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THEOREMS ON n-DIMENSIONAL LAPLACE TRANSFORMS AND THEIR APPLICATIONS

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Abstract. In the present paper we prove certain theorems involving the classical Laplace transform of n-variables. The theorems are then shown to yield a nice algorithm for evaluating n-dimensional Laplace transform pairs. In the second part, boundary value problems are solved by using the double Laplace transformation.

1. Introduction and Notation

The generalization of the well-known Laplace transform $L[f(t);s]=\int_0^\infty e^{-st}f(t)dt$ to n-dimensional is given by

$$L_n[f(\bar{t});\bar{s}]=\int_0^\infty\int_0^\infty\cdots\int_0^\infty \exp(-\bar{s}\cdot\bar{t})f(\bar{t})P_n(d\bar{t})$$

where $\bar{t}=(t_1,t_2,\dots,t_n)$, $\bar{s}=(s_1,s_2,\dots,s_n)$, $\bar{s}\cdot\bar{t}=\sum_{i=1}^n s_it_i$ and $P_n(d\bar{t})=\prod_{k=1}^n dt_k$.

In addition to the notations introduced above, we will use the following throughout this article. Let $\overline{t^v}=(t_1^v,t_2^v,\dots,t_n^v)$ for any real exponent v and let $p_k(\bar{t})$ be the k-th symmetric polynomial in the components t_k of $\bar{t}$. Then

(i) $p_1(\overline{t^v})=t_1^v+t_2^v+\cdots+t_n^v$
(ii) $p_n(\overline{t^v})=t_1^v\cdot t_2^v\dots t_n^v$.

2. Main Results

Theorem 2.1. *Suppose that $f(x)$ and $f(x^2)$ are functions of class Ω. Let*

(i) $\mathcal{L}\{f(x);s\}=\phi(s)$,
(ii) $\mathcal{L}\{x^{-\frac{3}{2}}\phi\left(\frac{1}{x}\right);s\}=\xi(s)$,
(iii) $\mathcal{L}\{x^{-\frac{1}{2}}\xi\left(\frac{1}{x^2}\right);s\}=\zeta(s)$, *and*
(iv) $\mathcal{L}\{f(x^2);s\}=G(s)$,

where $x^{-\frac{3}{2}}\phi\left(\frac{1}{x}\right)$ and $x^{-\frac{1}{2}}\xi\left(\frac{1}{x^2}\right)$ are also functions of class Ω, $x^{-\frac{3}{2}}\exp\left(-sx-\frac{u}{x}\right)f(u)$ and $u^{-\frac{1}{2}}x^{-\frac{3}{2}}\exp\left(-sx-\frac{2u^{\frac{1}{2}}}{x}\right)f(u)$ belong to $L_1[(0,\infty)\times(0,\infty)]$.

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Then

$$\mathcal{L}_n\left\{\frac{1}{p_n\left(\overline{x^{\frac{1}{2}}}\right)}G[2p_1(\overline{x^{-1}})];\overline{s}\right\}=\frac{\pi^{\frac{n-2}{2}}}{2}\cdot\frac{p_1\left(\overline{s^{\frac{1}{2}}}\right)}{p_n\left(\overline{s^{\frac{1}{2}}}\right)}\zeta\left[\left(P_1\left(\overline{s^{\frac{1}{2}}}\right)\right)^2\right], \tag{2.1}$$

where $\mathcal{R}e\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]>d$, *a constant, provided the integrals involved exist for* $n=2,3,\ldots,N$. *The existence conditions for two-dimensions are given in Ditkin and Prudnikov* [11; p.4] *and similar conditions hold for* N*-dimensions, we refer to Brychkov et al.* [2; Ch.2].

Proof. Using (i) and (ii), we obtain

$$\xi(s)=\int_0^\infty\left[\int_0^\infty x^{-\frac{3}{2}}\exp\left(-sx-\frac{u}{x}\right)f(u)du\right]dx. \tag{2.2}$$

Next we wish to interchange the order of the integral on the right side of (2.2). The integrand $x^{-\frac{3}{2}}\exp\left(-sx-\frac{u}{x}\right)f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$, that, by Fubini's Theorem, interchanging the order of the integral on the right of (2.2) is permissible. Therefore,

$$\xi(s)=\int_0^\infty f(u)\left[\int_0^\infty x^{-\frac{3}{2}}\exp\left[-sx-\frac{u}{x}\right]dx\right]du \tag{2.3}$$

We then use a well-known result in Robert and Kaufman [15] on the right side of (2.3) to obtain

$$\xi(s)=\pi^{\frac{1}{2}}\int_0^\infty u^{-\frac{1}{2}}\exp\left(-2u^{\frac{1}{2}}s^{\frac{1}{2}}\right)f(u)du. \tag{2.4}$$

Using (2.4) and (iii), it follows that

$$\zeta(s)=\pi^{\frac{1}{2}}\int_0^\infty\left[\int_0^\infty x^{-\frac{1}{2}}u^{-\frac{1}{2}}\exp\left(-sx-\frac{2u^{\frac{1}{2}}}{x}\right)f(u)du\right]dx.,\quad\text{where }\mathcal{R}e\ s>\lambda_1. \tag{2.5}$$

Since $x^{-\frac{1}{2}}u^{-\frac{1}{2}}\exp\left(-sx-\frac{2u^{\frac{1}{2}}}{x}\right)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$; therefore, according to Fubini's Theorem, (2.5) can be rewritten as

$$\zeta(s)=\pi^{\frac{1}{2}}\int_0^\infty u^{-\frac{1}{2}}f(u)\left[\int_0^\infty x^{-\frac{1}{2}}\exp\left(-sx-\frac{\left(2^{\frac{3}{2}}u^{\frac{1}{4}}\right)^2}{4x}\right)dx\right]du,\quad\text{where }\mathcal{R}e\,s>\lambda_1.$$

From the tables of Roberts and Kaufman [15], we obtain

$$s^{\frac{1}{2}}\zeta(s)=\pi\int_0^\infty u^{-\frac{1}{2}}f(u)\exp\left(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}}\right)du. \tag{2.6}$$

Next, we substitute $u=v^2$ in (2.6) to obtain

$$s^{\frac{1}{2}}\zeta(s)=2\pi\int_0^\infty\exp\left(-2^{\frac{3}{2}}s^{\frac{1}{2}}v^{\frac{1}{2}}\right)f(v^2)dv. \tag{2.7}$$

Replacing s by $\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]^2$, multiplying both sides of (2.7) by $p_n\left(\overline{s^{\frac{1}{2}}}\right)$, we obtain

$$p_1\left(\overline{s^{\frac{1}{2}}}\right)p_n\left(\overline{s^{\frac{1}{2}}}\right)\zeta\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right] = 2\pi\int_0^\infty p_n\left(\overline{s^{\frac{1}{2}}}\right)\exp\left[-2^{\frac{3}{2}}p_1\left(\overline{s^{\frac{1}{2}}}\right)v^{\frac{1}{2}}\right]f(v^2)dv \tag{2.8}$$

Now we use the operational relation given in Ditkin and Prudnikov [11]

$$s_i^{\frac{1}{2}}\exp\left(-as_i^{\frac{1}{2}}\right) \doteqdot (\pi x_i)^{-\frac{1}{2}}\exp\left(-\frac{a^2}{4x_i}\right) \quad \text{for } i = 1, 2, \ldots, n \tag{2.9}$$

Equation (2.8) reads as follows

$$p_n\left(\overline{s^{\frac{1}{2}}}\right)p_1\left(\overline{s^{\frac{1}{2}}}\right)\zeta\left[\left(p_1\left(\overline{s^{\frac{1}{2}}}\right)\right)^2\right] \underset{n}{\overset{n}{=}} \frac{2}{\pi^{\frac{n-2}{2}}p_n\left(\overline{x^{\frac{1}{2}}}\right)}\int_0^\infty \exp\left(-2vp_1\left(\overline{x^{-1}}\right)\right)f(v^2)dv \tag{2.10}$$

Applying (iv) in (2.10), we arrive at

$$p_n\left(\overline{s^{\frac{1}{2}}}\right)p_1\left(\overline{s^{\frac{1}{2}}}\right)\zeta\left[\left(p_1\left(\overline{s^{\frac{1}{2}}}\right)\right)^2\right] \underset{n}{\overset{n}{=}} \frac{2}{\pi^{\frac{n-2}{2}}p_n\left(\overline{x^{\frac{1}{2}}}\right)}\, G\left[2p_1\left(\overline{x^{-1}}\right)\right].$$

Therefore,

$$\mathcal{L}_n\left\{\frac{1}{p_n\left(\overline{x^{\frac{1}{2}}}\right)}\, G\left[2p_1\left(\overline{x^{\frac{1}{2}}}\right)\right];\overline{s}\right\} = \frac{\pi^{\frac{n-2}{2}}}{2}\cdot\frac{p_1\left(\overline{s^{\frac{1}{2}}}\right)}{p_n\left(\overline{s^{\frac{1}{2}}}\right)}\zeta\left[\left(p_1\left(\overline{s^{\frac{1}{2}}}\right)\right)^2\right],$$

where $n = 2, 3, \ldots, N$.

To show the applicability of Theorem 2.1, we will construct certain functions with n variables and calculate their Laplace transformation.

Example 2.1. Assume that $f(x) = x^{\frac{\tau-1}{4}}$. Then

$$\begin{aligned}
\phi(s) &= \frac{\Gamma\left(\frac{\tau+3}{4}\right)}{s^{\frac{\tau+3}{4}}}, \mathcal{R}e\ s > 0,\ \mathcal{R}e\ v > -3;\\
\xi(s) &= \frac{\Gamma\left(\frac{\tau+3}{4}\right)\Gamma\left(\frac{\tau+1}{4}\right)}{s^{\frac{\tau+1}{4}}},\ \mathcal{R}e\ \tau > -1, \mathcal{R}e\ s > 0,\ \text{and}\\
\zeta(s) &= \frac{\Gamma\left(\frac{\tau+3}{4}\right)\Gamma\left(\frac{\tau+1}{4}\right)\Gamma\left(\frac{\tau}{2}+1\right)}{s^{\frac{\tau}{2}+1}},\ \mathcal{R}e\ s > 0,\ \mathcal{R}e\ \tau > -1.\\
G(s) &= \frac{\Gamma\left(\frac{\tau+1}{2}\right)}{s^{\frac{\tau+1}{2}}},\ \mathcal{R}e\ \tau > -1.
\end{aligned}$$

Therefore,

$$\mathcal{L}_n\left\{\frac{1}{p_n\left(\overline{x^{\frac{1}{2}}}\right)\left[p_1\left(\overline{x^{-1}}\right)\right]^{\frac{\tau+2}{2}}};\ \overline{s}\right\} = \pi^{\frac{n-1}{2}}\Gamma\left(\frac{\tau}{2}+\frac{3}{2}\right)\cdot\frac{1}{p_n\left(\overline{s^{\frac{1}{2}}}\right)\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]^{\tau+2}} \tag{2.11}$$

where $\mathcal{R}e\ \tau > -1,\ \mathcal{R}e\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right] > 0$, and $n = 2, 3, \ldots, N$.

Example 2.2. Suppose that $f(x) = I_0(ax^{\frac{1}{2}})$. Then

$$\phi(s) = \frac{1}{s}\exp\left(\frac{a^2}{4s}\right),\ \mathcal{R}e\ s > 0,$$

$$\xi(s) = \frac{\pi^{\frac{1}{2}}}{\left(s-\frac{a^2}{4}\right)^{\frac{1}{2}}},\ \mathcal{R}e\ s > \mathcal{R}e\ \frac{a^2}{4},$$

$$\zeta(s) = \frac{4\pi}{a^{\frac{5}{2}}}\left\{\frac{4\pi}{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}\,{}_1F_2\left[\begin{matrix}\frac{3}{4} & ; \\ \frac{1}{2},\frac{5}{4}; \end{matrix}\frac{s^2}{a^2}\right] - \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2 s}{6\pi}\,{}_1F_2\left[\begin{matrix}\frac{3}{4} & ; \\ \frac{3}{2},\frac{7}{4}; \end{matrix}\frac{s^2}{a^2}\right]\right\}.$$

$$G(s) = \frac{1}{(s^2-a^2)^{\frac{1}{2}}},\ \mathcal{R}e\ s > |\mathcal{R}e\ a|.$$

So that

$$\mathcal{L}_n\left\{\frac{1}{p_n(x^{\frac{1}{2}})}\cdot\frac{1}{\left[4p_1^2\left(\overline{x^{-1}}\right)-a^2\right]^{\frac{1}{2}}};\overline{s}\right\}$$

$$= \frac{2\pi^{\frac{n}{2}}p_1\left(\overline{s^{\frac{1}{2}}}\right)}{a^{\frac{5}{2}}p_n\left(\overline{s^{\frac{1}{2}}}\right)}\left\{\frac{4\pi}{\left(\Gamma\left(\frac{1}{4}\right)\right]^2}\,{}_1F_2\left[\begin{matrix}\frac{3}{4} & ; \\ \frac{1}{2},\frac{5}{4}; \end{matrix}\frac{p_1^2\left(\overline{s^{\frac{1}{2}}}\right)}{a^2}\right] - \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2 p_1^2\left(\overline{s^{\frac{1}{2}}}\right)}{6\pi}\right. \tag{2.12}$$

$$\left.{}_1F_2\left[\begin{matrix}\frac{5}{4} & ; \\ \frac{3}{2},\frac{7}{4}; \end{matrix}\frac{p_1^2\left(\overline{s^{\frac{1}{2}}}\right)}{a^2}\right];\overline{s}\right\},$$

where $\mathcal{R}e\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right] > |\mathcal{R}e\ a|$.

Example 2.3. Consider $f(x) = {}_pF_q\left[\begin{matrix}(a)_p; \\ (b)_q; \end{matrix}cx\right]$. Then

$$\phi(s) = \tfrac{1}{s}\,{}_{p+1}F_q\left[\begin{matrix}(a)_p; \\ (b)_q; \end{matrix}cx\right],$$

where $p \le q,\ \mathcal{R}e\ s > |\mathcal{R}e\ c|$.

$$\xi(s) = \frac{\pi^{\frac{1}{2}}}{s^{\frac{1}{2}}}\,{}_{p+2}F_q\left[\begin{matrix}(a)_p, 1, \frac{1}{2}; \\ (b)_q \quad ; \end{matrix}\frac{c}{s}\right],$$

where $p \le q-1, \mathcal{R}e\ \tau > 0$ if $p+1 < q, \mathcal{R}e\ s > \mathcal{R}e\ c$ if $p+1 = q$,

$$\zeta(s) = \frac{\pi}{2s^{\frac{3}{2}}}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; \\ (b)_q \quad ; \end{matrix}\frac{4c}{s^2}\right],$$

where $p \le q-3; \mathcal{R}e\ s > 0$ if $p \le q-4$; and $\mathcal{R}e(s+2c^{\frac{1}{2}}\cos\pi r) > 0$ $(r = 0,1)$ if $p = q-3$.
Hence

$$\mathcal{L}_n\left\{\frac{1}{p_n\left(\overline{x^{\frac{1}{2}}}\right)\left[p_1\left(\overline{x^{-1}}\right)\right]}\,{}_{p+2}F_q\left[\begin{matrix}(a)_p, 1, \frac{1}{2}; \\ (b)_q \quad ; \end{matrix}\frac{c}{p_1^2\left(\overline{x^{-1}}\right)}\right];s\right\}$$

$$= \frac{\pi^{\frac{n}{2}}}{2p_n\left(\overline{s^{\frac{1}{2}}}\right)p_1^2\left(\overline{s^{\frac{1}{2}}}\right)}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}; \\ (b)_q \quad ; \end{matrix}\frac{4c}{p_1^4\left(\overline{s^{\frac{1}{2}}}\right)}\right], \tag{2.13}$$

where $p \leq q-3, \mathcal{R}e\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]>0$ if $p \leq q-4$; and $\mathcal{R}e\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)+2c^{\frac{1}{2}}\cos \pi r\right]$ $>0 (r=0,1)$ if $p=q-3$.

Example 2.4. Assume that $f(x)=x^{\frac{1}{2}} J_0(a x^{\frac{1}{2}})$. Then

$$\phi(s)=\frac{\pi^{\frac{1}{2}}}{2 s^{\frac{3}{2}}}\,{}_1F_1\left[\begin{matrix}\frac{3}{2}; \\ 1;\end{matrix}\ -\frac{a^2}{4 s}\right], \mathcal{R}e\ s>0,$$

$$\xi(s)=\frac{\pi^{\frac{1}{2}}}{2}\,{}_2F_1\left[\begin{matrix}\frac{3}{2}, 1; \\ 1 \quad ;\end{matrix}\ -\frac{a^2}{4 s}\right], \mathcal{R}e\ s>-\mathcal{R}e-\frac{a^2}{4},$$

$$\zeta(s)=\frac{1}{\pi^{\frac{1}{2}} s^{\frac{1}{2}}} \mathbf{G}_{4,2}^{1,4}\left(\frac{a^2}{s^2} \middle| \begin{matrix}\frac{1}{4}, \frac{3}{4}, 0,-\frac{1}{2} \\ 0,0\end{matrix}\right),$$

where $\mathcal{R}e\ s>0, |\arg\ a|<2 \pi$.

$$G(s)=\frac{\pi^{\frac{1}{2}}}{2}\,{}_2F_1\left[\begin{matrix}\frac{3}{2}, 1; \\ 1 \quad ;\end{matrix}\ -\frac{a^2}{4 s}\right], \mathcal{R}e\ s>\mathcal{R}e-\frac{a^2}{4}.$$

Hence

$$\begin{aligned} & \mathcal{L}_n\left\{\frac{p_1\left(\overline{x^{-1}}\right)}{p_n\left(\overline{x^{\frac{1}{2}}}\right)\left[4 p_1^2\left(\overline{x^{-1}}\right)+a^2\right]^{\frac{3}{2}}}, \bar{s}\right\} \\ & =\frac{\pi^{\frac{n-3}{2}} p_1\left(\overline{s^{\frac{1}{2}}}\right)}{p_n\left(\overline{s^{\frac{1}{2}}}\right)} \mathbf{G}_{4,2}^{1,4}\left(\frac{a^2}{p_1^4\left(\overline{s^{\frac{1}{2}}}\right)} \middle| \begin{matrix}\frac{1}{2}, \frac{3}{4}, 0,-\frac{1}{2} \\ 0,0\end{matrix}\right), \end{aligned} \tag{2.14}$$

where $\mathcal{R}e\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]>0, |\arg\ a|<2 \pi$ and $\mathbf{G}_{4,2}^{1,4}$ is a Meijer's G-function.

Example 2.5 (Two-Dimensions).

Upon substituting $n=2$ in Examples 2.1, 2.2, 2.3, and 2.5 we arrive at the following results, respectively

$$\mathcal{L}_2\left\{\frac{(x y)^{\frac{\tau}{2}}}{(x+y)^{\frac{\tau+1}{2}}} ; s_1, s_2\right\}=\pi^{\frac{1}{2}} \Gamma\left(\frac{\tau}{2}+1\right) \cdot \frac{1}{(s_1 s_2)^{\frac{1}{2}}\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^{\tau+1}} \tag{2.15}$$

where $\mathcal{R}e\ \tau>-1,\ \mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0$.

$$\begin{aligned} & \mathcal{L}_2\left\{\frac{(x y)^{\frac{1}{2}}}{\left[4(x+y)^2-(a x y)^2\right]^{\frac{1}{2}}} ; s_1, s_2\right\} \\ & =\frac{2 \pi\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)}{a^{\frac{5}{2}}(s_1 s_2)^{\frac{1}{2}}}\left\{\frac{4 \pi}{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}\,{}_1F_2\left[\begin{matrix}\frac{3}{4} \quad ; \\ \frac{1}{2}, \frac{5}{4};\end{matrix}\ \frac{\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2}{a^2}\right]\right. \\ & \left.-\frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2}{6 \pi}\,{}_1F_2\left[\begin{matrix}\frac{5}{4} \quad ; \\ \frac{3}{2}, \frac{7}{4};\end{matrix}\ \frac{\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2}{a^2}\right]\right\} \end{aligned} \tag{2.16}$$

where $\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0, |\arg\ a|<2\pi.$

$$\mathcal{L}_2\left\{\frac{(xy)^{\frac{1}{2}}}{x+y}\,{}_{p+2}F_q\left[\begin{matrix}(a)_p,1,\frac{1}{2}; & \dfrac{c(xy)^2}{(x+y)^2}\\ (b)_q \qquad ; & \end{matrix}\right];s_1,s_2\right\}$$
$$=\frac{\pi}{2(s_1s_2)^{\frac{1}{2}}\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p,\frac{1}{2},\frac{3}{4},\frac{5}{4}; & \dfrac{4c}{\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^4}\\ (b)_q \qquad ; & \end{matrix}\right], \tag{2.17}$$

where $p\le q-3$, $\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0$ if $p\le q-4$; and $\mathcal{R}e\left[\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)+2c^{\frac{1}{2}}\cos\pi r\right]$ $>0 (r=0,1)$ if $p=q-3$.

$$\mathcal{L}_2\left\{\frac{(xy)^{\frac{3}{2}}(x+y)}{[4(x+y)^2+(axy)^2]^{\frac{3}{2}}};s_1,s_2\right\}=\frac{s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}}{(\pi s_1s_2)^{\frac{1}{2}}}\mathbf{G}^{1,4}_{4,2}\left(\frac{a^2}{\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^4}\middle|\begin{matrix}\frac{1}{2},\frac{3}{2},0,-\frac{1}{2}\\ 0,0\end{matrix}\right), \tag{2.18}$$

where $\mathcal{R}\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>|\mathcal{R}e\ a|$ and $\mathbf{G}^{1,4}_{4,2}$ is a Meijer's G-function.

Remark 2.1. If we let $\tau=0$ in Relation (2.15), and then using Relation (1) in [24]. We deduced that

$$\mathcal{L}_2\left\{(x+y)^{\frac{1}{2}};s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}\left(s_1+s_2+s_1^{\frac{1}{2}}s_2^{\frac{1}{2}}\right)}{2(s_1s_2)^{\frac{3}{2}}\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)} \tag{2.19}$$

Theorem 2.2. *Assume that f belongs to class Ω and ϕ be the one-dimensional Laplace transformation of f. Let*

(i) $\mathcal{L}\left\{x^{-\frac{1}{2}}\phi\left(\frac{1}{x}\right);s\right\}=\gamma(s)$,
(ii) $-\frac{d}{ds}\left\{s^{-v}\gamma\left(\frac{1}{s^2}\right)\right\}=\zeta(s)$,
(iii) $\mathcal{L}\left\{xf(x);s\right\}=H(s)$,

and suppose that $x^{-\frac{1}{2}}\phi\left(\frac{1}{x}\right)$ belongs to Ω and $\frac{d}{ds}\left\{s^{-v}\gamma\left(\frac{1}{s^2}\right)\right\}$ exists for $\mathcal{R}e\ s>c_1$ where c_1 is a constant. Then

$$\mathcal{L}_n\left\{\frac{(v-1)\phi\left[p_1\left(\overline{x^{-1}}\right)\right]-2p_1\left(\overline{x^{-1}}\right)H\left[p_1\left(\overline{x^{-1}}\right)\right]}{p_n\left(\overline{x^{\frac{1}{2}}}\right)};\overline{s}\right\}=$$
$$\frac{\pi^{\frac{n-1}{2}}}{p_n\left(\overline{s^{\frac{1}{2}}}\right)\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]^v}\zeta\left[\left(p_1\left(\overline{s^{\frac{1}{2}}}\right)\right)^{-1}\right], \tag{2.20}$$

where $\mathcal{R}e\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]>d$, a constant $n=2,3,\ldots,N$. It is asssumed that the integral on the left exists.

Proof. A similar method to Theorem 2.2 can be used to prove this theorem. The outline of the proof is as follows.

Making use of our hpothesis and (i) yields

$$\gamma(s)=\pi^{\frac{1}{2}}\int_0^\infty\left[\int_0^\infty x^{-\frac{1}{2}}\exp\left(-sx-\frac{u}{x}\right)f(u)du\right]dx,\ \text{where } \mathcal{R}e\ s>c_1. \tag{2.21}$$

Using Fubini's Theorem to interchange the order of the integral on the right side of (2.21) and applying a result in Roberts and Kaufman [15] we obtain

$$\gamma(s)=\pi^{\frac{1}{2}}\int_0^\infty f(u)s^{-\frac{1}{2}}\exp\left(-2u^{\frac{1}{2}}s^{\frac{1}{2}}\right)du. \tag{2.22}$$

Taking into account the condition (ii) we see that the equation (2.22) implies that

$$s^v\zeta(s)=\pi^{\frac{1}{2}}(v-1)\int_0^\infty f(u)\exp\left(-\frac{2u^{\frac{1}{2}}}{s}\right)du-2\pi^{\frac{1}{2}}s^{-1}\int_0^\infty u^{\frac{1}{2}}f(u)$$
$$\exp\left(\frac{-2u^{\frac{1}{2}}}{s}\right)du,\ \text{where } \mathcal{R}e\ s>c_1. \tag{2.23}$$

Now. replacing s by $\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]^{-1}$ and then multiplying each side of (2.23) by $p_n\left(\overline{s^{\frac{1}{2}}}\right)$ and then making use of operational relations (2.9) and (2.24)

$$s_i\exp\left(-as_i^{\frac{1}{2}}\right)\doteqdot\frac{a}{2}\pi^{-\frac{1}{2}}x_i^{-\frac{3}{2}}\exp\left(-\frac{a^2}{4x_i}\right)\ \text{for } i=1,2,\dots,n, \tag{2.24}$$

equation (2.23) reads

$$p_n\left(\overline{s^{\frac{1}{2}}}\right)\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]^{-v}\zeta\left[\left(p_1\left(\overline{s^{\frac{1}{2}}}\right)\right)^{-1}\right]\overset{n}{\underset{n}{=}}\frac{1}{\pi^{\frac{n-1}{2}}p_n\left(\overline{x^{\frac{1}{2}}}\right)}\left[(v-1)\int_0^\infty f(u)\right.$$
$$\left.\exp\left[-up_1\left(\overline{x^{-1}}\right)\right]dx-2p_1\left(\overline{x^{-1}}\right)\int_0^\infty uf(u)\exp\left[-up_1\left(\overline{x^{-1}}\right)\right]du.\right. \tag{2.25}$$

Equation (2.25) with (iii) and the definition of the one-dimensional Laplace transform, leads to

$$p_n\left(\overline{s^{\frac{1}{2}}}\right)\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]^{-v}\zeta\left[\left(p_1\left(\overline{s^{\frac{1}{2}}}\right)\right)^{-1}\right]$$
$$\overset{n}{\underset{n}{=}}\frac{1}{\pi^{\frac{n-1}{2}}p_n\left(\overline{x^{\frac{1}{2}}}\right)}\left\{(v-1)\phi\left[p_1\left(\overline{x^{-1}}\right)\right]-2p_1\left(\overline{x^{-1}}\right)H\left[p_1\left(\overline{x^{-1}}\right)\right]\right\}.$$

Therefore,

$$\mathcal{L}_n\left\{\frac{(v-1)\phi\left[p_1\left(\overline{x^{-1}}\right)\right]-2p_1\left(\overline{x^{-1}}\right)H\left[p_1\left(\overline{x^{-1}}\right)\right]}{p_n\left(\overline{x^{\frac{1}{2}}}\right)};\overline{s}\right\}$$
$$=\frac{\pi^{\frac{n-1}{2}}}{p_n\left(\overline{s^{\frac{1}{2}}}\right)\left[p_n\left(\overline{s^{\frac{1}{2}}}\right)\right]^v}\zeta\left[\left(p_1\left(\overline{s^{\frac{1}{2}}}\right)\right)^{-1}\right],$$

where $\mathcal{R}e\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]>d$ for some constant $d, n=2,3,\dots,N$.

Example 2.6. Let $f(x) = J_0\left(2x^{\frac{1}{2}}\right)$. Then

$$\phi(s) = \frac{1}{s}\exp\left(-\frac{1}{s}\right), \ \mathcal{R}e\ s > 0,$$

$$\gamma(s) = \frac{\pi^{\frac{1}{2}}}{2(s+1)^{\frac{3}{2}}}, \mathcal{R}e\ s > -1.$$

Thus

$$\zeta(s) = \frac{\pi^{\frac{1}{2}}(vs^2+v-3)}{2s^{v-2}(1+s^2)^{\frac{5}{2}}}, \ \mathcal{R}e\ s > -1.$$

Therefore

$$\mathcal{L}_n\left\{\frac{1}{p_1\left(\overline{x^{-1}}\right)p_n\left(\overline{x^{\frac{1}{2}}}\right)}\exp\left(-\frac{1}{p_1\left(\overline{x^{-1}}\right)}\right)\left\{(v-1)-2{}_1F_1\left[\begin{matrix}-1; \\ 1;\end{matrix}\ \frac{1}{p_1\left(\overline{x^{-1}}\right)}\right]\right\};\overline{s}\right\}$$

$$= \frac{\pi^{\frac{n}{2}}p_1\left(\overline{s^{\frac{1}{2}}}\right)\left[v+(v-3)p_1^2\left(\overline{s^{\frac{1}{2}}}\right)\right]}{2p_n\left(\overline{s^{\frac{1}{2}}}\right)\left[1+p_1^2\left(\overline{s^{\frac{1}{2}}}\right)\right]^{\frac{5}{2}}}, \text{ where } \mathcal{R}e\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right] > -1. \tag{2.26}$$

Remark 2.2. If we let $n = 2$ and $v = 1$ or $v = 3$, from the equation (2.26) we deduce the following results, respectively

$$(i) \quad \mathcal{L}_2\left\{\frac{(xy)^{\frac{1}{2}}}{(x+y)}\exp\left(-\frac{xy}{x+y}\right){}_1F_1\left[\begin{matrix}-1; \\ 1;\end{matrix}\ \frac{xy}{x+y}\right];s_1,s_2\right\} =$$

$$\frac{\pi\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)\left[2\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2-1\right]}{4(s_1s_2)^{\frac{1}{2}}\left[1+\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2\right]^{\frac{5}{2}}}, \tag{2.26$'$}$$

where $\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right] > -1$.

$$(ii) \quad \mathcal{L}_2\left\{\frac{(xy)^{\frac{1}{2}}}{(x+y)}\exp\left(-\frac{xy}{x+y}\right)\left\{1-{}_1F_1\left[\begin{matrix}-1; \\ 1;\end{matrix}\ \frac{xy}{x+y}\right]\right\};s_1,s_2\right\} =$$

$$\frac{3\pi\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)}{2(s_1s_2)^{\frac{1}{2}}\left[1+\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2\right]^{\frac{5}{2}}}, \tag{2.26$''$}$$

where $\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right] > 1$.
Notice, with the help of (2.26′) from (2.26″) we arrive at the following result

$$\mathcal{L}_2\left\{\frac{(xy)^{\frac{1}{2}}}{(x+y)}\exp\left(-\frac{xy}{x+y}\right);s_1,s_2\right\} = \frac{\pi}{2}\frac{\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)}{(s_1s_2)^{\frac{1}{2}}\left[1+\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2\right]^{\frac{2}{3}}}. \tag{2.26$'''$}$$

This is the same as the result (2.109) in Ditkin and Prudnokov [11; p. 140].

Furthermore, with the help of (2.26‴) and the operational relation (47) in Voelker and Doetsch [24; p. 159], we derive the following new results.

$$\mathcal{L}_2\left\{\left(\frac{y}{x}\right)^{\frac{1}{2}}\cdot\frac{1}{x+y}\exp\left(-\frac{xy}{x+y}\right);s_1,s_2\right\}=\frac{\pi}{s_2^{\frac{1}{2}}\left[1+\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}},$$

$$\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>1. \tag{2.26^{iv}}$$

$$\mathcal{L}_2\left\{\left(\frac{x}{y}\right)^{\frac{1}{2}}\cdot\frac{1}{x+y}\exp\left(-\frac{xy}{x+y}\right);s_1,s_2\right\}=\frac{\pi}{s_1^{\frac{1}{2}}\left[1+\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}}, \tag{2.26^{v}}$$

where $\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>1$.

Example 2.7. Assuming $x^{-\frac{1}{2}}\cos 2x^{\frac{1}{2}}$, we obtain

$$\begin{aligned}
\phi(x)&=\left(\frac{\pi}{s}\right)^{\frac{1}{2}}\exp\left(-\frac{1}{x}\right),\ \mathcal{R}e\ s>0.\\
\gamma(s)&=\frac{\pi}{s+1},\ \mathcal{R}e\ s>-1.\\
\zeta(s)&=\left(vs^2+v-2\right)\frac{\pi^{\frac{1}{2}}s^{-v+1}}{(s^2+1)^2},\ \mathcal{R}e\ s>-1.\\
H(s)&=\frac{\pi^{\frac{1}{2}}}{s^{\frac{5}{2}}}\left(\frac{s}{2}-1\right)\exp\left(-\frac{1}{s}\right),\ \mathcal{R}e(s)>0.
\end{aligned}$$

Therefore,

$$\mathcal{L}_n\left\{\frac{(v-1)p_1\left(\overline{x^{-1}}\right)-\left(p_1\left(\overline{x^{-1}}\right)-2\right)\exp\left(-\frac{1}{p_1\left(\left(\overline{x^{-1}}\right)\right)}\right)}{\left[p_1\left(\overline{x^{-1}}\right)\right]^{\frac{3}{2}}p_n\left(\overline{x^{-1}}\right)};\overline{s}\right\}$$

$$=\frac{\pi^{\frac{n-1}{2}}p_1\left(\overline{s^{\frac{1}{2}}}\right)}{p_n\left(\overline{s^{\frac{1}{2}}}\right)\left[1+p_1^2\left(\overline{s^{\frac{1}{2}}}\right)\right]^2}\left\{(v-2)p_1^2\left(\overline{s^{\frac{1}{2}}}\right)+v\right\},\mathcal{R}e\left[p_1\left(\overline{s^{\frac{1}{2}}}\right)\right]>0. \tag{2.27}$$

Remark 2.3. If we let $n=2$ and $v=1$ in (2.27) by using the following well-known formula

$$\mathcal{L}_2\left\{\frac{xy}{(x+y)^{\frac{3}{2}}}\exp\left(-\frac{xy}{x+y}\right);s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)}{(s_1s_2)^{\frac{1}{2}}\left[1+\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2\right]^2},$$

$$\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0.$$

we derive that

$$\mathcal{L}_2\left\{\frac{1}{(x+y)^{\frac{1}{2}}}\exp\left(-\frac{xy}{x+y}\right); s_1, s_2\right\} = \frac{\pi^{\frac{1}{2}}\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)}{(s_1s_2)^{\frac{1}{2}}\left[1+\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2\right]}, \tag{2.27'}$$

$\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right] > 0.$

3. Non-Homogeneous Second Order Partial Differential Equations of Parabolic type

In this section we solved a few partial differential equations of the type

$$u_{xx}+2u_{xy}+u_{yy}+u = f(x,y), 0<x<\infty, 0<y<\infty, \tag{3.1}$$

under the following initial and boundary conditions

$$u(x,0) = u(0,y) = u_y(x,0) = u_x(0,y) = u(0,0) = 0 \tag{3.2}$$

by means of some of our results established in Section 2 using the double Laplace transformation.

Example 3.1. Determination of a solution $u = u(x,y)$ of (3.1) and (3.2) for

(a) $f(x,y) = (x+y)^{\frac{1}{2}}$

(b) $f(x,y) = \frac{(xy)^{\frac{\tau}{2}}}{(x+y)^{\frac{\tau+1}{2}}}$

(c) $f(x,y) = \frac{(xy)^{\frac{\tau}{2}}}{(x+y)^{\frac{\tau+2}{2}}}$

We will use the following for the rest of this section. If

$$u(x,0) = f(x), u(0,y) = g(y),$$
$$u_y(x,y)|_{y=0} = u_y(x,0) = f_1(x), u_x(x,y)|_{x=0} = u_x(0,y) = g_1(y)$$

and if their one-dimensional Laplace transformations are $F(s_1), G(s_2), F_1(s_1)$ and $G_1(s_2)$, respectively, then

$$\mathcal{L}_2\{u(x,y); s_1, s_2\} = \int_0^\infty\int_0^\infty \exp(-s_1x-s_2y)u(x,y)dxdy = U(s_1,s_2) \tag{3.3}$$

$$\mathcal{L}_2\{u_{xx}; s_1, s_2\} = s_1^2U(s_1,s_2) - s_1G(s_2) - G_1(s_2) \tag{3.4}$$

$$\mathcal{L}_2\{u_{yy}; s_1, s_2\} = s_2^2U(s_1,s_2) - s_2F(s_1) - F_1(s_1) \tag{3.5}$$

$$\mathcal{L}_2\{u_{xy}; s_1, s_2\} = s_1s_2U(s_1,s_2) - s_1F(s_1) - s_2G(s_2) + u(0,0) \tag{3.6}$$

(a) By applying the double Laplace transformation termwise to partial differential equation and the initial-boundary condition in (3.1) and (3.2), using (3.3)–(3.6), and with the aid of Relation (2.19) in Remark 2.1, we obtain the transformed problem

$$U(s_1,s_2)=\frac{1}{(s_1+s_2)^2+1}\cdot\frac{\pi^{\frac{1}{2}}\left[s_1+s_2+(s_1s_2)^{\frac{1}{2}}\right]}{2(s_1s_2)^{\frac{3}{2}}\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)},\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0. \tag{3.7}$$

The inversion of (3.7) will provide us with the solution of (3.1) and (3.2). So that, the inverse transform of (3.7) can be obtained using formula (133) in [24]

$$u(x,y)=\int_0^x (x+y-2\xi)^{\frac{1}{2}}\sin\xi d\xi \tag{3.8}$$

By a simple change of variable $x+y-2\xi=2t^2$ in (3.8), we obtain

$$u(x,y)=2^{+\frac{3}{2}}\int_{\left(\frac{x+y}{2}\right)^{\frac{1}{2}}}^{\left(\frac{y-x}{2}\right)^{\frac{1}{2}}} t^2\sin\left(t^2-\frac{x+y}{2}\right)dt \quad \text{if} \quad y>x.$$

Expanding the sine and making some simplification, we deduce that

$$u(x,y)=2^{+\frac{3}{2}}\left[\cos\left(\frac{x+y}{2}\right)\int_{\left(\frac{x+y}{2}\right)^{\frac{1}{2}}}^{\left(\frac{y-x}{2}\right)^{\frac{1}{2}}} t^2\sin t^2dt-\sin\left(\frac{x+y}{2}\right)\right.$$
$$\left.\int_{\left(\frac{x+y}{2}\right)^{\frac{1}{2}}}^{\left(\frac{x-y}{2}\right)^{\frac{1}{2}}} t^2\cos t^2dt\right] \tag{3.9}$$

Calculating the integrals involved in (3.9), we arrive at

$$u(x,y)=\frac{1}{4}\left\{\begin{array}{r}\left(\frac{x+y}{2}\right)^{\frac{1}{2}}-\left(\frac{y-x}{2}\right)^{\frac{1}{2}}\cos x+\pi^{\frac{1}{2}}\cos\left(\frac{x+y}{2}\right)\left[C\left(\frac{y-x}{2}\right)-D\left(\frac{x+y}{2}\right)\right]\\ +\pi^{\frac{1}{2}}\sin\left(\frac{x+y}{2}\right)\left[S\left(\frac{y-x}{2}\right)-S\left(\frac{x+y}{2}\right)\right]\end{array}\right\}$$
$$\text{if } y>x,$$

where $C(\cdot)$ and $S(\cdot)$ are Fresnel integrals.

Similarly, to obtain the transform equations for parts (b) and (c). We replace Relation (2.19) in Example (2.5) for part (b) and (2.15) in the same example with Formula 181 in Brychkov et al. [2; p.300] for part (c) to arrive at

$$U(s_1,s_2)=\frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{\tau}{2}+1\right)}{\left[(s_1+s_2)^2+1\right](s_1s_2)^{\frac{\tau}{2}}\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^{\tau+1}} \tag{3.10}$$

$$U(s_1,s_2)=\frac{\pi\Gamma(\tau+1)}{2^{\tau}\Gamma\left(\frac{\tau+3}{2}\right)\left[(s_1+s_2)^2+1\right]\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^{\tau+1}}, \tag{3.11}$$

where $\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0$. Thus, we obtain the following solutions, respectively

$$u(x,y)=\frac{1}{2^{\tau+1}}\int_{x+y}^{y-x}t^{-\frac{\tau+1}{2}}\left[t^2-(x-y)^2\right]^{\frac{\tau}{2}}\sin\left(\frac{t-(x+y)}{2}\right)dt$$
$$\text{if } y>x\mathcal{R}e\ v>-1. \tag{3.12}$$

$$u(x,y)=\frac{1}{2^{\tau+1}}\int_{x+y}^{y-x}t^{-\frac{\tau+3}{2}}\left[t^2-(x-y)^2\right]^{\frac{\tau}{2}}\sin\left(\frac{t-(x+y)}{2}\right)dt$$
$$\text{if } y>x, \mathcal{R}e\ v>-1. \tag{3.13}$$

Remark 2.4. Substituting $\tau=0$ in parts (b) and (c), lead to

$$u_{xx}+2u_{xy}+u_{yy}+u=\frac{1}{(x+y)^{\frac{1}{2}}} \tag{3.14}$$

$$u_{xx}+2u_{xy}+u_{yy}+u=\frac{1}{(x+y)^{\frac{3}{2}}} \tag{3.15}$$

Next, using (3.14) and (3.15), we arrive at the following explicit solutions for the equations (3.14) and (3.15) respectively

$$u(x,y)=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left\{\cos\left(\frac{x+y}{2}\right)\left[S\left(\frac{y-x}{2}\right)-S\left(\frac{x+y}{2}\right)\right]-\right.$$
$$\left.\sin\left(\frac{x+y}{2}\right)\left[C\left(\frac{y-x}{2}\right)-C\left(\frac{x+y}{2}\right)\right]\right\} \text{ if } y>x.$$

$$u(x,y)=\frac{1}{(x+y)^{\frac{1}{2}}}\cos(x+y)-\frac{1}{(y-x)^{\frac{1}{2}}}\cos y-\pi^{\frac{1}{2}}\left\{\cos\left(\frac{x+y}{2}\right)\left[S\left(\frac{y-x}{2}\right)\right.\right.$$
$$\left.\left.-S\left(\frac{x+y}{2}\right)\right]-\sin\left(\frac{x+y}{2}\right)\left[C\left(\frac{y-x}{2}\right)-C\left(\frac{x+y}{2}\right)\right]\right\} \text{ if } y>x.$$

Example 3.2. Solve the following Parabolic differential equation described by

$$u_{xx}+2u_{xy}+u_{yy}+u=\frac{1}{(x+y)^{\frac{1}{2}}}\exp\left(-\frac{xy}{x+y}\right),\ 0<x<\infty,\ 0<y<\infty, \tag{3.16}$$

under the initial and boundary conditions

$$u(x,0)=u(0,y)=u_y(x,0)=u_x(0,y)=u(0,0)=0. \tag{3.17}$$

With the aid of (2.27') in Remark 2.3 and the similar procedure we have followed for Example 3.1, the transformed problems reads

$$U(s_1,s_2)=\frac{\pi\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)}{\left[(s_1+s_2)^2+1\right]\left[1+\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}}, \tag{3.18}$$

where $\mathcal{R}e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0$. Using formula (133) in [24] the inverse transform of (3.18) leads to the following integral representation.

$$u(x,y)=\int_{x+y}^{y-x}t^{-\frac{1}{2}}\exp\left[\frac{t^2-(x+y)^2}{4t}\right]\sin\left(\frac{t-(x+y)}{2}\right)dt \text{ if } y>x.$$

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