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Remarks on the relativistic self-dual Maxwell-Chern-Simons-Higgs system *

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Abstract

In this note we present some recent developments on the topological multi-vortex solutions of the self-dual Maxwell-Chern-Simons-Higgs system in $\mathbb{R}^2$. We find that all the topological solutions are admissible in the sense defined in [2]. We also discover that the convergence of the topological solution to the solutions of the self-dual Chern-Simons equations can be improved to be strong.

1 Introduction

We are concerned with the semilinear elliptic system in $\mathbb{R}^2$:

$$\Delta u = 2q^2(e^u - 1) - 2q\kappa A_0 + 4\pi \sum_{j=1}^{m} \delta(z - z_j) \tag{1}$$

$$\Delta A_0 = \kappa q(1 - e^u) + (\kappa^2 + 2q^2 e^u)A_0, \tag{2}$$

where $q > 0$ is the charge of electron, and $\kappa > 0$ is the Chern-Simons coupling constant. Each point of the prescribed set $\mathbf{Z} = \{z_1, \cdots, z_m\}$ is called a vortex point. We consider two different physically-meaningful sets of boundary conditions related to finiteness of total energy ([5], [2]). One is the topological boundary condition,

$$\lim_{|z|\to\infty} u = 0, \quad \lim_{|z|\to\infty} A_0 = 0, \tag{3}$$

and the other is the nontopological boundary condition,

$$\lim_{|z|\to\infty} u = -\infty, \quad \lim_{|z|\to\infty} A_0 = -\frac{q}{\kappa}, \tag{4}$$

The system (1)-(2) arises from a Jaffe-Taubes[4] reduction of the Bogomolnyi equations of the self-dual Maxwell-Chern-Simons-Higgs system[5]. One major motivation for introducing the system was unification of two apparently separate

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systems, namely, the Abelian Higgs system and the Chern-Simons system. The Abelian Higgs system reduces to

$$\Delta u = 2q^2(e^u - 1) + 4\pi \sum_{j=1}^{m} \delta(z - z_j), \tag{5}$$

while the Chern-Simons system reduces to

$$\Delta u = 4l^2 e^u(e^u - 1) + 4\pi \sum_{j=1}^{m} \delta(z - z_j), \tag{6}$$

where l is a physical parameter related to the Chern-Simons constant. The Abelian Higgs system (5) has finite-energy solutions under only the topological boundary conditions (3), while the Chern-Simons system has finite-energy solutions under either boundary conditions (3) or (4). For the precise meaning of the finite energy of these systems, we refer to [4].

The mathematical analysis of the topological multi-vortex solutions for the Abelian Higgs system and the Chern-Simons system is studied in [4] and [7], [6]. We note that recently existence of non-topological multi-vortex solutions of (6) is established in [1]. The existence of topological multi-vortex solutions of (1)-(2) was established by a simple variational argument in [2]. Moreover, the existence of so-called admissible solutions was established by an iteration scheme in [2]. We recall the definition of admissible topological solution of (1)-(3).

Definition A topological solution pair (u, A_0) of (1)-(2) is called admissible if it satisfies one of the following inequalities.

(i) $A_0 \le 0$

(ii) $u \le 0$

(iii) $A_0 \ge \frac{q}{\kappa}(e^u - 1)$

(iv) $v \le u_a^q$, where u_a^q is the solution the (topological) Abelian Higgs equation (5).

We remark that the conditions (i)-(iv) are shown to be equivalent to each other[2]. In [2] we also considered the two convergence problems of the *admissible* topological solutions of (1)-(2) . One is the Abelian Higgs limit, namely the problem of identifying the behavior of the solution $u^{\kappa,q}$ of (1)-(2) as $\kappa \to 0$ with q kept fixed. The other is the Chern-Simons limit, the similar problem as both κ and q go to infinity with the ratio $l = q^2/\kappa$ kept fixed. In the Abelian Higgs limit we proved in [2] that admissible topological solution of (1)-(2) converges strongly to the solution of the Abelian Higgs solution, while in the Chern-Simons limit we could just show that our solution is weakly consistent to the Chern-Simons equation. For the precise statements of these results see [2]. The natural open questions raised were

1. Is any topological solution of (1)-(2) , which is smooth except at the points $z_1, \cdots, z_m$, admissible?

2. Can we strengthen the sense of convergence in the Chern-Simons limit problem?

Question 1 is concerned with the physical validity of the model system, (1)-(2), since $u \leq 0$ is equivalent to the condition $|\phi|^2 = e^u \leq 1$ for the Higgs field ϕ. Question 2, combined with the already established strong convergence in the Abelian Higgs limit, is concerned with the rigorous verification of the physical argument that the Maxwell-Chern-Simons-Higgs model is a unification of the Abelian Higgs model and the Chern-Simons model[5]. In [3] we answer these two questions in the affirmative. In the next section we state our results and their implications.

For further discussion, we introduce the background function u_0 defined by

$$u_0 = \sum_{j=1}^{m} \ln\left(\frac{|z-z_j|^2}{1+|z-z_j|^2}\right),$$

and we set $u = v + u_0$ to remove the singular inhomogeneous term in (1). Then (1) and (2) become

$$\begin{aligned} \Delta v &= 2q^2(e^{v+u_0} - 1) - 2q\kappa A_0 + g, && (7) \\ \Delta A_0 &= \kappa q(1 - e^{v+u_0}) + (\kappa^2 + 2q^2 e^{v+u_0})A_0 && (8) \end{aligned}$$

with the topological boundary condition

$$\lim_{|z|\to\infty} v = 0, \quad \lim_{|z|\to\infty} A_0 = 0, \tag{9}$$

where

$$g = \sum_{j=1}^{m} \frac{4}{(1+|z-z_j|^2)^2}.$$

In this setting the Chern-Simons equation (6) becomes

$$\Delta v = 4l^2 e^{v+u_0}(e^{v+u_0} - 1) + g. \tag{10}$$

2 Main Results

The following result is proved in [3].

Theorem 1 *Suppose $\mathbf{Z} = \{z_1, \cdots, z_m\} \subset \mathbb{R}^2$ is given as before. Then any topological solution (u, A_0) in $C^2(\mathbb{R}^2 \setminus \mathbf{Z})$ is admissible.*

One immediate consequence of Theorem 1 and the argument of the construction is that the solution constructed in Section 3 is maximal. On the other hand, due to the monotonicity of the minimizing functional $\mathcal{F}$,

$$\begin{aligned}\mathcal{F}(v) \quad = \quad & \int\Bigg[\frac{1}{2}|\Delta v|^2-(\Delta g-\kappa^2 g)v+2q^4(e^{v+u_0}-1)^2\\ & +\frac{1}{2}\kappa^2|\nabla v|^2+2q^2e^{v+u_0}|\nabla(v+u_0)|^2\Bigg]dx,\end{aligned} \tag{11}$$

as established in Section 4 of [2], the solution constructed in Section 2 of [2] by the variational method is minimal. Thus we have, as a result, constructed the maximal and the minimal solutions of the system. As remarked in [3], the strong convergence in the Chern-Simons limit for admissible solutions in the periodic boundary condition can be extended to the case of our solutions of the system (7)-(8). In particular, due to Theorem 1 we can remove the condition of admissibility, and obtain:

Theorem 2 *Let $(v^{\kappa,q}, A_0^{\kappa,q})$ be any topological solution of (7)-(8). Then $v^{\kappa,q}\to v_{cs}^l$, and $\frac{\kappa}{q}A_0^{\kappa,q}\to e^{v_{cs}^l+u_0}-1$, both in $H^1(\mathbb{R}^2)$ as $\kappa\to\infty$ with $\frac{q^2}{\kappa}=l$ kept fixed, where v_{cs}^l denotes a topological solution of (10).*

We now have further open problems to consider for the system (1)- (2):

1. Prove uniqueness, or multiplicity of topological solution.
2. Prove existence of non-topological multi-vortex solutions.

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References

[1] D. Chae and O. Yu. Imanuvilov, *The existence of non- topological multi-vortex solutions in the relativistic self-dual Chern-Simons theory*, IMA Preprint Series **1629**, (1999).

[2] D. Chae and N. Kim, *Topological multi-vortex solutions of the self-dual Maxwell-Chern-Simons-Higgs system* , J. Differential Equations **134** , no. 1, pp. 154–182(1997).

[3] D. Chae and N. Kim, *Vortex condensates in the relativistic self-dual Maxwell-Chern-Simons-Higgs system*, RIM-GARC Preprint Series **97-50**, (1997).

[4] A. Jaffe and C. Taubes, Vortices and Monopoles, Birkhäuser, Boston, (1980).

[5] C. Lee, K. Lee and H. Min, *Self-Dual Maxwell Chern-Simons Solitons*, Phys. Lett. B, **252**, pp. 79-83(1990).

[6] J. Spruck and Y. Yang, *Topological Solutions in the Self-Dual Chern-Simons Theory: Existence and Approximation*, Ann. Inst. Henri Poincaré, **12**, pp. 75-97(1995).

[7] R. Wang, *The Existence of Chern-Simons Vortices*, Comm. Math. Phys., **137**, pp. 587-597(1991).

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