = \emptyset$. In particular, $\overline{\Gamma}_V \cap E \hookrightarrow (\pi_2 \circ p_2)^{-1}(\overline{V}_2 \cap (X_2 \times (\{0\})^r \times \square^{n_2})) = \emptyset$. This finishes the proof. $\square$

Lemma 5.8. *Let $X \in \mathbf{Sch}_k^{\mathrm{ess}}$ and let V be a cycle on $X \times \mathbb{A}^1 \times \mathbb{A}^r \times \square^n$ with modulus $(|\underline{m}|, \underline{m})$, where $\underline{m} = (m_1, \cdots, m_r) \geq 1$. Suppose $\mu|_V$ is finite. Then the closed subscheme $\mu(V)$ on $X \times \mathbb{A}^r \times \square^n$ has modulus $\underline{m}$.*

Proof. This is a straightforward generalization of [19, Proposition 3.8] and is a simple application of Lemma 2.7. We skip the detail. We only remark that it is crucial for the proof that the $\mathbb{A}^1$-component of the modulus is at least $|\underline{m}|$. $\square$

Definition 5.9. For any irreducible closed subscheme $V \subset X \times \mathbb{A}^1 \times \mathbb{A}^r \times \square^n$ such that $\mu|_V : V \to \mu(V)$ is finite, where μ is as in Definition 5.6, define $\mu_*(V)$ as the push-forward $\mu_*(V) = \deg(\mu|_V) \cdot [\mu(V)]$. Extend it $\mathbb{Z}$-linearly.
If V_1 is a cycle on $X_1 \times \mathbb{A}^1 \times \square^{n_1}$ and V_2 is a cycle on $X_2 \times \mathbb{A}^r \times \square^{n_2}$ such that $\mu|_{V_1 \times V_2}$ is finite, we define the external product $V_1 \times_\mu V_2 := \mu_*(V_1 \times V_2)$. If $p_i = \dim V_i$, then $\dim(V_1 \times_\mu V_2) = p_1 + p_2$. If $X_1 \times X_2$ is equidimensional and if q_i is the codimension of V_i, then $V_1 \times_\mu V_2$ has codimension $q_1 + q_2 - 1$.

Lemma 5.10. *Let $V_1 \in z^{q_1}(X_1[1]|D_m, n_1)$ and $V_2 \in z^{q_2}(X_2[r]|D_{\underline{m}}, n_2)$ with $X_1, X_2 \in \mathbf{Sch}_k^{\mathrm{ess}}$ and $m, \underline{m} \geq 1$. Then $V_1 \times_\mu V_2$ intersects all faces of $X_1 \times X_2 \times \mathbb{A}^r \times \square^{n_1+n_2}$ properly.*

Proof. We may assume that V_1 and V_2 are irreducible. $V_1 \times V_2$ clearly intersects all faces of $X_1 \times X_2 \times \mathbb{A}^1 \times \mathbb{A}^r \times \square^{n_1+n_2}$ properly. It follows from Proposition 5.7 that $\mu|_{V_1 \times V_2}$ is finite. In this case, the proper intersection property of $\mu(V_1 \times_\mu V_2)$ follows exactly like that of the finite push-forwards of Bloch's higher Chow cycles. $\square$

Corollary 5.11. *Let $X_1, X_2, X_3 \in \mathbf{Sch}_k^{\mathrm{ess}}$ be equidimensional and let $\underline{m} \geq 1$. Then there is a product*

$$\times_\mu : z^{q_1}(X_1[1]|D_{|\underline{m}|}, n_1) \otimes z^{q_2}(X_2[r]|D_{\underline{m}}, n_2) \to \\ \to z^{q_1+q_2-1}((X_1 \times X_2)[r]|D_{\underline{m}}, n_1+n_2)$$

which satisfies the relation $\partial(\xi \times_\mu \eta) = \partial(\xi) \times_\mu \eta + (-1)^{n_1}\xi \times_\mu \partial(\eta)$. It is associative in the sense that $(\alpha_1 \times_\mu \alpha_2) \times_\mu \beta = \alpha_1 \times_\mu (\alpha_2 \times_\mu \beta)$ for $\alpha_i \in z^{q_i}(X_i[1]|D_{|\underline{m}|}, n_i)$ for $i = 1, 2$ and $\beta \in z^{q_3}(X_3[r]|D_{\underline{m}}, n_3)$. In particular, it induces operations $\times_\mu : \mathrm{CH}^{q_1}(X_1[1]|D_{|\underline{m}|}, n_1) \otimes \mathrm{CH}^{q_2}(X_2[r]|D_{\underline{m}}, n_2) \to \mathrm{CH}^{q_1+q_2-1}((X_1 \times X_2)[r]|D_{\underline{m}}, n_1+n_2)$.

Proof. The existence of $\times_\mu$ on the level of cycle complexes follows from the combination of Proposition 5.7, Lemma 5.8 and Lemma 5.10. The associativity follows from that of the Cartesian product $\times$ and the product $\mu : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$. By definition, one checks $\partial(\xi \times \eta) = \partial(\xi) \times \eta + (-1)^{n_1}\xi \times \partial(\eta)$. So, by applying μ_*, we get the required relation. That $\times_\mu$ descends to the homology follows. $\square$

Definition 5.12. Let $\underline{m} = (m_1, \cdots, m_r) \geq 1$ and let X be in $\mathbf{SmAff}_k^{\mathrm{ess}}$ or in $\mathbf{SmProj}_k$. For cycle classes $\alpha_1 \in \mathrm{CH}^{q_1}(X[1]|D_{|\underline{m}|}, n_1)$ and $\alpha_2 \in \mathrm{CH}^{q_2}(X[r]|D_{\underline{m}}, n_2)$, define the internal product $\alpha_1 \wedge_X \alpha_2$ to be $\Delta_X^*(\alpha_1 \times_\mu \alpha_2)$ via the diagonal pull-back $\Delta_X^* : \mathrm{CH}^{q_1+q_2-1}((X \times X)[r]|D_{\underline{m}}, n_1+n_2) \to \mathrm{CH}^{q_1+q_2-1}(X[r]|D_{\underline{m}}, n_1+n_2)$. This map exists by Theorem 4.5 and Corollary 4.15.

Lemma 5.13. *$\wedge_X$ is associative in the sense that $(\alpha_1 \wedge_X \alpha_2) \wedge_X \beta = \alpha_1 \wedge_X (\alpha_2 \wedge_X \beta)$ for $\alpha_1, \alpha_2 \in \mathrm{CH}(X[1]|D_{|\underline{m}|})$ and $\beta \in \mathrm{CH}(X[r]|D_{\underline{m}})$. $\wedge_X$ is also graded-commutative on $\mathrm{CH}(X[1]|D_{|\underline{m}|})$.*

Proof. The associativity holds by Corollary 5.11. For the graded-commutativity, first note by Theorem 3.2 that we can find representatives α_1 and α_2 of the given cycle classes whose codimension 1 faces are all trivial. Let σ be the permutation that sends $(1, \cdots, n_1, n_1+1, \cdots, n_1+n_2)$ to $(n_1+1, \cdots, n_1+n_2, 1, \cdots, n_1)$ so that $\mathrm{sgn}(\sigma) = (-1)^{n_1+n_2}$. It follows from Lemma 5.3 that $\alpha_1 \wedge_X \alpha_2 = (-1)^{n_1+n_2}\alpha_2 \wedge_X \alpha_1 + \partial(W)$ for some admissible cycle W, as desired. $\square$

Proof of Theorem 5.4. The proof of (1) and (2) is just a combination of the above discussion under the observation that $\mathrm{TCH}^q(X, n; m) = \mathrm{CH}^q(X[1]|D_{m+1}, n-1)$ for $m \geq 0$ and $n \geq 1$. To prove (3) for f^*, consider the commutative diagram

$$
\begin{tikzcd}
Y[r] \times \square^n \arrow[r, "\Delta_Y"] \arrow[d, "f"] & (Y \times Y)[r] \times \square^n \arrow[d, "f\times f"] & (Y \times Y)[r+1] \times \square^n \arrow[l, "\mu_Y"'] \arrow[d, "f\times f"] \\
X[r] \times \square^n \arrow[r, "\Delta_X"] & (X \times X)[r] \times \square^n & (X \times X)[r+1] \times \square^n. \arrow[l, "\mu_X"']
\end{tikzcd}
\tag{5.3}
$$

There is a finite set $\mathcal{W}$ of locally closed subsets of X such that $f^* : z^{q_1}_{\mathcal{W}}(X[1]|D_{|\underline{m}|}, \bullet) \to z^{q_1}(Y[1]|D_{|\underline{m}|}, \bullet)$ and $f^* : z^{q_2}_{\mathcal{W}}(X[r]|D_{\underline{m}}, \bullet) \to z^{q_2}(Y[r]|D_{\underline{m}}, \bullet)$ can be defined as taking cycles associated to the inverse images. Moreover, it is enough to consider the product of cycles in $z^{q_1}_{\mathcal{W}}(X[1]|D_{|\underline{m}|}, \bullet)$ and $z^{q_2}_{\mathcal{W}}(X[r]|D_{\underline{m}}, \bullet)$ by the moving lemmas Theorems 4.1 and 4.4. For irreducible cycles $V_1 \in z^{q_1}(X[1]|D_{|\underline{m}|}, n_1)$ and $V_2 \in z^{q_2}(X[r]|D_{\underline{m}}, n_2)$, the map μ_Y is finite when restricted to $f^*(V_1) \times f^*(V_2)$ by Lemma 5.7. In particular, $\mu_Y(f^*(V_1) \times f^*(V_2)) \in z^{q_1+q_2-1}((Y \times Y)[r]|D_{\underline{m}}, n_1 + n_2)$.
Since the right square in the diagram (5.3) is transverse, it follows that $f^*(\mu_X(V_2 \times V_2)) = \mu_Y(f^*(V_1) \times f^*(V_2))$ as cycles. The desired commutativity of the product with f^* now follows from the commutativity of the left square in (5.3) and the composition law of Theorem 4.5.
The proof of (3) for f_* is just the projection formula, whose proof is identical to the one given in [19, Theorem 3.19] in the case when $X_1, X_2 \in \mathbf{SmProj}_k$. $\square$

As applications, we obtain:

Corollary 5.14. *Let X be in $\mathbf{SmAff}^{\rm ess}_k$ or in $\mathbf{SmProj}_k$. Then for $q, n \geq 0$ and $\underline{m} \geq 1$, the group $\mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$ is a $\mathbb{W}_{(|\underline{m}|-1)}(k)$-module.*

Proof. Applying Theorem 5.4 to X and the structure map $X \to \mathrm{Spec}\,(k)$, it follows that $\mathrm{CH}(X[r]|D_{\underline{m}})$ is a graded module over $\mathrm{TCH}(k; |\underline{m}| - 1)$. By Corollary 5.11, this yields a $\mathrm{TCH}^1(k, 1; |\underline{m}| - 1)$-module structure on each $\mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$. The corollary now follows from the fact that there is a ring isomorphism $\mathbb{W}_m(k) \xrightarrow{\simeq} \mathrm{TCH}^1(k, 1; m)$ for every $m \geq 1$ by [28, Corollary 3.7]. $\square$

We can explain the homotopy invariance of the groups $\mathrm{CH}^q(X, n)$ in terms of additive higher Chow groups as follows.

Corollary 5.15. *For $X \in \mathbf{Sch}^{\rm ess}_k$ which is equidimensional and for $q, n \geq 0$, we have $\mathrm{CH}^q(X[1]|D_1, n) = 0$.*

Proof. By Corollary 5.11, we have a map $\times_\mu : \mathrm{CH}^1(pt[1]|D_1, 0) \otimes \mathrm{CH}^q(X[1]|D_1, n) \to \mathrm{CH}^q(X[1]|D_1, n)$ and it follows from the definition of $\times_\mu$ that $[1] \times_\mu \alpha = \alpha$ for every $\alpha \in \mathrm{CH}^q(X[1]|D_1, n)$, where $[1] \in \mathrm{CH}^1(pt[1]|D_1, 0)$ is the cycle given by the closed point $1 \in \mathbb{A}^1(k)$. It therefore suffices to show that the homology class of 1 is zero. To do so, we may use the identification $(\square, \{\infty, 0\}) \simeq (\mathbb{A}^1, \{0, 1\})$ given by $y \mapsto 1/(1-y)$ again. Then the cycle $C \subset \mathbb{A}^2$ given by $\{(t, y) \in \mathbb{A}^2 | ty = 1\}$ is an admissible cycle in $z^1(pt[1]|D_1, 1)$ such that $\partial_1([C]) = [1]$ and $\partial_0([C]) = 0$. $\square$

6. The structure of differential graded modules

In this section, we construct a differential operator on the graded module of §5 of multivariate additive higher Chow groups over the univariate additive higher Chow groups, generalizing [19, §4]. We assume that k is perfect and $\mathrm{char}(k) \neq 2$.

6.1. Differential. Let X be a smooth quasi-projective scheme essentially of finite type over k. Let $r \geq 1$ and let $\underline{m} = (m_1, \cdots, m_r) \geq 1$. Let $(\mathbb{G}_m^r)^\times := \{(t_1, \cdots, t_r) \in \mathbb{G}_m^r \mid t_1 \cdots t_r \neq 1\}$. Consider the morphism $\delta_n : (\mathbb{G}_m^r)^\times \times \square^n \to \mathbb{G}_m^r \times \square^{n+1}$, $(t_1, \cdots, t_r, y_1, \cdots, y_n) \mapsto (t_1, \cdots, t_r, \frac{1}{t_1 \cdots t_r}, y_1, \cdots, y_n)$. It induces $\delta_n : X \times (\mathbb{G}_m^r)^\times \times \square^n \to X \times \mathbb{G}_m^r \times \square^{n+1}$.
Recall a closed subscheme $Z \subset X \times \mathbb{A}^r \times \square^n$ with modulus $\underline{m}$ does not intersect the divisor $\{t_1 \cdots t_r = 0\}$. So, it is closed in $X \times \mathbb{G}_m^r \times \square^n$. For such Z, we define $Z^\times := Z|_{X \times (\mathbb{G}_m^r)^\times \times \square^n}$.

Lemma 6.1. *For a closed subscheme $Z \subset X \times \mathbb{A}^r \times \square^n$ with modulus $\underline{m}$, the image $\delta_n(Z^\times)$ is closed in $X \times \mathbb{G}_m^r \times \square^{n+1}$.*

Proof. It is enough to show that $\delta_n : X \times (\mathbb{G}_m^r)^\times \times \square^n \to X \times \mathbb{G}_m^r \times \square^{n+1}$ is a closed immersion. It reduces to show that the map $(\mathbb{G}_m^r)^\times \to \mathbb{G}_m^r \times (\mathbb{P}^1 \setminus \{1\})$ given by $(t_1, \cdots, t_r) \mapsto (t_1, \cdots, t_r, 1/(t_1 \cdots t_r))$ is a closed immersion. This is obvious because the image coincides with the closed subscheme given by the equation $t_1 \cdots t_r y = 1$, where $(t_1, \cdots, t_r, y) \in \mathbb{G}_m^r \times \square$ are the coordinates. $\square$

Definition 6.2 (*cf.* [19, Definition 4.3]). For a closed subscheme $Z \subset X \times \mathbb{A}^r \times \square^n$ with modulus $\underline{m}$, we write $\delta_n(Z) := \delta_n(Z^\times)$. If Z is a cycle, we define $\delta_n(Z)$ by extending it $\mathbb{Z}$-linearly. We may often write $\delta(Z)$ if no confusion arises.

Lemma 6.3. *Let Z be a cycle on $X \times \mathbb{A}^r \times \square^n$ with modulus $\underline{m}$. Then $\delta_n(Z)$ is a cycle on $X \times \mathbb{A}^r \times \square^{n+1}$ with modulus $\underline{m}$.*

Proof. We may suppose that Z is irreducible. Let $V = \delta_n(Z)$, which is *a priori* closed in $X \times \mathbb{G}_m^r \times \square^{n+1}$. If the closure V' of V in $X \times \mathbb{A}^r \times \square^{n+1}$ has modulus $\underline{m}$, then it does not intersect the divisor $\{t_1 \cdots t_r = 0\}$ of $X \times \mathbb{A}^r \times \square^{n+1}$, so $V = V'$, and V is closed in $X \times \mathbb{A}^r \times \square^{n+1}$ with modulus $\underline{m}$. So, we reduce to show that V' has modulus $\underline{m}$.
Let $\overline{Z}$ and $\overline{V}$ be the Zariski closures of Z and V' in $X \times \mathbb{A}^r \times \overline{\square}^n$ and $X \times \mathbb{A}^r \times \overline{\square}^{n+1}$, respectively. Observe that δ_n extends to $\overline{\delta}_n : X \times \mathbb{A}^r \times \overline{\square}^n \to X \times \mathbb{A}^r \times \overline{\square}^{n+1}$, which is induced from $\mathbb{A}^r \overset{\Gamma}{\to} \mathbb{A}^r \times \overline{\square} \overset{\mathrm{Id} \times \sigma}{\to} \mathbb{A}^r \times \overline{\square}$, where Γ is the graph morphism of the composite $\mathbb{A}^r \to \mathbb{A}^1 \hookrightarrow \overline{\square}$ of the product map followed by the open inclusion, $(t_1, \cdots, t_r) \mapsto (t_1 \cdots t_r) \mapsto (t_1 \cdots t_r; 1)$, while $\sigma : \overline{\square} \to \overline{\square}$ is the antipodal automorphism $(a; b) \mapsto (b; a)$, where $(a; b) \in \overline{\square} = \mathbb{P}^1$ are the homogeneous coordinates. Since Γ is a closed immersion and $\mathrm{Id} \times \sigma$ is an isomorphism, the morphism $\overline{\delta}_n$ is projective. Hence, the dominant map $\delta_n|_{Z^\times} : Z^\times \to V$ induces $\overline{\delta}_n|_{\overline{Z}} : \overline{Z} \to \overline{V}$. In particular, we have a commutative

diagram

$$\begin{tikzcd} \overline{Z}^N \arrow[r, "\nu_Z"] \arrow[d, "\widetilde{\delta}_n"] & \overline{Z} \arrow[r, hook, "\iota_Z"] \arrow[d, "\overline{\delta}_n|_{\overline{Z}}"] & X \times \mathbb{A}^r \times \overline{\square}^n \arrow[d, "\overline{\delta}_n"] \\ \overline{V}^N \arrow[r, "\nu_V"] & \overline{V} \arrow[r, hook, "\iota_V"] & X \times \mathbb{A}^r \times \overline{\square}^{n+1}, \end{tikzcd} \tag{6.1}$$

where ι_Z, ι_V are the closed immersions, ν_Z, ν_V are normalizations, and $\widetilde{\delta}_n$ is given by the universal property of normalization for dominant maps.

By definition, $\overline{\delta}_n^*\{t_j = 0\} = \{t_j = 0\}$ for $1 \le j \le r$. First consider the case $n \ge 1$. Then $\overline{\delta}_n^* F^1_{n+1,i} = F^1_{n,i-1}$ for $2 \le i \le n+1$. Now, $\widetilde{\delta}_n^*\nu_V^*\iota_V^*(\sum_{i=1}^{n+1} F^1_{n+1,i} - \sum_{j=1}^r m_j\{t_j = 0\}) \ge \widetilde{\delta}_n^*\nu_V^*\iota_V^*(\sum_{i=2}^{n+1} F^1_{n+1,i} - \sum_{j=1}^r m_j\{t_j = 0\}) =^\dagger \nu_Z^*\iota_Z^*\overline{\delta}_n^*(\sum_{i=2}^{n+1} F^1_{n+1,i} - \sum_{j=1}^r m_j\{t_j = 0\}) = \nu_Z^*\iota_Z^*(\sum_{i=2}^{n+1} F^1_{n,i-1} - \sum_{j=1}^r m_j\{t_j = 0\}) = \nu_Z^*\iota_Z^*(\sum_{i=1}^{n} F^1_{n,i} - \sum_{j=1}^r m_j\{t_j = 0\}) \ge^\ddagger 0$, where † holds by the commutativity of (6.1) and ‡ holds as Z has modulus $\underline{m}$. Using Lemma 2.7, we can drop $\widetilde{\delta}_n^*$, i.e., V' has modulus $\underline{m}$.

When $n = 0$, we have for $1 \le j \le r$, $\widetilde{\delta}_0^*\nu_V^*\iota_V^*\{t_j = 0\} = \nu_Z^*\iota_Z^*\overline{\delta}_0^*\{t_j = 0\} = \nu_Z^*\iota_Z^*\{t_j = 0\}$, which is 0 because $\overline{Z} \cap \{t_j = 0\} = \emptyset$. Hence, $\widetilde{\delta}_0^*\nu_V^*\iota_V^*(F^1_{1,1} - \sum_{j=1}^r m_j\{t_j = 0\}) = \widetilde{\delta}_0^*\nu_V^*\iota_V^* F^1_{1,1} \ge 0$. Dropping $\widetilde{\delta}_0^*$, we get V' has modulus $\underline{m}$. ∎

Proposition 6.4. *Let $Z \in z^q(X[r]|D_{\underline{m}}, n)$. Then $\delta(Z) \in z^{q+1}(X[r]|D_{\underline{m}}, n{+}1)$. Furthermore, δ and ∂ satisfy the equality $\delta\partial + \partial\delta = 0$.*

Proof. We may assume that Z is an irreducible cycle. Let $\partial^\epsilon_{n,i}$ be the boundary given by the face $F^\epsilon_{n,i}$ on $X \times \mathbb{A}^r \times \square^n$, for $1 \le i \le n$ and $\epsilon = 0, \infty$.

Claim: For $\epsilon = 0, \infty$, (i) $\partial^\epsilon_{n+1,1} \circ \delta_n = 0$, (ii) $\partial^\epsilon_{n+1,i} \circ \delta_n = \delta_{n-1} \circ \partial^\epsilon_{n,i-1}$ for $2 \le i \le n+1$.

For (i), we show that $\delta_n(Z) \cap \{y_1 = \epsilon\} = \emptyset$ for $\epsilon = 0, \infty$. Since $\delta_n(Z) \subset V(t_1 \cdots t_r y_1 = 1)$, we have $\delta_n(Z) \cap \{y_1 = 0\} = \emptyset$. On the other hand, if $\delta_n(Z)$ intersects $\{y_1 = \infty\}$, then some t_i must be zero on Z, i.e., Z intersects $\{t_i = 0\}$ for some $1 \le i \le r$. However, since Z has modulus $\underline{m}$, this can not happen. Thus, $\delta_n(Z) \cap \{y_1 = \infty\} = \emptyset$. This shows (i). For (ii), by the definition of δ_n, the diagram

$$\begin{tikzcd} (\mathbb{G}_m^r)^\times \times \square^{n-1} \arrow[r, "\iota^\epsilon_{i-1}"] \arrow[d, "\delta_{n-1}"] & (\mathbb{G}_m^r)^\times \times \square^n \arrow[d, "\delta_n"] \\ \mathbb{G}_m^r \times \square^n \arrow[r, "\iota^\epsilon_i"] & \mathbb{G}_m^r \times \square^{n+1} \end{tikzcd}$$

is Cartesian. Thus, $\delta_{n-1}((\iota^*_{i-1}(Z)) = (\iota^\epsilon_i)^*(\delta_n(Z))$ by [6, Proposition 1.7], i.e., (ii) holds. This proves the claim.

By Lemma 6.3, we know $\delta_n(Z)$ has modulus $\underline{m}$. Since Z intersects all faces properly, so does $\delta_n(Z)$ by applying (i) and (ii) of the above claim repeatedly. For $\partial\delta + \delta\partial = 0$, note that $\partial\delta_n(Z) = \sum_{i=1}^{n+1}(-1)^i(\partial^\infty_{n+1,i}\delta_n(Z) - \partial^0_{n+1,i}\delta_n(Z)) =^\dagger$

$\sum_{i=2}^{n+1}(-1)^i(\delta_{n-1}\partial^{\infty}_{n,i-1}(Z) - \delta_{n-1}\partial^{0}_{n,i-1}(Z)) = -\sum_{i=1}^{n}(-1)^i(\delta_{n-1}\partial^{\infty}_{n,i}(Z) - \delta_{n-1}\partial^{0}_{n,i-1}(Z)) = -\delta_{n-1}\sum_{i=1}^{n}(-1)^i(\partial^{\infty}_{n,i}(Z) - \partial^{0}_{n,i}(Z)) = -\delta_{n-1}\circ\partial(Z)$, where † holds by the claim. □

Lemma 6.5 and Corollary 6.6 below, which generalize [19, §4.2], have much simpler proofs than *loc.cit.*

Lemma 6.5. *Let $Z \in z^q(X[r]|D_{\underline{m}}, n)$ be such that $\partial^{\epsilon}_i(Z) = 0$ for $1 \le i \le n$ and $\epsilon = 0, \infty$. Then $2\delta^2(Z)$ is the boundary of an admissible cycle with modulus $\underline{m}$.*

Proof. Note that $\delta^2(Z)$ is an admissible cycle on $X \times \mathbb{A}^r \times \square^{n+2}$ with modulus $\underline{m}$, by Proposition 6.4. For the transposition $\tau = (1, 2)$ on the set $\{1, \cdots, n+2\}$, we have $\tau \cdot \delta^2(Z) = \delta^2(Z)$, by the definition of δ. On the other hand, we have $\tau \cdot \delta^2(Z) = -\delta^2(Z) + \partial(\gamma)$ for some admissible cycle γ, by Lemma 5.3. Hence, we have $-\delta^2(Z) + \partial(\gamma) = \delta^2(Z)$, i.e., $2\delta^2(Z) = \partial(\gamma)$, as desired. □

Corollary 6.6. *Let k be a perfect field of characteristic $\neq 2$ and let X be in $\mathbf{SmAff}^{\mathrm{ess}}_k$ or in $\mathbf{SmProj}_k$. Let $\underline{m} \ge 1$. Then $\delta^2 = 0$ on $\mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$.*

Proof. If $r = \underline{m} = 1$, by Corollary 5.15, there is nothing to prove. So, suppose either $r \ge 2$ or $|\underline{m}| \ge 2$. But, if $r \ge 2$, then we automatically have $|\underline{m}| \ge 2$, so we just consider the latter case.
Given $\alpha \in \mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$, by Theorem 3.2, we can find a representative $Z \in z^q(X[r]|D_{\underline{m}}, n)$ such that $\partial^{\epsilon}_i(Z) = 0$ for $1 \le i \le n$ and $\epsilon = 0, \infty$. Then by Lemma 6.5, we have $2\delta^2(\alpha) = 0$.
On the other hand, by Corollary 5.14, the group $\mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$ is a $\mathbb{W}_{(|\underline{m}|-1)}(k)$-module. As $|\underline{m}| \ge 2$ and $\mathrm{char}(k) \neq 2$, it follows that $2 \in (\mathbb{W}_{(|\underline{m}|-1)}(k))^{\times}$. In particular, $\delta^2(\alpha) = 0$. □

6.2. Leibniz rule. We now discuss the Leibniz rule, generalizing [19, §4.3]. Let $X \in \mathbf{Sch}^{\mathrm{ess}}_k$. Let $(x, t, t_1, \cdots, t_r, y_1, \cdots, y_{n+2}) \in X \times \mathbb{A}^{r+1} \times \square^{n+2}$ be the coordinates. Let $T \subset X \times \mathbb{A}^{r+1} \times \square^{n+2}$ be the closed subscheme defined by the equation $ty_{n+1} = y_{n+2}(tt_1 \cdots t_r y_{n+1} - 1)$.

Definition 6.7 (*cf.* [19, Definition 4.9]). Given a closed subscheme $Z \subset X \times \mathbb{A}^{r+1} \times \square^n$, define $C_Z := T \cdot (Z \times \square^2)$ on $X \times \mathbb{A}^{r+1} \times \square^{n+2}$. This is extended $\mathbb{Z}$-linearly to cycles.

Lemma 6.8. *Let Z be a cycle on $X \times \mathbb{A}^{r+1} \times \square^n$ with modulus $\underline{m} = (m_1, \cdots, m_{r+1})$. Then C_Z has modulus $\underline{m}$ on $X \times \mathbb{A}^{r+1} \times \square^{n+2}$.*

Proof. We may assume Z is irreducible. We show that each irreducible component $V \subset C_Z$ has modulus $\underline{m}$. Let $\overline{Z}$ and $\overline{V}$ be the Zariski closures of Z and V in $X \times \mathbb{A}^{r+1} \times \overline{\square}^n$ and $X \times \mathbb{A}^{r+1} \times \overline{\square}^{n+2}$, respectively. The projection $\mathrm{pr} : X \times \mathbb{A}^{r+1} \times \overline{\square}^{n+2} \to X \times \mathbb{A}^{r+1} \times \overline{\square}^n$ that ignores the last two $\overline{\square}^2$ is projective, while its restriction to $X \times \mathbb{A}^{r+1} \times \square^{n+2}$ maps V into Z. So, pr maps

$\overline{V}$ to $\overline{Z}$, giving a commutative diagram

(6.2)
$$\begin{array}{ccccc} \overline{V}^N & \xrightarrow{\nu_V} & \overline{V} & \xhookrightarrow{\iota_V} & X \times \mathbb{A}^{r+1} \times \overline{\square}^{n+2} \\ \downarrow {\scriptstyle \mathrm{pr}^N} & & \downarrow {\scriptstyle \mathrm{pr}|_{\overline{V}}} & & \downarrow {\scriptstyle \mathrm{pr}} \\ \overline{Z}^N & \xrightarrow{\nu_Z} & \overline{Z} & \xhookrightarrow{\iota_Z} & X \times \mathbb{A}^{r+1} \times \overline{\square}^{n}, \end{array}$$

where ι_V and ι_Z are the closed immersions, ν_V and ν_Z are normalizations, and pr^N is induced by the universal property of normalization for dominant maps. The modulus condition for V is now easily verified using the pull-back of the modulus condition for Z on $\overline{Z}^N$ and the fact that $\mathrm{pr}^*\{t_j = 0\} = \{t_j = 0\}$ for all j and $\mathrm{pr}^* F^1_{n,i} = F^1_{n+2,i}$ for all i. □

COROLLARY 6.9. *Let $X_1, X_2 \in \mathbf{Sch}^{\mathrm{ess}}_k$. Let $V_1 \subset X_1 \times \mathbb{A}^1 \times \square^{n_1}$ and $V_2 \subset X_2 \times \mathbb{A}^r \times \square^{n_2}$ be closed subschemes with moduli $|\underline{m}|$ and $\underline{m}$, respectively with $\underline{m} \geq 1$.*
Under the exchange of factors $X_1 \times \mathbb{A}^1 \times \square^{n_1} \times X_2 \times \mathbb{A}^r \times \square^{n_2} \simeq X_1 \times X_2 \times \mathbb{A}^{r+1} \times \square^n$, where $n = n_1 + n_2$, consider the cycle $C_{V_1 \times V_2}$ on $X_1 \times X_2 \times \mathbb{A}^{r+1} \times \square^{n+2}$. Then $\mu|_{C_{V_1 \times V_2}}$ is finite. In particular, $\mu_*(C_{V_1 \times V_2})$ as in Definition 5.9 is well-defined, and has modulus $\underline{m}$.

Proof. We set $V = V_1 \times V_2$. From the definition of μ, the map $\mu : V \times \square^2 \to X_1 \times X_2 \times \mathbb{A}^r \times \square^{n+2}$ is of the form $\mu|_V \times \mathrm{Id}_{\square^2}$. By Proposition 5.7, the map $\mu|_V$ is finite, thus so is $\mu|_V \times \mathrm{Id}_{\square^2} : V \times \square^2 \to X_1 \times X_2 \times \mathbb{A}^r \times \square^{n+2}$. Hence, its restriction to $C_V = T \cdot (V \times \square^2)$ is also finite. The modulus condition for $\mu_*(C_V)$ follows from Lemmas 5.8 and 6.8. □

DEFINITION 6.10 (*cf.* [19, Definition 4.12]). Let $V_1 \in z^{q_1}(X_1[1]|D_{|\underline{m}|}, n_1)$ and $V_2 \in z^{q_2}(X_2[r]|D_{\underline{m}}, n_2)$ with $X_1, X_2 \in \mathbf{Sch}^{\mathrm{ess}}_k$. Let $n = n_1 + n_2$ and define $V_1 \times_{\mu'} V_2$ be the cycle $\sigma \cdot \mu_*(C_{V_1 \times V_2})$, where $\sigma = (n+2, n+1, \cdots, 1)^2 \in \mathfrak{S}_{n+2}$.

LEMMA 6.11. *Let V_1, V_2 be as in Definition 6.10. Then $V_1 \times_{\mu'} V_2 \in z^{q_1+q_2-1}((X_1 \times X_2)[r]|D_{\underline{m}}, n_1 + n_2 + 2)$.*

Proof. By Corollary 6.9, the cycle $\mu_*(C_{V_1 \times V_2})$ has modulus $\underline{m}$, thus so does $W := V_1 \times_{\mu'} V_2$. It remains to prove that W intersects all faces properly. Let $\sigma_{n_1} = (n_1 + 1, n_1, \cdots, 1) \in \mathfrak{S}_{n+1}$. Then by direct calculations, we have

(6.3)
$$\begin{cases} \partial_1^\infty W = \sigma_{n_1}(V_1 \times_\mu \delta(V_2)), \partial_1^0 W = 0, \partial_2^\infty W = \delta(V_1 \times_\mu V_2), \\ \partial_2^0 W = \delta(V_1) \times_\mu V_2, \\ \partial_i^\epsilon W = \begin{cases} \partial^\epsilon_{i-2}(V_1) \times_{\mu'} V_2, & \text{for } 3 \leq i \leq n_1 + 2, \\ V_1 \times_{\mu'} \partial^\epsilon_{i-n_1-2}(V_2), & \text{for } n_1 + 3 \leq i \leq n+2, \end{cases} \quad \epsilon \in \{0, \infty\}. \end{cases}$$

Since each V_i is admissible, using (6.3), Lemma 5.10, Proposition 6.4 and induction on the codimension of faces, we deduce that W intersects all faces properly. □

PROPOSITION 6.12. *Let $X_1, X_2 \in \mathbf{Sm}^{\mathrm{ess}}_k$. Let $\xi \in z^{q_1}(X_1[1]|D_{|\underline{m}|}, n_1)$ and $\eta \in z^{q_2}(X_2[r]|D_{\underline{m}}, n_2)$. Let $n = n_1 + n_2$ and $q = q_1 + q_2$. Suppose that* *all codimension one faces of ξ and η vanish. Then in the group $z^{q-1}((X_1 \times X_2)[r]|D_{\underline{m}}, n+1)$, the cycle $\delta(\xi \times_\mu \eta) - \delta\xi \times_\mu \eta - (-1)^{n_1}\xi \times_\mu \delta\eta$ is the boundary of an admissible cycle.*

Proof. By (6.3), for $3 \le i \le n_1 + 2$, we have $\partial_i^\epsilon(\xi \times_{\mu'} \eta) = \partial_{i-2}^\epsilon(\xi) \times_{\mu'} \eta = 0$, while for $n_1+3 \le i \le n+2$, we have $\partial_i^\epsilon(\xi \times_{\mu'} \eta) = \xi \times_{\mu'} \partial_{i-n_1-2}^\epsilon(\eta) = 0$. Hence, $\partial(\xi \times_{\mu'} \eta) = \sum_{i=1}^{n+2}(-1)^i(\partial_i^\infty - \partial_i^0)(\xi \times_{\mu'} \eta) = \delta(\xi \times_\mu \eta) - \{\sigma_{n_1} \cdot (\xi \times_\mu \delta\eta) + \delta\xi \times_\mu \eta\}$ by (6.3) for $i = 1, 2$. Equivalently,

$$\delta(\xi \times_\mu \eta) - \delta\xi \times_\mu \eta - \sigma_{n_1} \cdot (\xi \times_\mu \delta\eta) = \partial(\xi \times_{\mu'} \eta). \tag{6.4}$$

But, for $\xi \times_\mu \delta\eta$, notice that

$$\partial_i^\epsilon(\xi \times_\mu \delta\eta) = \begin{cases} \partial_i^\epsilon \xi \times_\mu \delta\eta = 0, & \text{for } 1 \le i \le n_1, \\ \xi \times_\mu \partial_{i-n_1}^\epsilon(\delta\eta), & \text{for } n_1 + 1 \le i \le n+1, \end{cases} \quad \epsilon \in \{0, \infty\}. \tag{6.5}$$

We have $\partial_1^\epsilon(\delta\eta) = 0$ when $i = n_1 + 1$ by Claim (i) of Proposition 6.4, and $\partial_{i-n_1}^\epsilon(\delta\eta) = \delta(\partial_{i-n_1-1}^\epsilon \eta) = \delta(0) = 0$ when $n_1 + 2 \le i \le n+1$ by Claim (ii) of Proposition 6.4. Hence, $\xi \times_\mu \delta\eta$ is a cycle with trivial codimension 1 faces, so, by Lemma 5.3, for some admissible cycle γ, we have $\sigma_{n_1} \cdot (\xi \times_\mu \delta\eta) = \mathrm{sgn}(\sigma_{n_1})(\xi \times_\mu \delta\eta) + \partial(\gamma) = (-1)^{n_1}\xi \times_\mu \delta\eta + \partial(\gamma)$. Putting this back in (6.4), we obtain $\delta(\xi \times_\mu \eta) - \delta\xi \times_\mu \eta - (-1)^{n_1}\xi \times_\mu \delta\eta = \partial(\xi \times_{\mu'} \eta) - \partial(\gamma)$, as desired. $\square$

The above discussion summarizes as follows:

Theorem 6.13. *Let X be in $\mathbf{SmAff}_k^{\mathrm{ess}}$ or in $\mathbf{SmProj}_k$ over a perfect field k with $\mathrm{char}(k) \neq 2$. Let $r \ge 1$ and $\underline{m} = (m_1, \cdots, m_r) \ge 1$. Then the following hold:*

1. *$(\mathrm{CH}(X[1]|D_{|\underline{m}|}), \wedge_X, \delta)$ forms a commutative differential graded $\mathbb{W}_{(|\underline{m}|-1)}\Omega_k^\bullet$-algebra.*
2. *$(\mathrm{CH}(X[r]|D_{\underline{m}}), \delta)$ forms a differential graded $(\mathrm{CH}(X[1]|D_{|\underline{m}|}), \wedge_X, \delta)$-module.*

In particular, $(\mathrm{CH}(X[r]|D_{\underline{m}}), \delta)$ is a differential graded $\mathbb{W}_{(|\underline{m}|-1)}\Omega_k^\bullet$-module.

Proof. The commutative differential graded algebra structure on $\mathrm{CH}(X[1]|D_{|\underline{m}|})$ and the differential graded module structure on $\mathrm{CH}(X[r]|D_{\underline{m}})$ over $\mathrm{CH}(X[1]|D_{|\underline{m}|})$ follows by combining Theorem 5.4, Corollary 6.6 and Proposition 6.12 using Theorem 3.2.
The structure map $p : X \to \mathrm{Spec}(k)$ turns $(\mathrm{CH}(X[1]|D_{|\underline{m}|}), \wedge_X, \delta)$ into a differential graded algebra over $(\mathrm{CH}(pt[1]|D_{|\underline{m}|}), \wedge_{pt}, \delta)$ via p^*. Since $\oplus_{n\ge 0}\mathrm{CH}^{n+1}(pt[1]|D_{|\underline{m}|}, n)$ forms a differential graded sub-algebra of $(\mathrm{CH}(pt[1]|D_{|\underline{m}|}), \wedge_{pt}, \delta)$. The map of commutative differential graded algebras $\mathbb{W}_{(|\underline{m}|-1)}\Omega_k^\bullet \to \oplus_{n\ge 0}\mathrm{CH}^{n+1}(pt[1]|D_{|\underline{m}|}, n)$ (see [28]) finishes the proof of the theorem. $\square$

As a consequence of Theorem 6.13 (use Corollary 5.15 when $|\underline{m}| = 1$), we obtain the following property of multivariate additive higher Chow groups.

Corollary 6.14. *Let $r \geq 1$ and $\underline{m} \geq 1$ and let X be in $\mathbf{SmAff}_k^{\mathrm{ess}}$ or in $\mathbf{SmProj}_k$. Then each $\mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$ is a k-vector space provided* $\mathrm{char}(k) = 0$.

7. Witt-complex structure on additive higher Chow groups

Let k be a perfect field of characteristic $\neq 2$. In this section, a *smooth affine k-scheme* means an object in $\mathbf{SmAff}_k^{\mathrm{ess}}$, i.e., an object of either $\mathbf{SmAff}_k$ or $\mathbf{SmLoc}_k$.

Rülling proved in [28] that the additive higher Chow groups of 0-cycles over $\mathrm{Spec}\,(k)$ form a restricted Witt-complex over k. When X is a smooth projective variety over k, it was proven in [19] that additive higher Chow groups of X form a restricted Witt-complex over k. Our objective is to prove the stronger assertion that the additive higher Chow groups of $\mathrm{Spec}\,(R) \in \mathbf{SmAff}_k^{\mathrm{ess}}$ have the structure of a restricted Witt-complex over R.

Since we exclusively use the case $r = 1$ only, we use the older notations $\mathrm{Tz}^q(X, n; m)$ and $\mathrm{TCH}^q(X, n; m)$ instead of $z^q(X[1]|D_{m+1}, n-1)$ and $\mathrm{CH}^q(X[1]|D_{m+1}, n-1)$. For $X \in \mathbf{Sch}_k^{\mathrm{ess}}$, we let $\mathrm{TCH}(X; m) := \oplus_{n,q}\mathrm{TCH}^q(X, n; m)$ and $\mathrm{TCH}^M(X; m) := \oplus_n \mathrm{TCH}^n(X, n; m)$. The superscript M is for *Milnor*. Let $\mathrm{TCH}(X) := \oplus_m \mathrm{TCH}(X; m)$ and $\mathrm{TCH}^M(X) := \oplus_m \mathrm{TCH}^M(X; m)$. We similarly define $\mathcal{TCH}(X; m)$, $\mathcal{TCH}^M(X; m)$, $\mathcal{TCH}(X)$, and $\mathcal{TCH}^M(X)$ for $X \in \mathbf{Sch}_k$ using Definition 4.7.

7.1. Witt-complex structure over k.

In this section, we show that the additive higher Chow groups for an object of $\mathbf{SmAff}_k^{\mathrm{ess}}$ form a functorial restricted Witt-complex over k. For $r \geq 1$, let $\phi_r : \mathbb{A}^1 \to \mathbb{A}^1$ be the morphism $x \mapsto x^r$, which induces $\phi_r : \mathrm{Spec}\,(R) \times B_n \to \mathrm{Spec}\,(R) \times B_n$. By [19, §5.1, 5.2], we have the Frobenius $F_r : \mathrm{TCH}^q(R, n; rm+r-1) \to \mathrm{TCH}^q(R, n; m)$ and the Verschiebung $V_r : \mathrm{TCH}^q(R, n; m) \to \mathrm{TCH}^q(R, n; rm+r-1)$ given by $F_r = \phi_{r*}$ and $V_r = \phi_r^*$. We also have a natural inclusion $\mathfrak{R} : \mathrm{Tz}^q(R, \bullet; m+1) \to \mathrm{Tz}^q(R, \bullet; m)$ for any $m \geq 1$, which induces $\mathfrak{R} : \mathrm{TCH}^q(R, n; m+1) \to \mathrm{TCH}^q(R, n; m)$, called the *restriction*. Finally, by Theorem 6.13, there is a differential $\delta : \mathrm{Tz}^q(R, \bullet; m) \to \mathrm{Tz}^q(R, \bullet + 1; m)$, which induces $\delta : \mathrm{TCH}^q(R, n; m) \to \mathrm{TCH}^q(R, n+1; m)$.

Theorem 7.1. *Let $X \in \mathbf{SmAff}_k^{\mathrm{ess}}$ and $m \geq 1$. Then $\mathrm{TCH}(X; m)$ is a DGA and $\mathrm{TCH}^M(X; m)$ is its sub-DGA. Furthermore, with respect to the operations $\delta, \mathfrak{R}, F_r, V_r$ in the above together with $\lambda = f^* : \mathbb{W}_m(k) = \mathrm{TCH}^1(k, 1; m) \to \mathrm{TCH}^1(X, 1; m)$ for the structure morphism $f : X \to \mathrm{Spec}\,(k)$, $\mathrm{TCH}(X)$ is a restricted Witt-complex over k and $\mathrm{TCH}^M(X)$ is a restricted sub-Witt-complex over k. These structures are functorial.*

Proof. In [19, Theorem 1.1, Scholium 1.2], it was stated that $\mathrm{TCH}(X; m)$ and $\mathrm{TCH}^M(X; m)$ are DGAs, and that $\mathrm{TCH}(X)$ and $\mathrm{TCH}^M(X)$ are restricted Witt-complexes over k with respect to the above $\delta, \mathfrak{R}, F_r, V_r$, provided the moving lemma holds for X. But this is now shown in Theorems 4.1 and 4.10. We give a very brief sketch of this structure and its functoriality.

The functoriality of the restriction operator $\mathfrak{R}$ recalled above, was stated in [19, Corollary 5.19], which we easily check here: let $f : X \to Y$ be a morphism in $\mathbf{SmAff}_k^{\mathrm{ess}}$ and consider the following commutative diagram:

$$\begin{tikzcd} \mathrm{Tz}^q_{\mathcal{W}}(Y, \bullet; m+1) \arrow[r, "f^*"] \arrow[d, hook] & \mathrm{Tz}^q(X, \bullet; m+1) \arrow[d, hook] \\ \mathrm{Tz}^q_{\mathcal{W}}(Y, \bullet; m) \arrow[r, "f^*"] & \mathrm{Tz}^q(X, \bullet; m), \end{tikzcd}$$

where $\mathcal{W}$ is a finite set of locally closed subsets of Y, and the horizontal maps are chain maps given by the inverse images as in the proof of Theorem 4.5 and Corollary 4.15. The diagram and Theorems 4.1 and 4.10 imply that $f^*\mathfrak{R} = \mathfrak{R}f^*$ because the vertical inclusions induce $\mathfrak{R}$ by definition.
For each $r \geq 1$, the Frobenius F_r and Verschiebung V_r recalled in the above are functorial as proven in [19, Lemmas 5.4, 5.9], and that F_r is a graded ring homomorphism is proven in [19, Corollary 5.6].
Finally, the properties (i), (ii), (iii), (iv), (v) in Section 2.2.2, are all proven in [19, Theorem 5.13], where none requires the projectivity assumption. $\square$

Corollary 7.2. *Let $m \geq 1$ be an integer. Then $\mathcal{TCH}(-; m)$ and $\mathcal{TCH}^M(-; m)$ define presheaves of DGAs on $\mathbf{Sch}_k$, and the pro-systems $\mathcal{TCH}(-)$ and $\mathcal{TCH}^M(-)$ define presheaves of restricted Witt-complexes over k on $\mathbf{Sch}_k$.*

Proof. Let $X \in \mathbf{Sch}_k$. By definition, $\mathcal{TCH}(X; m)$ is the colimit over all $(X \to A) \in (X \downarrow \mathbf{SmAff}_k)^{\mathrm{op}}$ of $\mathrm{TCH}(A; m)$. But the category of DGAs is closed under filtered colimits (see [13]) so that $\mathcal{TCH}(X; m)$ is a DGA. For each morphism $f : X \to Y$ in $\mathbf{Sch}_k$, one checks $f^* : \mathcal{TCH}(Y; m) \to \mathcal{TCH}(X; m)$ is a morphism of DGAs. The other assertions follow easily using Theorem 7.1. $\square$

Before we discuss Witt-complexes over R, we state the following behavior of various operators under finite push-forward maps.

Proposition 7.3. *Let $f : X \to Y$ be a finite map in $\mathbf{SmAff}_k^{\mathrm{ess}}$. Then for $r \geq 1$, we have: (a) $f_*\mathfrak{R} = \mathfrak{R}f_*$; (b) $f_*\delta = \delta f_*$; (c) $f_*F_r = F_r f_*$; (d) $f_*V_r = V_r f_*$.*

Proof. The item (a) is obvious and (b) and (c) follow at once from the fact that these operators are defined as push-forward under closed immersion and finite maps and they preserve the faces. For (d), we consider the commutative diagram

$$\begin{tikzcd} X \times \mathbb{A}^1 \arrow[r, "\mathrm{Id}\times\phi_r"] \arrow[d, "f\times\mathrm{Id}"] & X \times \mathbb{A}^1 \arrow[d, "f\times\mathrm{Id}"] \\ Y \times \mathbb{A}^1 \arrow[r, "\mathrm{Id}\times\phi_r"] & Y \times \mathbb{A}^1. \end{tikzcd} \tag{7.1}$$

Since this diagram is Cartesian and f as well as ϕ preserve the faces, we conclude from [6, Proposition 1.7] that $f_* \circ \phi_r^* = \phi_r^* \circ f_*$. $\square$

7.2. Witt-complex structure over R. Let $X = \mathrm{Spec}\,(R) \in \mathbf{SmAff}_k^{\mathrm{ess}}$. The objective of this section is to strengthen Theorem 7.1 by showing that $\mathrm{TCH}(X)$ is a restricted Witt-complex over R.

Let $m \geq 1$ be an integer. We first define a group homomorphism $\tau_R : \mathbb{W}_m(R) \to \mathrm{TCH}^1(R, 1; m)$ for any k-algebra R. Recall that the underlying abelian group of $\mathbb{W}_m(R)$ identifies with the multiplicative group $(1+tR[[t]])^\times/(1+t^{m+1}R[[t]])^\times$. For each polynomial $p(t) \in (1 + R[[t]])^\times$, consider the closed subscheme of $\mathrm{Spec}\,(R[t])$ given by the ideal $(p(t))$, and let $\Gamma_{(p(t))}$ be its associated cycle. By definition, $\Gamma_{(p(t))} \cap \{t = 0\} = \emptyset$ so that $\Gamma_{(p(t))} \in \mathrm{Tz}^1(R, 1; m)$. We set $\Gamma_{a,n} = \Gamma_{(1-at^n)}$ for $n \geq 1$ and $a \in R$.

Lemma 7.4. *Let $f(t), g(t)$ be polynomials in $R[t]$, and let $h(t) \in R[t]$ be the unique polynomial such that $(1-tf(t))(1-tg(t)) = 1-th(t)$. Then $\Gamma_{(1-th(t))} = \Gamma_{(1-tf(t))} + \Gamma_{(1-tg(t))}$ in $\mathrm{Tz}^1(R, 1; m)$.*

Proof. This is obvious by $(1 - tf(t))(1 - tg(t)) = 1 - th(t)$. □

Lemma 7.5. *For $n \geq m + 1$, we have $\Gamma_{(1-t^n f(t))} \equiv 0$ in $\mathrm{TCH}^1(R, 1; m)$.*

Proof. Consider the closed subscheme $\Gamma \subset X \times \mathbb{A}^1 \times \square$ given by $y_1 = 1-t^n f(t)$. Let $\nu : \overline{\Gamma}^N \to \overline{\Gamma} \hookrightarrow X \times \mathbb{A}^1 \times \mathbb{P}^1$ be the normalization of the Zariski closure $\overline{\Gamma}$ in $X \times \mathbb{A}^1 \times \mathbb{P}^1$. Since $f(t)t^n = 1 - y_1$ on $\overline{\Gamma}$, we see that $n\nu^*\{t = 0\} \leq \nu^*\{y_1 = 1\}$ on $\overline{\Gamma}^N$. Since $n \geq m + 1$, this shows that Γ satisfies the modulus m condition. Since $\partial_1^\infty(\Gamma) = 0$ and $\partial_1^0(\Gamma) = \Gamma_{(1-t^n f(t))}$ (which is of codimension 1), the cycle Γ is an admissible cycle in $\mathrm{Tz}^1(R, 2; m)$ such that $\partial\Gamma = \Gamma_{(1-t^n f(t))}$. This shows that $\Gamma_{(1-t^n f(t))} \equiv 0$ in $\mathrm{TCH}^1(R, 1; m)$. □

Proposition 7.6. *Let R be a k-algebra. Then the map $\tau_R : (1 + tR[t]) \to \mathrm{Tz}^1(R, 1; m)$ that sends a polynomial $1 - tf(t)$ to $\Gamma_{(1-tf(t))}$, defines a group homomorphism $\tau_R : \mathbb{W}_m(R) \to \mathrm{TCH}^1(R, 1; m)$.*

Proof. Every element $p(t) \in (1 + tR[[t]])^\times$ has a unique expression $p(t) = \prod_{n\geq 1}(1 - a_n t^n)$ for $a_n \in R$. For any such $p(t)$, set $p^{\leq m}(t) = \prod_{n=1}^m (1 - a_n t^n)$. We define $\tau_R(p(t)) = \Gamma_{(p^{\leq m}(t))}$. It follows from Lemmas 7.4 and 7.5 that this map descends to a group homomorphism from $\mathbb{W}_m(R)$. □

Recall from [28, Appendix A] that for each $r \geq 1$, we have the Frobenius $F_r : \mathbb{W}_{rm+r-1}(R) \to \mathbb{W}_m(R)$ and the Verschiebung $V_r : \mathbb{W}_m(R) \to \mathbb{W}_{rm+r-1}(R)$. They are given by $F_r(1 - at^n) = (1 - a^{\frac{r}{s}} t^{\frac{n}{s}})^s$, where $s = \gcd(r, n)$ and $V_r(1 - at^n) = 1 - at^{rn}$. On the other hand, as seen in Section 7.1, we have operations F_r and V_r on $\{\mathrm{TCH}^1(R, 1; m)\}_{m\in\mathbb{N}}$.

Lemma 7.7. *Let R be a k-algebra. Then the maps $\tau_R : \mathbb{W}_m(R) \to \mathrm{TCH}^1(R, 1; m)$ of Proposition 7.6 commute with the F_r and V_r operators on both sides.*

Proof. That $\tau_R V_r = V_r \tau_R$, is easy: we have $V_r(\tau_R(1-at^n)) = V_r(\Gamma_{a,n}) = \Gamma_{a,rn}$, while $\tau_R(V_r(1-at^n)) = \Gamma_{(1-at^{rn})} = \Gamma_{a,rn}$.
That $\tau_R F_r = F_r \tau_R$, is slightly more involved. Recall that $F_r(1-at^n) = (1-a^{\frac{r}{s}} t^{\frac{n}{s}})^s$, where $s = \gcd(r,n)$. Write $n = n's$ and $r = r's$, where $1 = (r', n')$. Hence, we have $\tau_R F_r(1-at^n) = s\Gamma_{a^{\frac{r}{s}}, \frac{n}{s}} = sV_{\frac{n}{s}}(\Gamma_{a^{\frac{r}{s}},1}) = sV_{n'}(\Gamma_{a^{r'},1}) =: \clubsuit$, while $F_r \tau_R(1-at^n) = F_r \Gamma_{a,n} = F_r V_n(\Gamma_{a,1}) =: \heartsuit$.
First observe that when $n=1$, we have $s=1$, $r=r'$, $n=n'=1$, and we have $\heartsuit = F_r(\Gamma_{a,1}) = \Gamma_{a^r,1} = \clubsuit$, so that $\tau_R F_r(1-at) = F_r \tau_R(1-at)$, indeed.
For a general $n \geq 1$, we have $F_r V_n = F_{r'} F_s V_s V_{n'} = F_{r'} \circ (s \cdot \mathrm{Id}) \circ V_{n'} = sF_{r'} V_{n'} =^{\dagger} sV_{n'} F_{r'}$, where $\dagger$ holds because $(r', n') = 1$. Since $F_{r'}(\Gamma_{a,1}) = \Gamma_{a^{r'},1}$ (by the first case), we have $\heartsuit = F_r V_n(\Gamma_{a,1}) = sV_{n'} F_{r'}(\Gamma_{a,1}) = sV_{n'}(\Gamma_{a^{r'},1}) = \clubsuit$. This shows $\tau_R F_r = F_r \tau_R$. $\square$

Remark 7.8. In the proof of Lemma 7.7, we saw that for $s = (r,n)$,

$$F_r(\Gamma_{a,n}) = s\Gamma_{a^{\frac{r}{s}}, \frac{n}{s}}, \quad V_r(\Gamma_{a,n}) = \Gamma_{a,rn}. \tag{7.2}$$

Proposition 7.9. *For $X = \mathrm{Spec}\,(R) \in \mathbf{SmAff}_k^{\mathrm{ess}}$, the maps $\tau_R : \mathbb{W}_m(R) \to \mathrm{TCH}^1(R,1;m)$ form a morphism of pro-rings that commutes with F_r and V_r for $r \geq 1$.*

Proof. It is clear from the definition of τ_R in Proposition 7.6 that it commutes with $\mathfrak{R}$. We saw that τ_R commutes with F_r and V_r in Lemma 7.7. So, we only need to show that τ_R respects the products. By [2, Proposition (1.1)], it is enough to prove that for $a, b \in R$ and $u, v \geq 1$,

$$\Gamma_{a,u} \wedge \Gamma_{b,v} = w\Gamma_{a^{\frac{v}{w}} b^{\frac{u}{w}}, \frac{uv}{w}} \quad \text{in } \mathrm{TCH}^1(R,1;m), \tag{7.3}$$

where $w = \gcd(u,v)$ and $\wedge = \wedge_X$ is the product structure on the ring $\mathrm{TCH}^1(R,1;m)$ as in Theorem 7.1.
Step 1. First, consider the case when $u = v = 1$, *i.e.*, we prove $\Gamma_{a,1} \wedge \Gamma_{b,1} = \Gamma_{ab,1}$. Recall that $\wedge$ is defined as the composition $\Delta^* \circ \mu_* \circ \times$ in

$$X \times \mathbb{A}^1 \times X \times \mathbb{A}^1 \xrightarrow{\mu} X \times X \times \mathbb{A}^1 \xleftarrow{\Delta} X \times \mathbb{A}^1.$$

Under the identification $X \times X \simeq \mathrm{Spec}\,(R \otimes_k R)$, we have $\mu_*(\Gamma_{a,1} \times \Gamma_{b,1}) = \Gamma_{(a\otimes 1)(1 \otimes b),1}$, and $\Delta^*(\Gamma_{(a\otimes 1)(1\otimes b),1}) = \Gamma_{ab,1}$, because Δ is given by the multiplication $R \otimes_k R \to R$. This proves (7.3) for Step 1.
For the following remaining two steps, we use the projection formula: $x \wedge V_s(y) = V_s(F_s(x) \wedge y)$, which we can use by Theorem 7.1.
Step 2. Consider the case when $v = 1$, but $u \geq 1$ is any integer. We apply the projection formula to $x = \Gamma_{b,1}$ and $y = \Gamma_{a,1}$ with $s = u$. Since $\mathrm{TCH}^1(R,1;m)$ is a commutative ring, by the projection formula, we get $V_u(\Gamma_{a,1}) \wedge \Gamma_{b,1} = V_u(\Gamma_{a,1} \wedge F_u(\Gamma_{b,1}))$. Here, the left hand side is $\Gamma_{a,u} \wedge \Gamma_{b,1}$ by eqn:FV identity, while the right hand side is $=^1 V_u(\Gamma_{a,1} \wedge \Gamma_{b^u,1}) =^2 V_u(\Gamma_{ab^u,1}) =^3 \Gamma_{ab^u,u}$, where $=^1$ and $=^3$ hold by (7.2) and $=^2$ holds by Step 1. This proves (7.3) for Step 2.
Step 3. Finally, let $u, v \geq 1$ be any integers. Let $w = \gcd(u,v)$. We again apply the projection formula to $x = V_u(\Gamma_{a,1})$, $y = \Gamma_{b,1}$, $s = v$, so that $V_u(\Gamma_{a,1}) \wedge$

$V_v(\Gamma_{b,1}) = V_v(F_v(V_u(\Gamma_{a,1}))\wedge\Gamma_{b,1})$. Its left hand side coincides with that of (7.3) by (7.2). Its right hand side is $=^1 V_v(F_v(\Gamma_{a,u}) \wedge \Gamma_{b,1}) =^2 V_v(w\Gamma_{a^{\frac{v}{w}},\frac{u}{w}} \wedge \Gamma_{b,1})$, where $=^1$ and $=^2$ hold by (7.2). But, Step 2 says that $\Gamma_{a^{\frac{v}{w}},\frac{u}{w}} \wedge \Gamma_{b,1} = \Gamma_{a^{\frac{v}{w}}b^{\frac{u}{w}},\frac{u}{w}}$ so that $V_v(w\Gamma_{a^{\frac{v}{w}},\frac{u}{w}} \wedge \Gamma_{b,1}) = wV_v(\Gamma_{a^{\frac{v}{w}}b^{\frac{u}{w}},\frac{u}{w}}) =^\dagger w\Gamma_{a^{\frac{v}{w}}b^{\frac{u}{w}},\frac{uv}{w}}$, where $=\dagger$ holds by (7.2). This last expression is the right hand side of (7.3). Thus, we obtain the equality (7.3) and this finishes the proof. $\square$

Theorem 7.10. *For* $\mathrm{Spec}\,(R) \in \mathbf{SmAff}_k^{\mathrm{ess}}$*,* $\mathrm{TCH}(R)$ *is a restricted Witt-complex over* R*, and its sub-pro-system* $\mathrm{TCH}^M(R)$ *is a restricted sub-Witt-complex over* R*.*

Proof. As saw in the proof of Theorem 7.1, we already have the restriction $\mathfrak{R}$, the differential δ, the Frobenius F_r and the Verschiebung V_r defined by the same formulas. Furthermore, by Proposition 7.9, now we have ring homomorphisms $\lambda = \tau_R : \mathbb{W}_m(R) \to \mathrm{TCH}^1(R, 1; m)$ for $m \geq 1$. The properties (i), (ii), (iii), (iv) in Section 2.2.2 are independent of the choice of the ring, so that what we checked in Theorem 7.1 still work. To prove the theorem, the only thing left to be checked is the property (v) that for all $a \in R$ and $r \geq 1$,

$$F_r\delta\tau_R([a]) = \tau_R([a]^{r-1})\delta\tau_R([a]), \tag{7.4}$$

where we have shrunk the product notation $\wedge$ and taken the ring homomorphism λ to be τ_R. To check this, we identify $\mathbb{W}_m(R)$ with $(1 + tR[[t]])^\times/(1 + t^{m+1}R[[t]])^\times$.

If $a = 0$, then $\tau_R([a]) = \Gamma_{(1-0\cdot t)} = \emptyset$. So, both sides of (7.4) are zero.

If $a = 1$, then $\tau_R([a]) = \tau_R(1 - t) = \Gamma_{(1-t)}$. But, in our definition of δ, to compute it, we should first restrict the cycle $\Gamma_{(1-t)} \subset \mathrm{Spec}\,(R) \times \mathbb{G}_m$ onto $\mathrm{Spec}\,(R) \times (\mathbb{G}_m \setminus \{1\})$, which becomes empty. Hence, $\delta\tau_R([a]) = \delta\Gamma_{(1-t)} = 0$, so again both sides of (7.4) are zero.

Let $a \in R \setminus \{0, 1\}$. Then $\tau_R([a]) = \Gamma_{(1-at)} \subset \mathrm{Spec}\,(R) \times \mathbb{A}^1$, and $\delta\tau_R([a])$ is given by the ideal $(1 - at, 1 - ty_1)$ in $R[t, y_1]$. Since t is not a zero-divisor in $R[t, y_1]$, we have $(1 - at, 1 - ty_1) = (1 - at, y_1 - a)$ as ideals. Hence, $F_r\delta\tau_R([a])$ is given by the ideal $(1 - a^rt, y_1 - a)$ in $R[t, y_1]$. On the other hand,

$$\begin{aligned} &\tau_R([a]^{r-1})\delta\tau_R([a]) = \Gamma_{(1-a^{r-1}t)} \wedge \mathrm{Spec}\,\left(\tfrac{R[t,y_1]}{(1-at,y_1-a)}\right) \\ = \quad &\Delta^*\left(\tfrac{(R\otimes_k R)[t,y_1]}{(1-(a^{r-1}\otimes 1)(1\otimes a),y_1-(1\otimes a))}\right) =^\dagger \mathrm{Spec}\,\left(\tfrac{R[t,y_1]}{(1-a^rt,y_1-a)}\right), \end{aligned} \tag{7.5}$$

where $\dagger$ holds because Δ is induced by the product homomorphism $R\otimes_k R \to R$. Hence, both hand sides of (7.4) coincide. This completes the proof. $\square$

Theorem 7.11. *For* $\mathrm{Spec}\,(R) \in \mathbf{SmAff}_k^{\mathrm{ess}}$ *and* $n, m \geq 1$*, there is a unique homomorphism* $\tau_{n,m}^R : \mathbb{W}_m\Omega_R^{n-1} \to \mathrm{TCH}^n(R, n; m)$ *that defines a morphism of restricted Witt-complexes over* R*,* $\{\tau_{\bullet,m}^R : \mathbb{W}_m\Omega_R^{\bullet-1} \to \mathrm{TCH}^\bullet(R, \bullet; m)\}_m$*, such that* $\tau_{1,m}^R = \tau_R$*.*

Proof. The theorem follows from Theorem 7.10 and [28, Proposition 1.15]. We have $\tau_{1,m}^R = \tau_R$ because the map λ of §2.2.2 is given by τ_R in Theorem 7.10. $\square$

We have shown in Propositions 7.6 and 7.9 that τ_R is a group homomorphism for any k-algebra R and is a ring homomorphism if R is smooth. Here, we provide the following information on τ_R.

Theorem 7.12. *Let R be an integral domain which is an essentially of finite type k-algebra. Then τ_R is injective. It is an isomorphism if R is a UFD.*

Proof. Let $K := \mathrm{Frac}(R)$ and $\iota : R \hookrightarrow K$ be the inclusion. This induces a commutative diagram

$$\begin{tikzcd} \mathbb{W}_m(R) \arrow[r, "\mathbb{W}_m(\iota)"] \arrow[d, "\tau_R"'] & \mathbb{W}_m(K) \arrow[d, "\tau_K"', "\simeq"] \\ \mathrm{TCH}^1(R,1;m) \arrow[r] & \mathrm{TCH}^1(K,1;m), \end{tikzcd}$$

where the bottom map is the flat pull-back via $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(R)$, and τ_K is the isomorphism by [28, Corollary 3.7]. Since $\mathbb{W}_m(\iota)$ is clearly injective (see [28, Properties A.1.(i)]), it follows that τ_R is injective, too.
Suppose now R is a UFD and V is an irreducible admissible cycle in $\mathrm{Tz}^1(R,1;m)$. Then we must have $(I(V),t) = R[t]$, where $I(V)$ is the ideal of V. Since $R[t]$ is a UFD, using basic commutative algebra, one checks that $I(V) = (1 - tf(t))$ for some non-zero polynomial $f(t) \in R[t]$. In particular, the map τ_R is surjective and hence an isomorphism. $\square$

7.3. Étale descent. Finally:

Proof of Theorem 1.4. By Corollary 5.15, we can assume $|\underline{m}| \geq 2$. We set $Y = X/G$, $\lambda = |G|$ and consider the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\gamma"] \arrow[d, "p"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd} \tag{7.6}$$

where γ is the action map and p is the projection. Since G acts freely on X, this square is Cartesian and f is étale of degree λ. By [6, Proposition 1.7], we have $f^* \circ f_* = p_* \circ \gamma^* : \mathrm{CH}^q(X[r]|D_{\underline{m}}, n) \to \mathrm{CH}^q(X[r]|D_{\underline{m}}, n)$.
Since f is G-equivariant with respect to the trivial G-action on Y, we see that f^* induces a map $f^* : \mathrm{CH}^q(Y[r]|D_{\underline{m}}, n) \to \mathrm{CH}^q(X[r]|D_{\underline{m}}, n)^G$. Moreover, it follows from [21, Theorem 3.12] that $f_* \circ f^*$ is multiplication by λ.
On the other hand, it follows easily from the action map γ that $p_* \circ \gamma^*(\alpha) = \sum_{g \in G} g^*(\alpha)$. In particular, $p_* \circ \gamma^*(\alpha) = \lambda \cdot \alpha$ if $\alpha \in \mathrm{CH}^q(X[r]|D_{\underline{m}}, n)^G$.
Since $\lambda \in k^\times$ and the Teichmüller map is multiplicative with $|\underline{m}| \geq 2$, we see that $\lambda \in (\mathbb{W}_{(|\underline{m}|-1)}(k))^\times$. We conclude from Theorem 5.4(3) and Corollary 5.14 that the composite $\mathrm{CH}^q(Y[r]|D_{\underline{m}}, n) \xrightarrow{f^*} \mathrm{CH}^q(X[r]|D_{\underline{m}}, n)^G \xrightarrow{\lambda^{-1} f_*} \mathrm{CH}^q(Y[r]|D_{\underline{m}}, n)$ yields the desired isomorphism. $\square$

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