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# *An upper bound for the chromatic number of line graphs*

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It was conjectured by Reed [12] that for any graph $G$, the graph's chromatic number $\chi(G)$ is bounded above by $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$, where $\Delta(G)$ and $\omega(G)$ are the maximum degree and clique number of $G$, respectively. In this paper we prove that this bound holds if $G$ is the line graph of a multigraph. The proof yields a polynomial time algorithm that takes a line graph $G$ and produces a colouring that achieves our bound.

## 1 Introduction

The *chromatic number* of a graph $G$, denoted by $\chi(G)$, is the minimum number of colours required to colour the vertex set of $G$ so that no two adjacent vertices are assigned the same colour. That is, the vertices of a given colour form a *stable set*. Determining the exact chromatic number of a graph efficiently is very difficult, and for this reason it has proven fruitful to explore the relationships between $\chi(G)$ and other invariants of $G$. The *clique number* of $G$, denoted by $\omega(G)$, is the largest set of mutually adjacent vertices in $G$ and the *degree* of a vertex $v$, written $\deg(v)$, is the number of vertices to which $v$ is adjacent; the maximum degree over all vertices in $G$ is denoted by $\Delta(G)$. It is easy to see that $\omega(G) \leq \chi(G) \leq \Delta(G) + 1$. Brooks' Theorem (see [1]) tightens this:

**Brooks' Theorem** *$\chi(G) \leq \Delta(G)$ unless $G$ contains a clique of size $\Delta(G) + 1$ or $\Delta(G) = 2$ and $G$ contains an odd cycle.*

So for $\chi(G)$ we have a trivial upper bound in terms of $\Delta(G)$ and a trivial lower bound in terms of $\omega(G)$. We are interested in exploring upper bounds on $\chi(G)$ in terms of a convex combination of $\Delta(G) + 1$ and $\omega(G)$. In [12], Reed conjectured a bound on the chromatic number of any graph $G$:

**Conjecture 1** *For any graph $G$, $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.*

Several related results exist. In the same paper, Reed proved that the conjecture holds if $\Delta(G)$ is sufficiently large and $\omega(G)$ is sufficiently close to $\Delta(G)$. Using this, he proved that there exists a positive constant $\alpha$ such that $\chi(G) \leq \alpha(\omega(G)) + (1 - \alpha)(\Delta(G) + 1)$ for all graphs. Some results are also known for generalizations of the chromatic number.

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A *fractional vertex c-colouring* of a graph $G$ can be described as a set of stable sets $\{S_1, S_2, \dots S_l\}$ with weights $\{w_1, w_2, \dots w_l\}$ such that for every vertex $v$, $\sum_{S_i : v \in S_i} w_i = 1$ and $\sum_{i=1}^{l} w_i = c$. The *fractional chromatic number* of $G$, written $\chi^*(G)$, is the smallest $c$ for which $G$ has a *fractional vertex c-colouring*. Note that it is always bounded above by the chromatic number. The *list chromatic number* of a graph $G$, written $\chi_l(G)$, is the smallest $r$ such that if each vertex is assigned any list of $r$ colours, the graph has a colouring in which every vertex is coloured with a colour on its list. For any graph we clearly have $\chi^*(G) \le \chi(G) \le \chi_l(G)$.

In [10], Molloy and Reed proved the fractional analogue of Conjecture 1 for all graphs, i.e. that

$$\chi^*(G) \le \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil \text{ for any graph } G. \tag{1}$$

In fact, the round-up is not needed in the fractional case. In this paper we prove that Conjecture 1 holds for line graphs, which are defined in the next section.

## 2 Fractional and Integer Colourings in Line Graphs of Multigraphs

A *multigraph* is a graph in which multiple edges are permitted between any pair of vertices – all multigraphs in this paper are loopless. Given a multigraph $H = (V, E)$, the *line graph* of $H$, denoted by $L(H)$, is a graph with vertex set $E$; two vertices of $L(H)$ are adjacent if and only if their corresponding edges in $H$ share at least one endpoint. We say that $G$ is a line graph if there is a multigraph $H$ for which $G = L(H)$.

The *chromatic index* of $H$, written $\chi'(H)$, is the chromatic number of $L(H)$. Similarly, the *fractional chromatic index* $\chi'^*(H)$ is equal to the fractional chromatic number of $L(H)$. In [6], Holyer proved that determining the chromatic index of an arbitrary multigraph is NP-complete, so practically speaking we are bound to the task of approximating the chromatic index of multigraphs and hence the chromatic number of line graphs.

Vizing's Theorem (see [14]) bounds the chromatic index of a multigraph in terms of its maximum degree, stating that $\Delta(H) \le \chi'(H) \le \Delta(H) + d$, where $d$ is the maximum number of edges between any two vertices in $H$. Both bounds are achievable, but a more meaningful bound should consider other invariants of $H$. Of course, $\chi'(H)$ is always bounded below by $\chi'^*(H)$, and Edmond's theorem for matching polytopes (presented in [3], also mentioned in [8]) tells us that given

$$\Gamma(H) = \max\left\{ \frac{2|E(W)|}{|V(W)| - 1} : W \subseteq H, |V(W)| \text{ is odd} \right\},$$

$$\chi'^*(H) = \max\{\Delta(H), \Gamma(H)\}. \tag{2}$$

Does this necessarily translate into a good upper bound on the chromatic index of a multigraph? The following long-standing conjecture, proposed by Goldberg [4] and Seymour [13], implies that $\chi'^*(H) \le \chi'(H) \le \chi'^*(H) + 1$:

**Goldberg-Seymour Conjecture** *For a multigraph $H$ for which $\chi'(H) > \Delta(H) + 1$, $\chi'(H) = \lceil \Gamma(H) \rceil$.*

Asymptotic results are known: Kahn [7] proved that the fractional chromatic index asymptotically agrees with the integral chromatic index, i.e. that $\chi'(H) \le (1 + o(1))\chi'^*(H)$. This implies the Goldberg-Seymour Conjecture asymptotically. He later proved that in fact, the fractional chromatic index asymptotically agrees with the list chromatic index [8].

Another result that supports the Goldberg-Seymour Conjecture is the following theorem:

**Theorem 2 (Caprara and Rizzi [2])** *For any multigraph $H$, $\chi'(H) \leq \max\{\lfloor 1.1\Delta(H)+0.7\rfloor, \lceil\Gamma(H)\rceil\}$.*

This theorem is a slight improvement of an earlier result of Nishizeki and Kashiwagi [11], lowering the additive factor from 0.8 to 0.7. Note that this implies the Goldberg-Seymour Conjecture for any multigraph $H$ with $\Delta(H) \leq 12$, since in this case we have $\lfloor 1.1\Delta(H) + 0.7\rfloor \leq \Delta(H) + 1$.

# 3 The Main Result

We will now prove our main result:

**Theorem 3** *For any line graph $G$, $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.*

Consider a multigraph $H$ for which $G = L(H)$. The proof consists of two cases: the case where $\Delta(G)$ is large compared to $\Delta(H)$, and the case where $\Delta(G)$ is close to $\Delta(H)$. In both cases we use the fact that $\omega(G) \geq \Delta(H)$. The first case is given by the following lemma, which follows easily from Theorem 2.

**Lemma 4** *If $G$ is the line graph of a multigraph $H$, and $\Delta(G) \geq \frac{3}{2}\Delta(H)-1$, then $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.*

**Proof of Theorem 3:**

Consider a counterexample $G = L(H)$ such that the theorem holds for every line graph on fewer vertices. We know that $\Delta(G) < \frac{3}{2}\Delta(H) - 1$. Our approach is as follows: We find a maximal stable set $S \subset V(G)$ that has a vertex in every maximum clique in $G$, and let $G'$ be the subgraph of $G$ induced on $V(G) \setminus S$. We can see that $\Delta(G') \leq \Delta(G) - 1$ (since $S$ is maximal) and $\omega(G') = \omega(G) - 1$, and that the theorem holds for $G'$, as any induced subgraph of a line graph is clearly a line graph. So we know that $\chi(G') \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil - 1$. We can now construct a proper $\chi(G') + 1$-colouring of $V(G)$ by taking a proper $\chi(G')$-colouring of $G'$ and letting $S$ be the final colour class, hence $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$, a contradiction.

It suffices, then, to show the existence of such a stable set $S$ in $G$. We actually need only find a stable set that hits all the maximum cliques of $G$, as we can extend any such stable set until it is maximal. We will do this in terms of a *matching* in $H$, i.e. a set of edges in $E(H)$, no two of which share an endpoint – a matching in $H$ exactly represents a stable set in $G$. We need some notation first. For a pair of vertices $u, v \in V(H)$, the *multiplicity* of $uv$ is the number of edges in $E(H)$ between $u$ and $v$; we denote it by $\mu(u, v)$. A *triangle* in $H$ is a set of three mutually adjacent vertices, and we denote the maximum number of edges of any triangle in $H$ by tri$(H)$; the edges of a triangle are those edges in $E(H)$ joining the triangle's vertices. Note the following facts that relate invariants of $H$ and $G$:

**Fact 1** $\Delta(G) = \max_{uv\in E(H)}\{\deg(u) + \deg(v) - \mu(u, v) - 1\}$.

**Fact 2** $\omega(G) = \max\{\Delta(H), \text{tri}(H)\}$.

We say that a matching *hits* a vertex $v$ if $v$ is an endpoint of an edge in the matching. We will find a maximal matching $M$ in $H$ which corresponds to a desired stable set because it hits every vertex of maximum degree in $H$ and contains an edge of every triangle with $\max\{\Delta(H), \text{tri}(H)\}$ edges in $H$.

To this end, let $S_\Delta$ be the set of vertices of degree $\Delta(H)$ in $H$ and let $T$ be the set of triangles in $H$ that contain $\max\{\Delta(H), \text{tri}(H)\}$ edges. It is instructive to consider how the elements of $T$ interact; we omit the straightforward proofs of these observations from this extended abstract.

**Observation 1** *If two triangles of $T$ intersect in exactly the vertices $a$ and $b$ then $ab$ has multiplicity greater than $\Delta(H)/2$.*

**Observation 2** *If $abc$ is a triangle of $T$ intersecting another triangle $ade$ of $T$ in exactly the vertex $a$ then $\mu(b,c)$ is greater than $\Delta(H)/2$.*

**Observation 3** *If there is an edge of $H$ joining two vertices $a$ and $b$ of $S_\Delta$ then $\mu(a,b) > \Delta(H)/2$.*

Guided by these observations, we let $T'$ be those triangles in $T$ that contain no pair of vertices of multiplicity $> \Delta(H)/2$ and $S'_\Delta$ be those elements of $S_\Delta$ which are in no pair of vertices of multiplicity greater than $\Delta(H)/2$. We treat $T' \cup S'_\Delta$ and $(T \setminus T') \cup (S_\Delta \setminus S'_\Delta)$ separately. A few more observations regarding $S'_\Delta$ and $T'$ will serve us well. Again, we omit the proofs.

**Observation 4** *For any $S \subseteq S'_\Delta$, $|N(S)| \geq |S|$.*

**Observation 5** *If an edge $ab$ in $H$ has exactly one endpoint in a triangle $bcd$ of $T'$, then the degree of $a$ is less than $\Delta(H)$.*

**Observation 6** *If an edge $ab$ in $H$ has exactly one endpoint in a triangle $bcd$ of $T'$, then $\mu(a,b) \leq \Delta(H)/2$.*

**Observation 7** *For any vertex $v$ with two neighbours $u$ and $w$, $\deg(u) + \mu(vw) \leq \frac{3}{2}\Delta(H)$.*

It is now straightforward to show that the desired matching exists. We begin with a matching $M$ consisting of one edge between each vertex pair with multiplicity greater than $\Delta(H)/2$ – this hits $S_\Delta \setminus S'_\Delta$ and contains an edge of each triangle in $T \setminus T'$. Observation 4 tells us that we can apply Hall's Theorem (see [5]) to get a matching in $H$ that hits $S'_\Delta$; Observation 7 dictates that this matching cannot hit $M$, so the union $M'$ of these two matchings is a matching in $H$ that hits $S_\Delta$ and contains an edge of each triangle in $T \setminus T'$. Every edge in this matching either hits a maximum-degree vertex in $H$ or has endpoints with multiplicity greater than $\Delta(H)/2$.

What, then, can prevent us from extending this $M'$ to contain an edge of every triangle in $T'$? Observations 1 and 2 tell us that any two triangles in $T'$ are vertex-disjoint, so our only worry is that $M'$ hits two vertices of some triangle in $T'$. Observations 3, 5 and 6 guarantee that at most one such vertex in a given triangle is hit, and if there is such a vertex, it has degree $\Delta(H)$. We can therefore extend $M'$ to contain an edge of every triangle in $T'$. The result is a matching that satisfies all of our requirements, so the proof of the theorem is complete. □

## 4 Algorithmic Considerations

We have presented a new upper bound for the chromatic number of line graphs, i.e. $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$. Our proof of the bound yields an algorithm for constructing a colour class in $G$ but we have an initial condition in the proof (i.e. $\Delta(G) < \frac{3}{2}\Delta(H) - 1$) that does not necessarily remain if we remove these vertices. However, the bound given by Caprara and Rizzi in Theorem 2 can be achieved in $O(|E(H)|(|V(H)| + \Delta(H)))$ time [2]. It is easy to see that in the proof of Theorem 3 we can find our matching in polynomial time, so we can formulate a polytime algorithm for $\left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$-colouring a line graph $G$ with root graph $H$ as follows.

1. While $\Delta(L(H)) < \frac{3}{2}\Delta(H) - 1$, remove a matching $M$ from $H$ as in the proof of Theorem 3 (and let it be a colour class).

2. Employ Caprara and Rizzi's algorithm to complete the edge colouring of $H$.

This, of course, assumes that we have the root graph $H$ such that $G = L(H)$. Lehot provides an $O(|E(G)|)$ algorithm that detects whether or not $G$ is the line graph of a simple graph $H$ and outputs $H$ if possible [9]. Two vertices $u$ and $v$ in $G$ are *twins* if they are adjacent and their neighbourhoods are otherwise identical. We can extend Lehot's algorithm to line graphs of multigraphs by contracting each set of $k$ mutually twin vertices in $G$ into a single vertex, which we say has multiplicity $k$. This can be done trivially in $O(|E(G)|\Delta(G))$ time. The resulting graph $G'$ is the line graph of a simple graph $H'$ if and only if $G$ is the line graph of a multigraph $H$; we can generate $H$ from $H'$ by considering the multiplicities of the vertices in $G'$ and duplicating edges in $H'$ accordingly.

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