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# *Asymptotics for generalized Riordan arrays*

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The machinery of Riordan arrays has been used recently by several authors. We show how meromorphic singularity analysis can be used to provide uniform bivariate asymptotic expansions, in the central regime, for a generalization of these arrays. We show how to do this systematically, for various descriptions of the array. Several examples from recent literature are given.

**Keywords:** bivariate asymptotics, generating function

## 1 Introduction

A Riordan array is an infinite complex matrix $(a_{rs})$ of a certain type (see below for exact definitions). The Riordan array formalism has been much used recently to study combinatorial questions in analysis of algorithms and other areas. Most work has been concerned with "exact" results. In this article we discuss asymptotics of such arrays.

We apply general machinery for deriving asymptotics of bivariate generating functions, following the research programme begun in [17, 18]. Asymptotic expansions of special cases of Riordan arrays have been discussed by several authors [4, 6]. The main purposes of this article are to show how the work in [17] immediately yields strong results for (a generalization of) Riordan arrays, and to use this case as an introduction to the much more general results in [17, 18], the computations being simpler to understand. In addition we try to simplify and automate the process of extracting asymptotics as far as possible.

### *1.1 Riordan arrays*

We first recall some standard facts about Riordan arrays. See [12, 15] for more details and proofs.

**Definition 1.1.** A ***Riordan array*** is an infinite complex matrix $(a_{rs})$, with array indexing starting from $r = s = 0$, whose bivariate generating function has the form

$$F(z, w) = \sum_{r,s} a_{rs} z^r w^s = \frac{\phi(z)}{1 - wv(z)}, \tag{1}$$

with $v(0) = 0$, $\phi(0) \neq 0$.

The geometric series expansion shows that $a_{rs}$ is precisely the coefficient of $t^r$ in the convolution $\phi(t)v(t)^s$, and this could of course be used as a definition of Riordan array. It follows that $a_{rs} = 0$ if $r < s$, so such an array is lower triangular. It is not strictly lower triangular since $a_{00} = \phi(0)$.

Note that the component univariate generating functions $\phi(t), v(t)$ can be recaptured from the bivariate generating function $F(z, w)$ via $\phi(t) = F(t, 0)$ and $\phi(t)v(t) = F_w(t, 0)$, so we may assume that $v(t)$ and $\phi(t)$ are explicitly known.

The set of Riordan arrays forms a monoid under matrix multiplication. The group of invertible elements is defined by the equivalent conditions below. Only the last condition is non-obvious, giving a "row" recurrence where the definition supplies a "column" recurrence.

**Definition 1.2.** A Riordan array is ***proper*** if it satisfies any of the equivalent conditions of Proposition 1.3.

**Proposition 1.3.** *The following conditions on a Riordan array are equivalent.*

- *for each* $r$*,* $a_{rr} \neq 0$*;*
- *for some* $r > 0$*,* $a_{rr} \neq 0$*;*
- $v'(0) \neq 0$*;*

- *there is a sequence* $(c_j)$ *such that* $a_{r+1,s+1} = \sum_j c_j a_{r,s+j}$ *for each* $r, s$.

□

The sequence $(c_j)$ is usually known as the $A$-sequence of the array (in this author's opinion, a good example of how not to name a mathematical concept). Let $A(t) = \sum_j c_j t^j$. Column $0$ of a proper Riordan array is not determined by $A$, though the other columns are determined by $A$ once column $0$ is known. Of course we have $\phi(t) = \sum_r a_{r0} t^r$, so that column has generating function $\phi(t)$. It turns out we can express the first column via another recurrence. For each proper Riordan array, there is a sequence $(z_j)$ such that for each $r$, $a_{r+1,0} = \sum_j z_j a_{rj}$.

The following relationships hold between the "implicit" and "explicit" descriptions of the array:

- $v(t) = tA(v(t))$ (the Lagrange inversion equation);
- $\phi(t) = \frac{a_{00}}{1 - tZ(v(t))}$.

Thus given a description in terms of $(\phi, v)$, we can convert, in theory, to one in terms of $(a_{00}, A, Z)$, and vice versa. Often in practice one description (generating function or recurrence) is much more convenient than the other. Conversion between them is often computationally difficult.

### 1.2 A slight generalization

In [17] the authors presented a taxonomy applicable to multivariate meromorphic generating functions, and derived asymptotics in the most common cases. Bivariate generating functions of type (1) fall into the easiest case of the classification. In fact, in that framework it is just as easy to consider a small generalization of Riordan array. The condition $v(0) = 0$ in the definition (1) is often violated in examples of interest, as we shall see below. Note that $1 - wv(z)$ exists as a bivariate formal power series for each univariate formal power series $v(z)$, since $1 - wv(z)$ does not lie in the maximal ideal of the local ring $\mathbb{C}[[z, w]]$. The condition $v(0) = 0$ is clearly equivalent to lower triangularity of the corresponding coefficient array, so is necessary for some combinatorial interpretations, but is inessential for our analysis. In addition, we need not require $\phi(0) \neq 0$.

However, we do require convergence of the power series $F(z, w)$ in a neighbourhood of the origin, in order to derive asymptotics via complex analysis. Thus from now on we shall consider bivariate GFs of the form (1) where $\phi, v$ are analytic functions in a neighbourhood of $0$ and $v$ is nonconstant.

We note that such arrays (without the assumption of analyticity) were called ***improper Riordan arrays*** in [23], but this name is misleading. Such arrays are not Riordan arrays at all according to the standard definition. Furthermore, this usage causes a notational conflict — one might expect an improper Riordan array to be a Riordan array that is not proper, but such Riordan arrays have been called ***stretched*** in [3].

## 2 Asymptotics via meromorphic singularity analysis

The asymptotic analysis of a general two-dimensional array presents considerable difficulties. Loosely speaking, we may say that these difficulties arise from the singular structure of the bivariate GF and from boundary effects in the integer lattice. The arrays considered here avoid the first problem, at least in the nonnegative case. In this section we show how asymptotics for our generalized Riordan arrays follow immediately from previous work in [17].

### 2.1 The general framework

In [17] the following analytic framework is adopted. We deal with a generating function $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z})$ of $d$ complex variables, where $G$ and $H$ are analytic in a neighbourhood of the origin and are relatively prime in $\mathbb{C}[\mathbf{z}]$. The zero-set of $H$, denoted $\mathcal{V}$, is called the ***singular variety*** of $F$, and is a complex analytic variety of complex dimension $d - 1$.

A point $\mathbf{z}$ of $\mathcal{V}$ is ***strictly minimal*** if it is the only point of $\mathcal{V}$ on the closed polydisk centred at the origin and determined by $\mathbf{z}$. We assume that $G$ and $H$ are analytic in a neighbourhood of $\mathbf{z}$, so that $F$ continues analytically past the boundary of the domain of convergence. In particular this is satisfied by rational functions. Such a minimal point is called ***smooth*** if no coordinate is zero and the gradient of $H$ is nonzero. The strictly minimal smooth point case is generic, though much more complicated local geometry can occur in practice. In this generic case (the only one considered by almost all authors in analytic combinatorics), many of our results can probably be obtained by other methods. However the point of [17] is to develop from scratch a unified analytic approach that allows us to attack the harder cases, is simpler to apply than existing methods, and more likely to lead to automation. For more details of combinatorial applications of the theory of [17, 18], see [19].

To each smooth minimal point $\mathbf{z}$ we associate a certain direction $\boldsymbol{\delta}(\mathbf{z})$ in which asymptotics are furnished by our analysis. By reducing the problem to computing the asymptotics of certain Fourier-Laplace integrals, we can obtain complete effectively computable expansions in any dimension. When $d = 2$, our results yield the following explicit result [17, Thm 3.1].

**Theorem 2.1 (Generic smooth point asymptotics, dimension $2$).** *Suppose that $d = 2$ and let $F(z, w) = G(z, w)/H(z, w)$ be as described above. Then if $(z, w)$ is a smooth minimal point of $\mathcal{V}$ where $szH_z = rwH_w$, there is a complete asymptotic expansion*

$$a_{rs} \sim z^{-r} w^{-s} s^{-1/2} \sum_{k=0}^{\infty} b_k s^{-k}.$$

*The expansion is uniform as $(z, w)$ varies over a compact set of such points.*

*Define*

$$Q(z, w) := -w^2 H_w^2 z H_z - w H_w z^2 H_z^2 - w^2 z^2 \left( H_w^2 H_{zz} + H_z^2 H_{ww} - 2 H_z H_w H_{zw} \right).$$

*If $Q(z, w)$ and $G(z, w)$ are nonzero, then the leading coefficient in the expansion is given by*

$$b_0 = \frac{G(z, w)}{\sqrt{2\pi}} \sqrt{\frac{-wH_w}{Q(z, w)}}.$$

□

## 2.2 Specialization to generalized Riordan arrays

Throughout, we make the assumption that the radius of convergence of $\phi$ is at least as large as that of $v$. See Section 4 for brief comments on the other case.

We have globally that $G(z, w) = \phi(z), H(z, w) = 1 - wv(z)$. The gradient of $H$ is therefore $(-wv'(z), v(z))$, which cannot vanish on a (minimal) point of $\mathcal{V}$ since its second component is nonzero. Thus every strictly minimal point of $\mathcal{V}$ is smooth. We can of course parametrize $\mathcal{V}$ globally in terms of $z$. This leads to parametrized expressions, which we present below, for previously introduced quantities. We write $Q(z)$ instead of $Q(z, 1/v(z))$, etc.

For each univariate formal power series $f(t) \in \mathbb{C}[[t]]$ we define as usual

$$\mu(f; t) = \frac{tf'(t)}{f(t)} \qquad \text{and} \qquad \sigma^2(f; t) = \frac{t^2 f''(t)}{f(t)} + \frac{tf'(t)}{f(t)} - \left( \frac{tf'(t)}{f(t)} \right)^2$$

and these are well-defined formal power series even if $f(0) = 0$, and converge in a neighbourhood of $0$ if and only if $f$ does.

We collect a few standard definitions.

**Definition 2.2.** We write $f \geq 0$ to mean that every coefficient of $f(t)$ is nonnegative. We denote by $\rho(f)$ the radius of convergence of $f(t)$. Note that if $\rho(f) > 0$, then $f \geq 0$ if and only if $f(x) \geq 0$ for each $x$ with $0 < x < \rho(f)$.

**Definition 2.3.** We say that $f(t)$ is ***aperiodic*** if $\sigma^2(f; t) \neq 0$. Equivalently, the set of indices of nonzero coefficients of $f$ has at least two elements and greatest common divisor equal to 1.

Note that if a Riordan array is proper, the corresponding $v$ will be aperiodic unless $v(t) = ct$.

Theorem 2.1 can be used for generalized Riordan arrays of any type. However, there is no nice criterion for minimality of a critical point in general. Furthermore the periodic case can be reduced in some sense to the aperiodic one by a simple change of variable. Thus in this article we make the following (standard) assumptions (see Section 4 for more discussion).

Assume that $\phi \geq 0, v \geq 0$, that $v$ is aperiodic, and that $\rho(\phi) \geq \rho(v)$. (*)

Note that $\phi \geq 0$ and $v \geq 0$ if and only if $F \geq 0$. Straightforward computations show that $Q(z) = \sigma^2(v; z)$ and the stationary phase equation $szH_z = rwH_w$ becomes $\mu(v; z) = r/s$. We now intend to use Theorem 2.1 to describe the asymptotics of our generalized Riordan arrays.

**Proposition 2.4.** *Assuming* (*)*, the minimal points of* $\mathcal{V}$ *are precisely those of the form* $(x, 1/v(x))$ *for which* $0 < x < \rho(v)$*. All these points are strictly minimal.*

*Proof.* Since $v \geq 0$, it follows that for each fixed $x$ with $0 < x < \rho(v)$, the maximum of $|v(z)|$ on the disc $|z| \leq x$ is attained at $z = x$. Each point $z$ in the disc satisfies $|v(z)| = |z|^a |g(z^b)| \leq x^a g(x^a)$ and equality can happen only if $|g(z^b)| = |g(x^b)|$. The triangle inequality applied to the power series expansion of $g$ shows that if $g$ is nonconstant then $z^b = x^b$. Such points are clearly minimal, so $x$ is strictly minimal if and only if $b = 1$, which occurs if and only if $v$ is aperiodic. □

Note that the type of minimal point does not depend on $x$, nor on the type of singularity of $v$ at $z = \rho(v)$. Also $\phi$ never vanishes at a minimal point. Thus generic strictly minimal smooth point behaviour is guaranteed by (*).

Theorem 2.1 now applies and yields an expansion that applies in a set $\mathbf{\Delta}$ of directions defined by the stationary phase equation for all possible choices of minimal points $(z, w)$. In fact $\mathbf{\Delta}$ is an interval (this is a consequence of log-convexity of the domain of convergence of $F$ [17]). One question remains: is this interval as large as possible? The answer turns out to be yes, as we show below.

**Definition 2.5.** Let $k$ denote the order of vanishing of $v(t)$ at the origin. Note that a Riordan array always has $k \geq 1$ and is proper if and only if $k = 1$. Let $l$ denote the degree of $v$: that is, the polynomial degree when $v(t)$ is a polynomial, and $\infty$ otherwise. Finally, let $\mathbf{\Delta}'$ denote the interval $[k, l]$.

Note that $a_{rs} = 0$ if $r/s \notin \mathbf{\Delta}'$, so directions outside this latter interval are not of interest.

**Proposition 2.6.** *Under assumption* (*)*, for each* $\lambda$ *in the interior of* $\mathbf{\Delta}$*, the equation*

$$\mu(v; z) = \lambda, \qquad 0 < z < \rho(v) \tag{2}$$

*has a unique solution* $z(\lambda)$*.*

*Proof.* We note that $\mu(v; t) = k + \mu(\psi; t)$ and $\mu'(v; t) = \sigma^2(v; t)/t = \sigma^2(\psi; t)/t$. Thus $x \mapsto \mu(v; x)$ is increasing for $0 < x < \rho(v)$, and $\lim_{x \to 0^+} \mu(v; x) = k$. By (*), $x \mapsto \mu(v; x)$ is strictly increasing and $\bar{\mathbf{\Delta}} = (k, l^*)$ where $l^* = \lim_{x \to \rho(v)^-} \mu(v; x)$. It remains to show that $l^* = l$, so that $\mathbf{\Delta} = \mathbf{\Delta}'$.

First consider the case $\rho(v) < \infty, \lim_{x \to \rho(v)^-} v(x) = \infty$. Then $\log \mathcal{D}$ is given by $p \leq \log \rho(v), q + \log v(e^p) \leq 0$. Thus $q \to -\infty$ as $p \to \log \rho(v)$ inside $\log \mathcal{D}$ and so the vertical asymptote $p = \log \rho(v)$ is a support hyperplane for $\log \mathcal{D}$. Next consider the case $\rho(v) < \infty, \lim_{x \to \rho(v)^-} v(x) < \infty$. Then $\lim_{x \to \rho(v)-} v'(x) = \infty$ and so $l^* = \infty$. Finally consider the case where $v$ is entire. Then by L'Hôpital, $\lim_{x \to \infty} xv'(x)/v(x) = 1 + \lim_{x \to \infty} xv''(x)/v'(x)$, etc. If $v$ is a polynomial, $l^* = \deg v$. Otherwise, $v$ is entire and not a polynomial; by induction $l^*$ is arbitarily large since all derivatives of $v$ satisfy the same hypotheses as $v$. □

**Definition.** Define the following quantities

$$w(\lambda) = 1/v(z(\lambda)); \tag{3}$$

$$b_0(\lambda) = \frac{\phi(z(\lambda))}{\sqrt{2\pi\sigma^2(v; z(\lambda))}}. \tag{4}$$

**Corollary 2.7.** *Suppose* $F$ *is as in* (1) *and that* (*) *is satisfied. Then with* $\lambda = r/s$*, the asymptotic formula*

$$a_{rs} \sim z(\lambda)^{-r} w(\lambda)^{-s} s^{-1/2} \sum_{k=0}^{\infty} b_k(\lambda) s^{-k} \tag{5}$$

*holds uniformly in* $\lambda$ *over compact subsets of* $\mathbf{\Delta}$*, where* $z, w$ *and* $b_0$ *are given by formulae* (2), (3) *and* (4)*, and similar though more complicated formulae are computable for* $b_k, k > 0$*.* □

We discuss the practical use of this explicit but perhaps rather complicated-looking formula in Section 3.

**Tab. 1:** Some generalized Riordan arrays

| $Z(t)$ | $A(t)$ | $\phi(x)$ | $v(x)$ | Interpretation of $a_{rs}$ / reference |
|---|---|---|---|---|
| $1$ | $1+t$ | $\frac{1}{1-x}$ | $\frac{x}{1-x}$ | Pascal triangle $\binom{r}{s}$ |
| $2t$ | $1+t^2$ | $\frac{1}{\sqrt{1-4x^2}}$ | $\frac{1-\sqrt{1-4x^2}}{2x}$ | [22, Example 2B] |
| $1+2t$ | $1+t+t^2$ | $\frac{1}{\sqrt{1-2x-3x^2}}$ | $\frac{1-x-\sqrt{1-2x-3x^2}}{2x}$ | Motzkin triangle [22, Sec. 3] |
| $2+2t$ | $1+2t+t^2$ | $\frac{1}{\sqrt{1-4x}}$ | $\frac{1-2x-\sqrt{1-4x}}{2x}$ | [23] |
| $\frac{1}{1-t}$ | $\frac{1}{1-t}$ | $\frac{1-\sqrt{1-4x}}{2x}$ | $\frac{1-\sqrt{1-4x}}{2}$ | Catalan triangle [15, Sec. 4] |
| $\frac{t}{1-t}$ | $\frac{1-t+t^2}{1-t}$ | $\frac{1+x-\sqrt{1-2x-3x^2}}{2x(1+x)}$ | $\frac{1+x-\sqrt{1-2x-3x^2}}{2(1+x)}$ | [15, (4.8)] |
| $\frac{2}{1-t}$ | $\frac{1+t}{1-t}$ | $\frac{1+x-\sqrt{1-6x+x^2}}{2x}$ | $\frac{1+x-\sqrt{1-6x+x^2}}{2}$ | [11] |
| $\frac{2}{1-t}$ | $\frac{1}{1-t}$ | $\frac{1}{\sqrt{1-4x}}$ | $\frac{1-\sqrt{1-4x}}{2}$ | [23] |
| $\frac{t-1+\sqrt{1-2t+5t^2}}{2t}$ | $\frac{1+t+\sqrt{1-2t+5t^2}}{2}$ | $\frac{1-x}{1-x-x^2}$ | $\frac{x-x^2}{1-x-x^2}$ | [14] |
| $\frac{t}{(1-t)^2}$ | $\frac{1}{1-t}$ | $\frac{1-5x+(1-x)\sqrt{1-4x}}{2(1-4x-x^2)}$ | $\frac{1-\sqrt{1-4x}}{2}$ | [13, Sec 4.2] |
| $0$ | $\frac{2-t}{1-t}$ | $1$ | $\frac{1+x-\sqrt{1-6x+x^2}}{2}$ | [15, (4.9)] |
| $\frac{2t-3+\sqrt{1+4t-4t^2}}{4t(t-1)}$ | $1/(1-t)$ | $\frac{4}{2+\sqrt{1-4x}+\sqrt{1+4x}}$ | $\frac{1-\sqrt{1-4x}}{2}$ | tennis ball problem [10, Appendix A] |
| $1+t$ | $1-t$ | $\frac{1-3x-\sqrt{1+2x-3x^2}}{2x(3x-2)}$ | $\frac{1+x-\sqrt{1+2x-3x^2}}{2x}$ | [15, p. 177] |
| | | $1$ | $1+x$ | $\binom{s}{r}$ |
| | | $1/(1-x)$ | $1/(1-x)$ | $\binom{r+s}{s}$ |
| | | $\frac{1}{1-x}$ | $\frac{1+x}{1-x}$ | Delannoy numbers [2] |
| | | $1$ | $\cosh(\sqrt{x})$ | Ehrenfest model [5, ] |

**Tab. 2:** Asymptotics for subgroups of the Riordan group

| Case | Explicit leading term $a_{rs} \sim$ | Implicit leading term $a_{rs} \sim$ |
|---|---|---|
| Bell subgroup | $x^{-r}v^{s+1}\frac{1}{\sqrt{2\pi s\sigma^2(v;x)}}$ | $v^{s-r}A^{r+1}\frac{s}{\sqrt{2\pi r^3\sigma^2(A;v)}}$ |
| Hitting time subgroup | $x^{-r}v^{s}\frac{r}{\sqrt{2\pi s^3\sigma^2(v;x)}}$ | $v^{s-r}A^{r}\frac{1}{\sqrt{2\pi r\sigma^2(A;v)}}$ |
| Associated subgroup | $x^{-r}v^{s}\frac{1}{\sqrt{2\pi s\sigma^2(v;x)}}$ | $v^{s-r}A^{r}\frac{s}{\sqrt{2\pi r^3\sigma^2(A;v)}}$ |

## 3 Computing with the asymptotic formulae

While any pair $(\phi(t), v(t))$ can be studied, some occur much more often than others in applications. Table 1 lists some examples of generalized Riordan arrays, taken from recent research literature.

We assume (*) throughout. Equation (2) has a unique, positive real, solution for all directions of interest. This solution is a strictly minimal point controlling asymptotics in the given direction. This minimal point can be found numerically for given $r, s$. Moreover, in many cases one can solve symbolically for every quantity in terms of $r/s$, and then derive an explicit symbolic asymptotic formula, restoring the symmetry between $r$ and $s$ in the process.

**Example 3.1.** The "Delannoy square" has $v(t) = (1+t)/(1-t)$, $\phi(t) = 1/(1-t)$. The coefficients $a_{rs}$ count walks from the origin to $(r, s)$ using steps in $\{(1,0), (0,1), (1,1)\}$. The stationary phase equation is $2sz = r(1-z^2)$. Since we know the minimal point is positive real, we take $z = (D-s)/r$ where $D$ denotes $\sqrt{r^2+s^2}$. After some algebra we obtain the leading term asymptotic

$$a_{rs} \sim \frac{r^r s^s}{(D-s)^r (D-r)^s} \sqrt{\frac{rs}{2\pi D(r+s-D)^2}}$$

uniformly for every $a, b$ such that $0 < a \leq r/s \leq b < \infty$. In particular, the number of central Delannoy paths $(r = s)$ is asymptotically

$$(3+2\sqrt{2})^r \frac{(2^{1/4} + 2^{-1/4})}{2\sqrt{\pi r}}.$$

However, this type of direct symbolic computation becomes difficult for more complicated $v(t)$. Even with a computer algebra system, it is not easy (for this author at least) to obtain formulae simple enough to yield insight. On the positive side, it is easily shown that the leading term of the asymptotic formula is an algebraic function of $(r, s)$ if $v(t)$ and $\phi(t)$ are algebraic series. For the next result, recall the notation of Corollary 2.7.

**Proposition 3.2.** *Suppose that $(\phi, v)$ defines a proper Riordan array. If $v(t)$ is an algebraic series, then $z(r, s)$ is an algebraic expression in $r$ and $s$. Thus if $\phi(t)$ is also an algebraic series, the leading term asymptotic approximation* (5) *is an algebraic expression in $r$ and $s$.*

*Proof.* Since $\mu(v;t) = tv'(t)/v(t)$, and the set of algebraic series is closed under multiplication, differentiation and taking reciprocal, $\mu(v;t)$ is algebraic if $v(t)$ is. Now $\mu(v;0) = 0$ and so $\mu(v;t)$ is compositionally invertible with inverse $m(v;t)$. Let $P(t, \mu(v;t)) = 0$ be a polynomial equation witnessing algebraicity of $\mu(v;t)$. Then $0 = P(m(v;t), \mu(v; m(v;t)) = P(m(v;t), t)$ and so $m(v;t)$ is algebraic.

Thus $z(r/s)$ is an algebraic expression in $r/s$, and hence, clearing denominators, $z$ is an algebraic expression in $r$ and $s$. The leading term involves only algebraic operations on algebraic quantities and is hence algebraic. □

Since $A(t)$ is often of a much simpler form than $v(t)$ (the former is a quadratic polynomial in many applications), it often makes sense to carry out computations in terms of $A(t)$ and $Z(t)$ rather than in terms of $v(t)$ directly. We now discuss the necessary translation of the formula (5).

Since the map $t \to v(t)$ is an automorphism of $\mathbb{C}[[t]]$, we may equally well use $v$ as a variable. Suppose now that $v(t) = tA(v(t))$ as above. Differentiating this and rearranging we obtain expressions involving $\mu(A;v)$ and $\sigma^2(A;v)$. This leads to $\mu(v;t) = 1/(1-\mu(A;v))$ and $\sigma^2(v;t) = \sigma^2(A;v)/((1-\mu(A;v))^3$. If $\mu(v;t) = r/s$, then we have $\mu(A;v) = (r-s)/r$. Thus we obtain the following result.

**Theorem 3.3.** *Given $(a_{00}, Z(t), A(t))$ with $Z$ and $A$ analytic at $0$, $a_{00} \neq 0, A(0) \neq 0$, the following equations uniquely define functions $v(z)$ and $\phi(z)$, analytic at $0$, and a function $v_0(\lambda)$:*

$$v(z) = zA(v(z)); \tag{6}$$

$$\mu(A; v_0) = \lambda; \tag{7}$$

$$\phi(z) = a_{00}/(1 - zZ(v(z))). \tag{8}$$

*This gives rise to a proper Riordan array $(a_{rs})$ via* (1)*. Let $l := \deg v$. If $l > 1$ then, with $\lambda = (r-s)/r$, we have*

$$a_{rs} \sim \frac{v_0(\lambda)^{s-r} A(v_0(\lambda))^r}{\sqrt{2\pi r^3 \sigma^2(A; v_0(\lambda))}} s\phi(v_0(\lambda)/A(v_0(\lambda))) = \frac{v_0(\lambda)^{s-r} A(v_0(\lambda))^r}{\sqrt{2\pi r^3 \sigma^2(A; v_0(\lambda))}} \frac{a_{00}s}{1 - \frac{v_0(\lambda)Z(v_0(\lambda))}{A(v_0(\lambda))}} \tag{9}$$

*This approximation is uniform for every $a, b$ such that $1 < a \leq r/s \leq b < l$.*

**Example 3.4 (The linear or quadratic case).** Suppose now that $A(t) = at^2 + bt + c$ with $a \geq 0, b \geq 0, c \geq 0$ and at least one of $b$ and $c$ being nonzero. If $a = 0$, the implicit formula does not apply since $\sigma^2(A; v) \equiv 0$. Using the explicit formula, we readily obtain (when $c \neq 0$, otherwise aperiodicity is violated and the result is easily obtained by other means)

$$a_{rs} \sim b^s c^{r-s} \frac{r^r}{s^s (r-s)^{(r-s)}} \frac{s\phi(\frac{s(r-s)}{cr})}{\sqrt{2\pi rs(r-s)}}. \tag{10}$$

This reflects the fact that we have applied a linear change of variable to Pascal's triangle.

Otherwise, we use the implicit formula. The stationary phase equation is quadratic in $v$, namely $0 = a(r+s)v^2 + bsv - c(r-s)$. Solving this and the defining equation for $A(v)$ gives, with $D = \sqrt{4ac(r^2 - s^2) + b^2 s^2}$,

$$a_{rs} \sim \frac{2^s c^r a^{(s-r)} r^r}{(r-s)^{(r-s)}(r+s)^r} \frac{(D+bs)^r}{(D+bs)^s} \frac{s\phi(v/A(v))}{\sqrt{2\pi r(2ar^2 + (r-s)s)}}. \tag{11}$$

In several common cases the formulae (5) and (9) simplify further. In [21] there occur three subgroups of the Riordan group defined by a relation between $v(t)$ and $\phi(t)$. In these cases we may directly eliminate $\phi$ from (9). The computations are routine and the results are displayed in Table 2.

**Example 3.5 (Subgroups of the Riordan group).** The ***Bell subgroup*** is defined by $t\phi(t) = v(t)$. Its elements are called ***renewal arrays***. They were introduced under that name, before Riordan arrays, in [20]. Note that $(1, A(t), Z(t))$ represents an element of the Bell subgroup if and only if $A(t) = 1 + tZ(t)$. The coefficients of such an array are clearly just shifted versions of those when $\phi(t) = 1$, and this fact is reflected in the leading term asymptotic.

In [16] the subgroup $S$ of the Riordan group consisting of elements $(\phi(t), v(t))$ such that $\mu(v; t) = \phi(t)$ was considered (in [21] $S$ was called the ***hitting time subgroup***). Note that $(1, A(t), Z(t))$ represents an element of the hitting time subgroup if and only if $Z(t) = A'(t)$. This subgroup includes several well-known arrays. Note that if $(\phi(t), v(t))$ belongs to $S$ then so does $(\phi(-t), v(-t))$.

The ***associated subgroup*** is defined by $\phi(x) = 1$ (equivalently, $Z(t) = 0$). Two main ways in which elements of this subgroup arise are as follows. Let $\mathcal{C}$ be a combinatorial class with size generating function $v(t) = \sum_n v_n t^n$. Then the bivariate GF for all (finite) $\mathcal{C}$-sequences enumerated by total size and number of parts is $(1 - wv(z))^{-1}$. An asymptotic approximation to the number $a_{rs}$ of $r$-sequences with exactly $s$ parts is then obtainable from (5). This covers familiar and important examples such as compositions, alignments, surjections, ordered forests.

Another common way in which elements of the associated subgroup arise is as follows. Let $\{X_i \mid i \geq 1\}$ be a countable set of independent identically distributed random variables supported on $\mathbb{N}$ and let $v(t)$ be their common PGF. The grand PGF of all the sums $S_s := X_1 + \cdots + X_r$ is then $(1 - wv(z))^{-1}$. If $F$ is nonconstant (the random variables are not point masses at 0) then (5) applies.

**Example 3.6 (simple sequence example).** An ordered tree can be thought of as a root and an ordered forest of subtrees of the root. Such a forest is a sequence of ordered trees. The GF $F(z, w)$ that enumerates ordered trees by number of nodes and root degree is then determined by the functional equation

$$F(z, w) = \frac{z}{1 - wF(z, 1)}.$$

Letting $v(z) = F(z, 1)$ we see that $v(z)$ satisfies $z/(1-v) = v$ and we obtain

$$a_{rs} \sim \frac{(2r-s)^{2r-s}}{r^r (r-s)^{(r-s)}} \frac{s}{\sqrt{2\pi r(r-s)(2r-s)}}.$$

Various restrictions on sequences also lead to similar problems. Consider the following [7, 2.3.18].

**Example 3.7 (Skolem subsets).** For each fixed $p \geq 1$, the objects to be enumerated are sequences $0 = \sigma_0 < \sigma_1 < \cdots < \cdots < \sigma_s \leq r$ such that $\sigma_i - \sigma_{i-1} \equiv 1 \mod p$ when $1 \leq i \leq k$. The bivariate GF is of Riordan type with $\phi(x) = 1/(1-x)$ and $v(x) = x/(1-x^p)$. A simple explicit computation shows that

$$a_{rs} \sim \frac{[r - s + ps]^{r-s+ps}}{(ps)^s (r-s)^{\frac{r-s}{p}}} \sqrt{\frac{r-s+ps}{2\pi ps(r-s)}}.$$

One can also count sequences according to the number of terms of a given size, or with number of terms of size divisible by a fixed integer $p$ [7, 2.3.12]. In each case the corresponding array is of generalized Riordan type.

**Example 3.8 (walks).** Walks on the integer lattice $\mathbb{Z}^2$ have often given rise to Riordan arrays in the literature. Often the resulting arrays are square and various linear transformations have been used to fit them into the Riordan array framework. We discuss only "genuine" examples here.

All walks start at the origin. Walks on $\mathbb{Z}$ can be represented in the usual way as walks on $\mathbb{Z}^2$ of a special type: the walk $n_0, n_1, \ldots, n_t$ corresponding to the walk $(0, n_0), (1, n_1), \ldots, (t, n_t)$. A positive walk is one constrained to lie in the upper halfspace or in $\mathbb{N}$. We let $a_{nk}$ denote the number of walks of a certain type ending at $(n, k)$, which corresponds in the directed case above to walks with $n$ steps ending at $k$, and let $F(z, w) = \sum_{n,k} a_{nk} z^n w^k$.

A standard example is that of generalized Dyck paths. These are positive walks on $\mathbb{Z}^2$ defined by a finite set of allowed jumps $E = \{(r_i, s_i) \mid 1 \leq i \leq k\}$. The generating function is of Riordan type if $1 = -\min s_i = \max s_i$, which includes the classical cases $E = \{(1,-1),(1,1)\}$ (Dyck paths), $E = \{(1,-1),(1,0),(1,1)\}$ (Motzkin paths), and $E = \{(1,-1),(2,0),(1,1)\}$ (Schröder paths) or if $r_i = 1, \max s_i = 1$ (corresponding to walks on $\mathbb{N}$ with steps given by the $s_i$).

In this latter case it has also been shown [15] that the generating function is a Riordan array even for more general positive walks with an infinite set of negative jumps (so that, for example, we may jump to 1 from anywhere in $\mathbb{N}$). Indeed, every proper Riordan array with nonnegative coefficients corresponds to such a walk.

One way of interpreting weighted walks is in terms of colours, the weight of a jump corresponding to the number of colours available. This interpretation was given in [23, Sec. 4] in the case of the finite set $\{-1, 0, 1\}$ of jumps. Calculations in [23] show that the GF for positive walks is of Riordan type with $A(t) = at^2 + bt + c$, while the GF for unconstrained walks has the same $A$-sequence and belongs to the hitting time subgroup. Thus for example, in the unconstrained case we have

$$a_{rs} \sim \frac{2^s c^r a^{(s-r)} r^r}{(r-s)^{(r-s)}(r+s)^r} \frac{(D+bs)^r}{(D+bs)^s} \frac{\sqrt{r}}{\sqrt{2\pi(2ar^2 + (r-s)s)}}$$

where $D = \sqrt{4ac(r^2 - s^2) + b^2 s^2}$.

Furthermore, in [16], the above was generalized. It was shown that normalizing by taking the weight of the jump 1 to be 1, and allowing an infinite set of negative jumps, leads to a bijection between unconstrained coloured walks and the hitting time subgroup. Also the generating function for strictly positive walks belongs to the associated subgroup. In each case $A(t)$ is the same, being given by $A(t) = \sum_l a_{1-l} t^l$, where $a_l$ denotes the weight of the jump $l$.

# 4 Further comments

## 4.1 Another extension

If in (1), we allow $\phi$ to depend also on $w$, much of the above analysis carries over (though the combinatorial interpetation may be more complicated). Certainly in the case that $\phi \geq 0$ is entire and $v \geq 0$, the classification of minimal points remains the same and the formula (5) needs only the obvious modification of changing $\phi(z)$ to $\phi(z, w)$. However even if $F$ has the global form $\phi/(1 - wv(z))$ and $F \geq 0$, it need not be the case that $v \geq 0$, as the example $v = -1, \phi = (1-w)^{-1}$ shows. Note that if $F \geq 0$ and $v \geq 0$ then necessarily $\phi \geq 0$.

**Example 4.1 (Multi-avoidance of polyomino patterns).** This simple example is from [8], where the generating function

$$F(x, y) = \frac{2xy}{1 - 2(x + y - xy)}$$

is presented. The coefficient $a_{rs}$ counts $r \times s$ binary matrices avoiding certain patterns. This example is clearly a simple shift of a Riordan array, with $v(x) = (2 - 2x)/(1 - 2x)$, and the above analysis applies. The stationary phase equation is quadratic and explicit formulae are readily obtained as above. In particular, the number of binary square matrices avoiding the given patterns is asymptotically given by $c\alpha^r/\sqrt{\pi r}$ where $c = (\sqrt{2} - 1)/2^{5/4}, \alpha = 6 + 4\sqrt{2}$.

**Example 4.2 (Substring pattern occurrences).** Let $\sigma$ be a fixed word of length $k$ from an alphabet of size $d$. The ***autocorrelation polynomial*** of $\sigma$ is the polynomial $c(z)$ of degree $d-1$ whose $j$th coefficient is 1 if moving $\sigma$ by $j$ positions to the right creates an overlap and 0 otherwise. Then as in [5, III.6],

$$F(z,w) = \frac{(w-1)c(z) - w}{(1-dz)((w-1)c(z) - w) + (w-1)z^k}$$

enumerates words by length and (overlapping) occurrences of $\sigma$. Since $F$ is rational and its denominator is linear in $w$, the above analysis should apply. The same is true in the case where the letters have different probabilities of occurrence.

## 4.2 Removing hypothesis (*)

Here we discuss removing each of the three assumptions, while keeping the other two. As mentioned above, the aperiodicity assumption is not essential. The general case requires us to deal with, instead of strictly minimal points, so-called finitely minimal and toral points. The asymptotics in the former case are a trivial extension of the strictly minimal case, and the asymptotics in the toral case are also easy (but not yet published).

When we drop the condition that coefficients be nonnegative, life becomes much more difficult. Consider the following example.

**Example 4.3.** Consider the case $F(x,y) = 1/(3-3x+x^2-y) = v(x)/(1-yv(x))$ where $v(x) = 1/(3-3x+x^2)$. Since $v$ vanishes to order 0 and is not polynomial, we expect asymptotics for all possible directions. These asymptotics can indeed be provided by smooth point analysis, but they are quite different in character from those we have seen in the nonnegative case.

First, the type of minimal point can vary. It is routine to verify that points of the form $(x, 1/v(x))$ for $0 < x \le 1$ are strictly minimal, and these correspond to values of $r/s \in (0,1]$. However, the asymptotics in each direction above the diagonal are provided by a pair of complex conjugate finitely minimal points.

Moreover, $\sigma^2(v;x) = 0$ at $x = 1$ (recall that this cannot happen in the positive case unless $v$ is monomial), and $a_{rr}$ decays as $r^{-1/3}$ while the decay is like $r^{-1/2}$ in each other fixed direction. This is an example of "Airy phenomena" as discussed in [1]. Uniform asymptotics for all directions can indeed be obtained by detailed analysis of the Fourier-Laplace integral, as shown by Manuel Lladser's PhD thesis [9].

Note that by the Maximum Modulus Theorem, a solution of $\mu(v;z) = r/s$ will yield a (strictly) minimal pole of $F$ provided that $z$ (uniquely) maximizes $|v|$ on its torus. If $v \ge 0$ and $z$ is positive real then this later condition must always be satisfied. If $v = 1/u$ where $u \ge 0$, and $z$ is negative real, then it is also satisfied. In general, however, determining minimality must be approached on a case-by-case basis.

Finally, if we remove the condition $\rho(\phi) \ge \rho(v)$, much more work is required. We sketch the necessary modifications here. Now $\boldsymbol{\Delta}'$ is a larger interval $[k, \infty)$, but we can still find asymptotics for each given direction. The minimal points of $\mathcal{V}$ include a double point. If the singularity of $\phi$ at $z = \rho(\phi)$ is a pole, then smooth point analysis as carried out in the present article furnishes asymptotics for an initial subinterval of directions, while the other directions are taken care of by analysis near the double point as in [18]. See [19] for an example along these lines. If $\rho(\phi)$ is not a pole, our methods do not directly apply, and a case-by-case analysis based on univariate methods may be required.

## 4.3 Uniform asymptotics near coordinate axes

In the case $\phi = 1$, Drmota [4] derived a bivariate asymptotic expansion of the form in Corollary 2.7. Furthermore he showed that in many cases (those where $v$ has a dominant singularity of square root type) the expansion is uniform for $k/n \in [0, \varepsilon]$. It is very likely that a similar result is true for general $\phi$ and more general $v$. However to state and prove such a result naturally within our current framework would require the analysis of parameter-varying Fourier-Laplace integrals as recently carried out by M. Lladser [9], and we shall not attempt it here.

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