Nonlinear Differential Equations,
Electron. J. Diff. Eqns., Conf. 05, 2000, pp. 51–67.
http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu ejde.math.unt.edu (login: ftp)

EXISTENCE AND PERTURBATION OF PRINCIPAL EIGENVALUES FOR A PERIODIC-PARABOLIC PROBLEM

DANIEL DANERS

Dedicated to Alan Lazer on his 60th birthday

ABSTRACT. We give a necessary and sufficient condition for the existence of a positive principal eigenvalue for a periodic-parabolic problem with indefinite weight function. The condition was originally established by Beltramo and Hess [*Comm. Part. Diff. Eq.*, **9** (1984), 919–941] in the framework of the Schauder theory of classical solutions. In the present paper, the problem is considered in the framework of variational evolution equations on arbitrary bounded domains, assuming that the coefficients of the operator and the weight function are only bounded and measurable. We also establish a general perturbation theorem for the principal eigenvalue, which in particular allows quite singular perturbations of the domain. Motivation for the problem comes from population dynamics taking into account seasonal effects.

1. Introduction

Population models with diffusion taking into account seasonal effects are often described by a periodic-parabolic problem. The habitat of the population is represented by a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3 in a real model), and the diffusion by an elliptic operator, $\mathcal{A}(t)$, having time periodic coefficients of period $T > 0$ (the length of one cycle). The linearization of such a boundary value problem at a periodic solution leads to a periodic-parabolic eigenvalue problem of the form

$$\begin{aligned} \partial_t u + \mathcal{A}(t)u &= \lambda m u && \text{in } \Omega \times [0,T], \\ u(\cdot\,,t) &= 0 && \text{on } \partial\Omega \times [0,T], \\ u(\cdot,0) &= u(\cdot,T) && \text{in } \Omega, \end{aligned} \tag{1.1}$$

with weight function m. It is of particular importance to know the existence of a positive principal eigenvalue of (1.1), which, by definition, is a number λ such that (1.1) has a nontrivial nonnegative solution. The notion of a principal eigenvalue for periodic-parabolic problems was introduced and motivated in Lazer [13] (see also Castro & Lazer [4]). More applications can for instance be found in Hess [11].

Mathematics Subject Classification. 35K20, 35P05, 35B20, 47N20.
Key words. principal eigenvalues, periodic-parabolic problems,
parabolic boundary-value problems, domain perturbation.

Published October 24, 2000.

In this paper we prove two results. First, we establish a necessary and sufficient condition on the weight function m which guarantees the existence of a positive principal eigenvalue of (1.1). Second, we provide a general perturbation result for the eigenvalues of (1.1) allowing quite singular perturbations of the domain Ω. All results will be proved in the framework of weak solutions. This requires the principal part of $\mathcal{A}(t)$ to be in divergence from, but allows us to deal with arbitrary domains Ω, and only requires the coefficients of $\mathcal{A}(t)$ and the weight function m to be bounded and measurable. Note that, as a special case, the results apply to weighted elliptic eigenvalue problems (c.f. [11, Remark 16.5]).

Working in the framework of the Schauder theory of classical solutions Beltramo & Hess [3] (see also [2, 11]) found necessary and sufficient conditions for the existence of a positive principal eigenvalue. It was somewhat a surprise that, unlike in case of the corresponding elliptic problem, it is not sufficient that m be positive somewhere in Ω. The relevant condition turned out to be

$$\mathcal{P}(m) := \frac{1}{T}\int_0^T \sup_{x\in\Omega} m(x,t)\,dt > 0.$$

We will show that a similar result holds under our assumptions. As the weight function m is only assumed to be bounded and measurable, we will need to replace the supremum by the essential supremum. The problem was also considered in Daners [7], where, in addition to the hypotheses in the present paper, it was assumed that m is lower semi-continuous. Godoy, Lami Dozo & Paczka [9] were able to deal with bounded and measurable weight functions m. However they kept the smoothness assumptions on the coefficients of $\mathcal{A}(t)$ and the domain made in the original theorem by Beltramo and Hess. They moreover required the top order coefficients of $\mathcal{A}(t)$ to be continuously differentiable. The reason was that in the proof they needed to rewrite $\mathcal{A}(t)$ in divergence form. We find it more natural to assume from the beginning that the operator be in divergence form, and then to get rid of the smoothness assumptions all together.

We then prove two perturbation results. The first asserts that any finite set of eigenvalues of (1.1) is upper semi-continuous with respect to the domain, the coefficients of $\mathcal{A}(t)$, and the weight function m. The second determines the behaviour of the principal eigenvalue of a sequence of approximating problems. It turns out that the limit exists, and is the smallest positive principal eigenvalue. The perturbation theorems improve and complement similar results in Daners [7]. We relax the conditions on the domain convergence, and not necessarily assume that the limiting set Ω be connected.

An outline of the paper is as follows. In Section 2 we give the precise assumptions and state our main results. In Section 3 we discuss the main steps of the proof of the existence result. In Section 4 we prove our general perturbation results. The techniques introduced there also give rise to an approximation procedure, which allows to pass from results known in the smooth case to the non-smooth case. This procedure is described and exploited in Section 5. In Section 6 we prove some spectral estimates providing the key to establish the existence of a positive principal eigenvalue. We close the paper by two appendices, the first outlining the changes necessary in [6] to relax the notion of domain convergence, and the second to prove a technical result about convex functions.

2. Assumptions and Main Results

Throughout let $\Omega \subset \mathbb{R}^N$ be a bounded open set, and let T be a fixed positive number. Moreover suppose that $\mathcal{A}(t)$ satisfies the following assumptions.

Assumption 2.1. Suppose that $\mathcal{A}(t)$ is defined by

$$\mathcal{A}(t)u := -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\sum_{j=1}^{N} a_{ij}(\cdot,t)\frac{\partial}{\partial x_j}u\Big) + \sum_{i=1}^{N} b_i(\cdot,t)\frac{\partial}{\partial x_i}u + c_0(\cdot,t)u, \tag{2.1}$$

where $a_{ij} = a_{ji}, b_i, c_0 \in L_\infty(\Omega \times (0,T))$. Moreover we assume that there exists $\alpha > 0$, called the ellipticity constant, such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geq \alpha|\xi|^2 \tag{2.2}$$

for all $(x,t) \in \Omega \times (0,T)$ and $\xi \in \mathbb{R}^N$.

By a solution of (1.1) we always mean a weak solution (for a definition see e.g. [16]). It is well known that weak solutions are classical solutions if the domain, the coefficients of $\mathcal{A}(t)$ and the weight function are smooth enough. The set of $\lambda \in \mathbb{C}$ such that

$$\partial_t u + \mathcal{A}(t)u - \lambda m u = f$$

in $\Omega \times [0,T]$ subject to the boundary conditions in (1.1) has a bounded inverse on $L_2(\Omega \times (0,T))$ is called the *resolvent set* of (1.1). The complement of the resolvent set is called the spectrum of (1.1). We call λ a [*principal*] *eigenvalue* of (1.1) if (1.1) has a nontrivial [nonnegative] solution. Such a nontrivial solution is said to be a [*principal*] *eigenfunction* of (1.1) to the [*principal*] *eigenvalue* λ.

Existence of a positive principal eigenvalue. We next state our main result on the existence of a positive principal eigenvalue of (1.1). We define

$$\mathcal{P}(m) := \frac{1}{T}\int_0^T \operatorname*{ess-sup}_{x\in\Omega} m(x,t)\,dt > 0. \tag{2.3}$$

If Ω is a bounded domain (an open and connected set) we have the following theorem. The assertions are wrong in general if Ω is not connected (c.f. Remark 2.13).

Theorem 2.2. *Suppose that Ω is a bounded domain, that $\mathcal{A}(t)$ is as above with $c_0 \geq 0$, and that $m \in L_\infty(\Omega \times (0,T))$. Then the following assertions are equivalent:*

1. $\mathcal{P}(m) > 0$.
2. *Problem* (1.1) *has a positive principal eigenvalue.*
3. *Problem* (1.1) *has an eigenvalue with positive real part.*

In this case the positive principal eigenvalue, λ_1, is the only principal eigenvalue with positive real part, and

$$\lambda_1 = \inf\{\operatorname{Re}\lambda \colon \lambda \text{ is an eigenvalue of (1.1) with } \operatorname{Re}\lambda > 0\}. \tag{2.4}$$

Remark 2.3. Note that the above theorem can also be used to give necessary and sufficient conditions for the existence of a negative principal eigenvalue of (1.1). We only need to replace m by $-m$.

Perturbation of the spectrum. To state our perturbation results we need some additional definitions and assumptions. We first look at domain convergence (c.f. [5, Section 2]).

Definition 2.4. Suppose that Ω is a bounded open set (not necessarily connected), and that Ω_n are bounded domains (connected by definition). We say that Ω_n converges to Ω, in symbols $\Omega_n \to \Omega$, if

(i) $\lim_{n\to\infty} \operatorname{meas}(\Omega_n \cap \bar{\Omega}^{\complement}) = 0$.
(ii) There exists a compact set $K \subset \Omega$ of capacity zero such that for each compact set $\Omega' \subset \Omega \setminus K$ there exists $n_0 \in \mathbb{N}$ such that $\Omega' \subset \Omega_n$ for all $n \geq n_0$.

Remark 2.5. In the above definition we did not assume that Ω_n stays in the same bounded subset of $\mathbb{R}^N$. In fact the diameter of Ω_n may tend to infinity as long as the measure of $\Omega_n \cap \Omega^{\complement}$ converges to zero.

As usual we denote by $W_2^1(\Omega)$ the standard Sobolev space, and by $\mathring{W}_2^1(\Omega)$ the closure of the set of all smooth functions with compact support in $W_2^1(\Omega)$.

Definition 2.6. An open set $\Omega \subset \mathbb{R}^N$ is said to be *stable* if for each $u \in W_2^1(\mathbb{R}^N)$ with support in $\bar{\Omega}$ we have that $u \in \mathring{W}_2^1(\Omega)$.

The stability of an open set is a very weak regularity condition. It can be characterized by means of capacities (see e.g. Adams & Hedberg [1, Theorem 11.4.1]). Next we state our assumptions on the perturbed operators $\mathcal{A}_n(t)$. As we assume that the coefficients are only bounded and measurable we can always extend them to $\mathbb{R}^N$ in such a way that the ellipticity constant remains unchanged.

Assumption 2.7. For all $n \in \mathbb{N}$ let $\mathcal{A}_n(t)$ be an operator of the form (2.1) with coefficients $a_{ij}^{(n)} = a_{ji}^{(n)}, b_i^{(n)}, c_0^{(n)} \in L_\infty(\mathbb{R}^N \times (0,T))$. Suppose that

$$\sup_{\substack{i,j=1,\dots,N\\ n\in\mathbb{N}}} \bigl\{\|a_{ij}^{(n)}\|_\infty, \|b_i^{(n)}\|_\infty, \|c_0^{(n)}\|_\infty\bigr\} < \infty,$$

and that $a_{ij}^{(n)}, b_i^{(n)}$ and $c_0^{(n)}$ converge to the corresponding coefficients of $\mathcal{A}(t)$ in $L_{2,\mathrm{loc}}(\mathbb{R}^N \times (0,T))$. Finally suppose that the sequence of ellipticity constants of $\mathcal{A}_n(t)$ has a positive lower bound.

We finally consider the weight functions. Note that we can assume them to be defined on $\mathbb{R}^N$ by simply extending them by zero outside Ω.

Assumption 2.8. Let $m_n, m \in L_\infty(\mathbb{R}^N \times (0,T))$ for all $n \in \mathbb{N}$, assume that $\|m_n\|_\infty$ is a bounded sequence, and that m_n converges to m in $L_{2,\mathrm{loc}}(\mathbb{R}^N \times (0,T))$.

We are now in a position to state our main perturbation results. What we mean by the multiplicity of an eigenvalue of (1.1) we explain in Definition 4.2.

Theorem 2.9. *Suppose that $c_0 \geq 0$, and that Assumption 2.7 and 2.8 are satisfied. Further assume that $\Omega \subset \mathbb{R}^N$ is a stable bounded open set, and that $\Omega_n \to \Omega$ in the sense of Definition 2.4. Finally let $U \subset \mathbb{C}$ be an open set containing exactly r eigenvalues of* (1.1). *Then, counting multiplicity, the perturbed problem*

$$\begin{aligned} \partial_t u + \mathcal{A}_n(t)u &= \lambda m_n u && \text{in } \Omega_n \times [0,T] \\ u(\cdot\,,t) &= 0 && \text{on } \partial\Omega_n \times [0,T] \\ u(\cdot,0) &= u(\cdot,T) && \text{in } \Omega_n \end{aligned} \tag{2.5}$$

has exactly r eigenvalues in U for $n \in \mathbb{N}$ sufficiently large.

The proof of the above theorem is given in Section 4 (it follows from Proposition 4.3 and Theorem 4.4). The next theorem determines what happens to a sequence of positive principal eigenvalues if we pass to the limit. The main problem is that Ω is not assumed to be connected, and thus the limiting problem might have more than one positive principal eigenvalue.

Theorem 2.10. *Suppose the assumptions of Theorem 2.9 hold, and that* (1.1) *admits a positive principal eigenvalue. Then for all $n \in \mathbb{N}$ large enough* (2.5) *has a unique positive principal eigenvalue λ_n. The sequence (λ_n) converges to a positive principal eigenvalue of* (1.1)*, and this eigenvalue can be characterized by* (2.4).

The above is a consequence of Theorem 2.9 and some spectral estimates. The proof is given in Lemma 4.5 and 4.6.

Remark 2.11. If $\Omega_n \subset \Omega$ for all $n \in \mathbb{N}$ then the above results remains true without assuming that Ω is stable (c.f. [6, Remark 3.2(a)]).

Remark 2.12. If Ω is not connected the spectrum of (1.1) is the union of the spectra of the corresponding problems on the components of Ω. Hence, the limiting problem may have several principal eigenvalues, or one with higher algebraic multiplicity.

Remark 2.13. If Ω is not connected it is possible for (1.1) not to have a positive principal eigenvalue even though $\mathcal{P}(m) > 0$. As an example look at a domain with two connected components, Ω_1 and Ω_2. Then the spectrum of (1.1) is the union of the spectra of the corresponding problems on Ω_1 and Ω_2. If we set $m_i := m|_{\Omega_i}$ $(i = 1, 2)$, then one can easily arrange that $\mathcal{P}(m_i) \leq 0$ for $i = 1, 2$, but $\mathcal{P}(m) > 0$. The reason is that the location where the essential supremum of m occurs may shift from Ω_1 to Ω_2 as t increases from 0 to T. Suppose that we are in this situation, and that Ω_n are domains approximating Ω in the sense of Definition 2.4 (for instance connect Ω_1 and Ω_2 by a small strip shrinking to a line). If m_n is the weight function on Ω_n then $\mathcal{P}(m_n) > 0$ for large $n \in \mathbb{N}$, and, by Theorem 2.2, there exists a positive principal eigenvalue, λ_n, for the perturbed domain. However, as the limiting problem does not have a principal eigenvalue, and 0 is not an eigenvalue, λ_n must converge to infinity as n goes to infinity. In fact, the upper bound of λ_n established in Lemma 4.6 also goes to infinity. The reason is that the curve γ and the function φ_0 used there cannot be chosen the same for all $n \in \mathbb{N}$.

Remark 2.14. In Theorem 2.10 it can be shown that, if normalized to one in the space $L_2(\Omega \times (0, T))$, at least a subsequence of the eigenfunctions converges to an eigenfunction of the limiting problem in $L_2(\Omega \times (0, T))$ (see proof of Lemma 4.5). However, if λ_0 is of higher multiplicity we cannot expect the whole sequence to converge. For the convergence of eigenfunctions see also Daners [7, Theorem 3.2].

Remark 2.15. Note that, as a special case, our perturbation results can be applied to weighted elliptic boundary value problems of the from

$$\begin{aligned} \mathcal{A}u &= \lambda m u \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

(c.f. [11, Remark 16.5]). Domain perturbations of weighted elliptic eigenvalue problems were also considered in López-Gómez [15].

3. Main Steps of the Existence Proof

In this section we outline the main steps of the proof of Theorem 2.2. The basic idea, which was already exploited by Beltramo & Hess [3], is to look at the family of auxiliary eigenvalue problems

$$\begin{aligned} \partial_t u + \mathcal{A}(t)u - \lambda m u &= \mu u && \text{in } \Omega \times [0,T], \\ u(\cdot\,,t) &= 0 && \text{on } \partial\Omega \times [0,T], \\ u(\cdot,0) &= u(\cdot,T) && \text{in } \Omega, \end{aligned} \tag{3.1}$$

where the parameter λ ranges over $\mathbb{R}$. We throughout assume that m is a bounded and measurable function on $\Omega \times [0,T]$. Concerning the existence of a principal eigenvalue for (3.1) the following is known (see [7, Section 2]).

Lemma 3.1. *For each $\lambda \in \mathbb{R}$ the eigenvalue problem (3.1) has a unique principal eigenvalue. This eigenvalue is real, algebraically simple, and the corresponding eigenfunction can be chosen to be continuous and positive in $\Omega \times [0,T]$.*

The continuity of the eigenfunction follows from the regularity theory for weak solutions of parabolic equations, the positivity follows from the periodicity and the weak Harnack inequality for parabolic equations (e.g. [16]).

For every $\lambda \in \mathbb{R}$ denote the principal eigenvalue of (3.1) by $\mu(\lambda)$. Note that λ is a principal eigenvalue of (1.1) if and only if $\mu(\lambda) = 0$. Hence, to prove Theorem 2.2 we need criteria ensuring that $\mu(\cdot)$ has a unique positive zero. The properties of $\mu(\cdot)$ leading to this conclusion are summarized in the following proposition.

Proposition 3.2. *The function $\mu(\cdot)$ has the following properties:*

1. *$\mu(\cdot)\colon \mathbb{R} \to \mathbb{R}$ is concave.*
2. *If $c_0 \geq 0$ then $\mu(0) > 0$.*
3. *$\lim_{\lambda\to\infty} \mu(\lambda) = -\infty$ if and only if $\mathcal{P}(m) > 0$.*
4. *If $\lambda \in \mathbb{C}$ is an eigenvalue of (1.1) with $\operatorname{Re}\lambda > 0$ then $\mu(\operatorname{Re}\lambda) \leq 0$.*

The above proposition can be used as follows to prove Theorem 2.2.

Proof of Theorem 2.2. Assuming that $c_0 \geq 0$ we have from (2) that $\mu(0) > 0$. By (1) the function $\mu(\cdot)$ is concave and hence continuous. Thus, by (3), the first two assertions of Theorem 2.2 are equivalent. Next, due to (4) and (3), the first assertion of Theorem 2.2 is equivalent to the third one. Finally, the uniqueness of a positive principal eigenvalue of (1.1) follows from the concavity of $\mu(\cdot)$. The characterization (2.4) is a consequence of (4). This completes the proof of Theorem 2.2. □

It remains to prove Proposition 3.2. The first two properties, (1) and (2), are established in [7], the first as part of the proof of Theorem 2.1 on p. 391, and the second in Lemma 2.4. The proof of (4) will be given in Lemma 5.5 using the result in the smooth case and an approximation procedure. It remains to prove (3). The necessity of the condition $\mathcal{P}(m) > 0$ clearly follows from the lower estimate

$$\mu(\lambda) \geq \mu(0) - \lambda\mathcal{P}(m) \tag{3.2}$$

valid for all $\lambda \geq 0$. A proof is given in Lemma 5.4. The most difficult part is to show that $\mathcal{P}(m) > 0$ implies that

$$\lim_{\lambda\to\infty} \mu(\lambda) = -\infty \tag{3.3}$$

The proof of the above assertion is quite technical and requires an upper estimate for $\mu(\lambda)$. To state the estimate in a concise form we define

$$A := [a_{ij}]_{1\le i,j\le N} \quad \text{and} \quad b := [b_1,\dots,b_N]^{\mathsf{T}}. \tag{3.4}$$

Note that due to the ellipticity condition (2.2) the matrix $A(x,t)$ is invertible for almost all $(x,t)\in\Omega\times(0,T)$. Let $\mathcal{D}(\Omega)$ denote the set of smooth functions with compact support in Ω. Finally, denote the support of a function u by $\operatorname{supp} u$. The following result is an obvious consequence of Proposition 6.3.

Proposition 3.3. *Suppose that $\gamma\in C^1(\mathbb{R},\mathbb{R}^N)$ is T-periodic. Further assume that $\varphi_0\in\mathcal{D}(\Omega)$ is a nonnegative function such that*

$$T\int_\Omega \varphi_0^2\,dx = 1, \tag{3.5}$$

and suppose that $\varphi(x,t) := \varphi_0(x-\gamma(t))\in\Omega\times\mathbb{R}$ for all $(x,t)\in\operatorname{supp}(\varphi_0)\times\mathbb{R}$. Furthermore, let $w := \varphi(b - d\gamma/dt) + 2A(\nabla\varphi)^{\mathsf{T}}$. Then, for all $\lambda\in\mathbb{R}$

$$\mu(\lambda)\le \frac{1}{4}\int_0^T\int_\Omega w^{\mathsf{T}}A^{-1}w + \varphi^2 c_0\,dx\,dt - \lambda\int_0^T\int_\Omega \varphi^2 m\,dx\,dt. \tag{3.6}$$

Our claim (3.3) follows from Proposition 3.3 if γ and φ_0 can be chosen such that

$$\int_0^T\int_\Omega [\varphi_0(x-\gamma(t))]^2 m(x,t)\,dx\,dt > 0. \tag{3.7}$$

The idea is that $\mathcal{P}(m)>0$ implies that the integral of m over a tubular neighbourhood about a periodic curve is positive. Godoy, et. al [9, Lemma 4.4] showed that there exists a T-periodic curve $\gamma\in C^1(\mathbb{R},\mathbb{R}^N)$ and an open set $\Omega_0\subset\Omega$ with the property that $x-\gamma(t)\in\Omega$ for all $(x,t)\in\bar{\Omega}_0\times[0,T]$, and

$$\int_0^T\int_{\Omega_0} m(x-\gamma(t),t)\,dx\,dt > 0.$$

Choosing an appropriate function $\varphi_0\in\mathcal{D}(\Omega_0)$ normalized by (3.5) we easily get the following lemma.

Lemma 3.4. *If $\mathcal{P}(m)>0$, then in Proposition 3.3 the curve γ and the function φ_0 can be chosen such that* (3.7) *holds.*

The above lemma together with (3.6) shows that $\mathcal{P}(m)>0$ implies (3.3) and thus completes the proof of Proposition 3.2.

4. Perturbation Results

The main purpose of this section is to prove Theorem 2.9 and 2.10. We start by studying the periodic-parabolic problem

$$\begin{aligned} \frac{\partial}{\partial t}u + \mathcal{A}(t)u + \mu u &= f && \text{in } \Omega\times(0,T),\\ u &= 0 && \text{on } \partial\Omega\times(0,T),\\ u(\cdot,0) &= u(\cdot,T) && \text{in } \Omega. \end{aligned} \tag{4.1}$$

It can be shown that, for each $\mu\in\mathbb{R}$ large enough, the above problem has unique weak solution

$$u\in L_2((0,T),\mathring{W}_2^1(\Omega))\cap C([0,T],L_2(\Omega))$$

for all $f \in L_2((0,T), W_2^{-1}(\Omega))$ (see [6, Theorem 2.2] or [14, Theorem 3.6.1]). Define the resolvent operator R_μ by $R_\mu f := u$ for all $f \in L_2((0,T), W_2^{-1}(\Omega))$. Then for all $p \geq 2$

$$R_\mu \in \mathcal{L}\big(L_p(\Omega \times (0,T)\big) \cap \mathcal{L}\big(C([0,T], L_2(\Omega))\big)$$

is a compact operator (see [6, Section 5]). Suppose now that $p > N/2$, and that $f \in L_p((0,T)\Omega))$ is a nontrivial nonnegative function. If u is the corresponding solution of (4.1) with $f \in L_p((0,T) \times \Omega))$ then the weak Harnack inequality, the regularity theory for parabolic equations (see e.g. [16]) and periodicity show that $u \in C(\Omega \times [0,T])$, and $u(x,t) > 0$ for all $(x,t) \in \Omega \times [0,T]$. We next look at the perturbed periodic-parabolic problem

$$\begin{aligned} \frac{\partial}{\partial t} u + \mathcal{A}_n(t)u + \mu u = f_n \qquad & \text{in } \Omega_n \times (0,T), \\ u = 0 \qquad & \text{on } \partial\Omega_n \times (0,T), \\ u(\cdot,0) = u(\cdot,T) \qquad & \text{in } \Omega_n. \end{aligned} \tag{4.2}$$

We suppose that $\mathcal{A}_n(t)$ satisfies Assumption 2.7, and that $\Omega_n \to \Omega$ in the sense of Definition 2.4. Further denote by $R_{\mu,n}$ the resolvent operator of (4.2).

Theorem 4.1. *Suppose that the above assumptions are satisfied, and that $\mu \in \mathbb{R}$ is large enough. Then for all $p \leq 2 < \infty$ the resolvent $R_{\mu,n}$ converges to R_μ in $\mathcal{L}\big(L_p(\Omega \times (0,T)\big)$. Moreover, if $f_n \rightharpoonup f$ weakly in $L_p(\mathbb{R}^N \times (0,T))$, then the solutions of* (4.2) *converge to the solution of* (4.1) *strongly in $L_p(\mathbb{R}^N \times (0,T))$.*

Proof. For a slightly weaker notion of domain convergence the above theorem was proved in [6, Theorem 5.1]. Note that all results in that paper only depend on [6, Theorem 3.1], so we only need to generalize this theorem for our definition of domain convergence. The necessary modifications of the proof are given in Appendix A. □

Suppose now that M is the multiplication operator induced by $m \in L_\infty(\Omega \times (0,T))$ on $L_2(\Omega \times (0,T))$. If $c_0 \geq 0$ we know from Proposition 3.2(2) that $R := R_0$ exists. Hence, taking into account the compactness of R, the operator $R \circ M$ is compact on $L_2(\Omega \times (0,T))$. It easily follows that $\lambda \in \mathbb{C}$ is in the spectrum of (1.1) if and only if λ^{-1} is in the spectrum of $R \circ M$. By the spectral theory for compact operators (e.g. [12, Theorem III.6.26]) all eigenvalues are of finite algebraic multiplicity.

Definition 4.2. By the multiplicity of an eigenvalue of (1.1) we mean the multiplicity of λ^{-1} as an eigenvalue of $R \circ M$.

The above reasoning leads to the following proposition.

Proposition 4.3. *The spectrum of* (1.1) *consists of eigenvalues of finite algebraic multiplicity. Moreover, $\lambda \in \mathbb{C}$ is an eigenvalue of* (1.1) *if and only if λ^{-1} is an eigenvalue of $R \circ M$.*

We next look at perturbations of $R \circ M$. We we set $R_n := R_{0,n}$, and denote the multiplication operator induced by m_n by M_n. The following theorem is a reformulation and extension of Theorem 2.9.

Theorem 4.4. *Suppose that $c_0 \geq 0$, that $\mathcal{A}_n(t)$ and m_n satisfy Assumption 2.7 and 2.8, respectively, and that $\Omega_n \to \Omega$ in the sense of Definition 2.4. Then for all $p \leq 2 < \infty$ the operator $R_n \circ M_n$ converges to $R \circ M$ in $\mathcal{L}\big(L_p(\Omega \times (0,T)\big)$. If $U \subset \mathbb{C}$ is an open set containing exactly r eigenvalues of $R \circ M$ then, counting multiplicity, U contains exactly r eigenvalues of $R_n \circ M_n$ for all n sufficiently large.*

Proof. The first assertion of the theorem is a simple consequence of Theorem 4.1 applying similar arguments as in [6, Theorem 5.1]. The second assertion follows from the first by applying a general perturbation theorem (Kato [12, Section IV.3.5]). ☐

The remainder of this section is devoted to the proof of Theorem 2.10. We first show that the limit of a sequence of principal eigenvalues is a principal eigenvalue. The main difficulty in the proof is that Ω is not assumed to be connected.

Lemma 4.5. *For each $n \in \mathbb{N}$ let λ_n be a principal eigenvalue of* (2.5)*, and assume that the sequence (λ_n) converges to some $\lambda_1 \in \mathbb{R}$. Then λ_1 is a principal eigenvalue of* (1.1)*. If $\lambda_1 > 0$ then it can be characterized by* (2.4).

Proof. Let u_n denote an eigenfunction to the principal eigenvalue λ_n of (2.5), and assume that λ_n converges to λ_1 as n goes to infinity. We can assume that $u_n > 0$ in $\Omega \times (0,T)$, and normalize it in $L_2(\Omega \times (0,T))$ to norm one. Then, $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in a Hilbert space, and therefore has a weakly convergent subsequence $(u_{n_k})_{k\in\mathbb{N}}$ with limit u (e.g. [17, Section V.2]). By our hypotheses on m_n (see Assumption 2.8) it follows that $\lambda_{n_k} m_{n_k} u_{n_k}$ converges to $\lambda_1 m u$ weakly in $L_2(\Omega \times (0,T))$. But then [6, Theorem 5.1] and the results in Appendix A imply that u_{n_k} converges to u strongly in $L_2(\mathbb{R}^N \times (0,T))$. Hence u is nontrivial and nonnegative, proving that λ_1 is a principal eigenvalue of (1.1).

It remains to show that, if $\lambda_1 > 0$, then (1.1) has no eigenvalue with positive real part smaller than λ_1. Suppose, to the contrary, that (1.1) has an eigenvalue $\nu \in \mathbb{C}$ with $0 < \operatorname{Re}\nu < \lambda_1$. Then, by Theorem 4.4, it follows that (2.5) has an eigenvalue $\mu_n \in \mathbb{C}$ with $0 < \operatorname{Re}\nu_n < \lambda_n$ for all $n \in \mathbb{N}$ large enough. As Ω_n is connected λ_n can be characterized by (2.4), leading to a contradiction. Hence, λ_1 is also given by (2.4). ☐

Theorem 2.10 follows from the above lemma if we can show the existence and convergence of a positive principal eigenvalue of the perturbed problem (2.5).

Lemma 4.6. *Suppose the assumptions of Theorem 2.10 hold. Then, for n sufficiently large, the perturbed eigenvalue problem* (2.5) *has a unique positive principal eigenvalue converging to a positive principal eigenvalue of* (1.1).

Proof. We start by proving the existence of a positive principal eigenvalue of the perturbed eigenvalue problem (2.5). To do so we consider the family of auxiliary eigenvalue problems

$$\begin{aligned} \partial_t u + \mathcal{A}_n(t)u - \lambda m_n u &= \mu u && \text{in } \Omega_n \times [0,T], \\ u(\cdot\,,t) &= 0 && \text{on } \partial\Omega_n \times [0,T], \\ u(\cdot,0) &= u(\cdot,T) && \text{in } \Omega_n. \end{aligned} \tag{4.3}$$

By Lemma 3.1 the above eigenvalue problem has a unique principal eigenvalue, $\mu_n(\lambda)$, for all $\lambda \in \mathbb{R}$. We show that $\mu_n(0)$ is a bounded sequence. To do so fix a function $\varphi \in \mathcal{D}(\Omega \setminus K)$, where K is the set from Definition 2.4 of domain convergence. By Definition 2.4(ii) the support of φ is contained in Ω_n if n is sufficiently large. Applying Proposition 3.3 we therefore have that

$$\mu_n(\lambda) \leq \frac{1}{4}\int_0^T \int_{\Omega_n} w_n^\mathsf{T} A_n^{-1} w_n + \varphi^2 c_0^{(n)}\, dx\, dt$$

for all $n \in \mathbb{N}$ sufficiently large. Here w_n and A_n are the expressions corresponding to w and A for the perturbed problem. By our assumptions it is easy to see that the right hand converges. Thus the sequence $\mu_n(\lambda)$ is bounded from above. It is bounded from below as $\mu_n(0) > 0$ for all $n \in \mathbb{N}$ by Proposition 3.2(2). Hence, there exists a subsequence $\big(\mu_{n_k}(0)\big)_{k\in\mathbb{N}}$ converging to $\mu_0 \geq 0$ as k goes to infinity. By Lemma 4.5 the limit μ_0 is a principal eigenvalue of (3.1) with $\lambda = 0$. Next we note that zero cannot be an eigenvalue of (1.1) as otherwise it would be an eigenvalue of (1.1) on a component of Ω. Since we assumed that $c_0 \geq 0$ this is not possible by Proposition 3.2(2). Hence $\mu_0 > 0$. By Lemma 4.5 the eigenvalue μ_0 is characterized as the one with the smallest positive real part. Hence, every convergent subsequence of $\big(\mu_n(0)\big)_{n\in\mathbb{N}}$ tends to μ_0 and thus the whole sequence converges. As $\mu_0 > 0$ we also have that $\mu_n(0) > 0$ for $n \in \mathbb{N}$ large enough.

Next we show that, for n sufficiently large, (2.5) has a positive principal eigenvalue. We assumed in Theorem 2.10 that (1.1) has a positive principal eigenvalue. Note that Ω is not necessarily connected, so the spectrum of (1.1) is the union of the spectra of the corresponding problems on the components. Hence we can select a connected component $\Omega_1 \subset \Omega$ such that (1.1) has a positive principal eigenvalue on Ω_1. By Theorem 2.2 it follows that $\mathcal{P}(m|_{\Omega_1}) > 0$. Due to Lemma 3.4 there exists a T-periodic curve $\gamma \in C^1(\mathbb{R},\mathbb{R}^N)$, a function $\varphi_0 \in \mathcal{D}(\Omega_1)$ satisfying (3.5) such that (3.7) holds. Setting $\varphi(x,t) := \varphi_0(x-\gamma(t))$ we see that

$$\lim_{n\to\infty}\int_0^T\int_{\Omega_n}\varphi^2 m_n\,dx\,dt = \int_0^T\int_{\Omega_n}\varphi^2 m_n\,dx\,dt.$$

As the right hand side of the above equation is positive there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_0^T\int_{\Omega_n}\varphi^2 m_n\,dx\,dt > \delta \tag{4.4}$$

for all $n \geq n_0$. Next observe that by Proposition 3.3

$$\mu_n(\lambda) \leq \frac{1}{4}\int_0^T\int_{\Omega_1} w_n^{\mathsf{T}}A_n^{-1}w_n + \varphi^2 c_0^{(n)}\,dx\,dt - \lambda\int_0^T\int_{\Omega_n}\varphi^2 m_n\,dx\,dt \tag{4.5}$$

for all $n \in \mathbb{N}$ and all $\lambda \in \mathbb{R}$. For each $n \geq n_0$ we thus have $\mu_n(\lambda) < 0$ if only λ is large enough. We showed already that $\mu_n(0) > 0$, so by Proposition 3.2(1) the function $\mu_n(\cdot)$ has a unique positive zero, λ_n, whenever $n \in \mathbb{N}$ is large enough. This proves the existence of a unique positive principal eigenvalue for (2.5) if n is large.

It remains to show that λ_n converges to a principal eigenvalue of (1.1). To do so we first establish a bound on λ_n. From (4.5) and (4.4) we conclude that

$$\lambda_n \leq \frac{1}{4\delta}\int_0^T\int_{\Omega_n} w_n^{\mathsf{T}}A_n^{-1}w_n + \varphi^2 c_0^{(n)}\,dx\,dt$$

for all $n \in \mathbb{N}$ large enough. It is easy to see that the right hand side of the above inequality converges. Hence, the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is bounded from above. On the other hand we know already that $\lambda_n > 0$ for all $n \in \mathbb{N}$. Thus the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is bounded. We next show that it converges. Due to the boundedness we can extract a subsequence converging to some $\lambda_1 \geq 0$. By Lemma 4.5 λ_1 is a principal eigenvalue of (1.1). We already showed that zero is no principal eigenvalue, so $\lambda_1 > 0$. Moreover, Lemma 4.5 asserts that λ_1 is the eigenvalue of (1.1) with the smallest positive real part. Hence all convergent subsequences of (λ_n) tend to λ_1, and thus the whole sequence converges. This completes the proof of the lemma. ☐

5. Approximation Procedures

We now want to introduce an approximation procedure which allows to pass from results known for smooth data to the case of non-smooth data. The idea is to regularize $\mathcal{A}(t)$, m and Ω. We start with $\mathcal{A}(t)$, and assume that it satisfies Assumption 2.1. In a first step we extend the coefficients of $\mathcal{A}(t)$ from $\Omega \times (0,T)$ periodically to $\Omega \times \mathbb{R}$, and then extend its first and zero order coefficients b_i and c_0 by zero outside $\Omega \times \mathbb{R}$. Next we extend a_{ij} by $\alpha\delta_{ij}$ to $\mathbb{R}^{N+1}$, where δ_{ij} is the Kronecker symbol and $\alpha > 0$ the ellipticity constant of $\mathcal{A}(t)$. In abuse of notation we denote this new operator again by $\mathcal{A}(t)$. It has the same ellipticity constant as the original one. We then fix nonnegative functions $\varphi \in \mathcal{D}(\mathbb{R}^N)$ and $\psi \in \mathcal{D}(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}^N} \varphi(x)\,dx = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \psi(t)\,dt = 1. \tag{5.1}$$

For all $n \in \mathbb{N}$ define φ_n and ψ_n by $\varphi_n(x) := n^N\varphi(nx)$ and $\psi_n(t) := n\varphi(nt)$, respectively. Then $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ are mollifiers on $\mathbb{R}^N$ and $\mathbb{R}$, respectively. Clearly $\Phi_n(x,t) := \varphi_n(x)\psi_n(t)$ defines a mollifier on $\mathbb{R}^{N+1}$. For all $n \in \mathbb{N}$ and $i, j = 1, \dots, N$ we set

$$a_{ij}^{(n)} := \Phi_n * a_{ij}, \quad b_i^{(n)} := \Phi_n * b_i \quad \text{and} \quad c_0^{(n)} := \Phi_n * c_0,$$

and define $\mathcal{A}_n(t)$ to be the operator of the form (2.1) with these coefficients. Using the definition of the convolution and the properties of the mollifiers (e.g. [8, Section 8.2]) it is straightforward to check that $\mathcal{A}_n(t)$ satisfies Assumption 2.7.

We next look at the weight function $m \in L_\infty(\Omega \times (0,T))$. We first extend it periodically to $\Omega\times\mathbb{R}$, and then by $-\|m\|_\infty$ outside $\Omega\times\mathbb{R}$. Then the approximations m_n defined by $m_n := \Phi_n * m$ clearly satisfy Assumption 2.8.

To approximate the bounded domain Ω let Ω_n be a sequence of sub-domains of class C^∞ exhausting Ω. Then $\Omega_n \to \Omega$ in the sense of Definition 2.4. Note that in this case we do not need to assume that Ω is stable in order to apply the results from Section 4 (c.f. Remark 2.11). Finally define

$$\mathcal{P}(m_n) := \frac{1}{T}\int_0^T \operatorname*{ess-sup}_{x\in\Omega_n} m(x,t)\,dt.$$

We then have the following lemma.

Lemma 5.1. *Under the above assumptions $\mathcal{P}(m_n)$ converges, and*

$$\lim_{n\to\infty} \mathcal{P}(m_n) \le \mathcal{P}(m). \tag{5.2}$$

Proof. By the definition of m_n we have that

$$\operatorname*{ess-sup}_{y\in\Omega_n} m_n(y,t) \le \int_{\mathbb{R}^N} \varphi_n(x-z) \int_{-\infty}^{\infty} \psi_n(t-s) \operatorname*{ess-sup}_{y\in\Omega} m(y,s)\,ds\,dz.$$

As we extended m by $-\|m\|_\infty$ outside $\Omega \times \mathbb{R}$ the essential supremum on the right hand side is the same as the essential supremum over $\mathbb{R}^N$. Taking into account (5.1) the above inequality reduces to

$$\operatorname*{ess-sup}_{y\in\Omega_n} m_n(y,t) \le \psi_n * \operatorname*{ess-sup}_{y\in\Omega} m(y\,,\cdot)(t)$$

for all $t \in [0,T]$. As $(\psi_n)_{n\in\mathbb{N}}$ is a mollifier the right hand side of the above inequality converges to $\operatorname{ess-sup}_{y\in\Omega} m(y\,,\cdot)$ almost everywhere in $(0,T)$. As all functions involved are bounded uniformly with respect to $n \in \mathbb{N}$, an application of the dominated convergence theorem yields (5.2). □

Remark 5.2. It can also be shown that $P(m) \leq \liminf_{n\to\infty} P(m_n)$. The proof is based on the trivial inequality $m_n(x,t) \leq \text{ess-sup}_{x\in\Omega_n} m_n(x,t)$ and Fatou's lemma. The above inequality is true for every sequence m_n approaching m pointwise almost everywhere, whereas (5.2) requires more properties of m_n.

For every $\lambda \in \mathbb{R}$ let $\mu_n(\lambda)$ and $\mu(\lambda)$ denote the unique principal eigenvalues of (4.3) and (3.1), respectively.

Proposition 5.3. *Under the above assumptions $\mu_n(\lambda)$ converges to $\mu(\lambda)$ uniformly with respect to λ in bounded sets of $\mathbb{R}$ as n goes to infinity.*

Proof. By Lemma 3.1 the eigenvalues $\mu_n(\lambda)$ and $\mu(\lambda)$ are algebraically simple. Hence Theorem 4.4 with $m = 1$ implies that $\mu_n(\lambda)$ converges to $\mu(\lambda)$ for all $\lambda \in \mathbb{R}$. By Proposition 3.2(1) the functions $\mu_n \colon \mathbb{R} \to \mathbb{R}$ are concave, and thus by the results in Appendix B local uniform convergence on $\mathbb{R}$ follows. □

Using the approximation procedure just introduced we next establish the lower estimate (3.2) for the principal eigenvalue of (1.1).

Lemma 5.4. *For all $\lambda \geq 0$ the inequality* (3.2) *holds.*

Proof. As before let $\mu_n(\lambda)$ denote the principal eigenvalue of (4.3). As all data are smooth we can apply [11, Lemma 15.6], which asserts that

$$\mu_n(\lambda) \geq \mu_n(0) - \lambda\mathcal{P}(m_n)$$

for all $\lambda \geq 0$. (Note that we used a slightly different definition of $\mathcal{P}(m_n)$.) Hence, an application of Proposition 5.3 and Lemma 5.1 shows that

$$\mu(\lambda) \geq \mu(0) - \lambda \lim_{n\to\infty} \mathcal{P}(m_n) \geq \mu(0) - \lambda\mathcal{P}(m),$$

proving the assertion of the lemma. □

Finally, we apply the approximation procedure to get Proposition 3.2(4).

Lemma 5.5. *If $\lambda \in \mathbb{C}$ is an eigenvalue of* (1.1) *with* $\operatorname{Re}\lambda > 0$, *then* $\mu(\operatorname{Re}\lambda) \leq 0$.

Proof. Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of (1.1) with $\operatorname{Re}\lambda > 0$. Then, by Theorem 4.4, there exists a sequence (λ_n) of eigenvalues to the perturbed eigenvalue problems (2.5) converging to λ as n tends to infinity. (The sequence (λ_n) is not necessarily unique.) Hence, $\operatorname{Re}\lambda_n > 0$ for large $n \in \mathbb{N}$. As all data are smooth, we can apply [3, Lemma 3.6] to conclude that $\mu_n(\operatorname{Re}\lambda_n) \leq 0$ for all $n \in \mathbb{N}$ sufficiently large. By Proposition 5.3 μ_n converges to μ locally uniformly, and thus $0 \geq \lim_{n\to\infty} \mu_n(\operatorname{Re}\lambda_n) = \mu(\operatorname{Re}\lambda)$. This concludes the proof of the lemma. □

6. Upper Estimates for the Principal Eigenvalue

In this section we provide an upper bound for the principal eigenvalue of

$$\begin{aligned} \partial_t u + \mathcal{A}(t)u &= \mu u && \text{in } \Omega \times [0,T] \\ u(\cdot\,,t) &= 0 && \text{on } \partial\Omega \times [0,T] \\ u(\cdot,0) &= u(\cdot,T) && \text{in } \Omega \end{aligned} \tag{6.1}$$

which leads to Proposition 3.3. Throughout we suppose that Assumption 2.1 holds, and that Ω is a bounded domain. Moreover, we define A and b as in (3.4). Then we can rewrite $\mathcal{A}(t)u$ by

$$\mathcal{A}(t)u = -\operatorname{div}\bigl((\nabla u)A\bigr) + (\nabla u)b + c_0 u.$$

For $k \in \mathbb{N} \cup \{\infty\}$ we define

$$C_T^k(\bar\Omega \times \mathbb{R}) := \bigl\{u \in C^k(\bar\Omega \times \mathbb{R})\colon u(x,t+T) = u(x,t) \text{ for all } (x,t) \in \bar\Omega \times \mathbb{R}\bigr\}.$$

The following lemma is a variation of Hess [10, Proposition 3.1]. The main difference is that in our case $\mathcal{A}(t)$ is in divergence form. Our aim is to give a version for arbitrary bounded domains and operators $\mathcal{A}(t)$ with bounded and measurable coefficients. To achieve this we first look at the corresponding problem in the smooth case and then pass to the general case by the approximation procedure established in Section 5 and the perturbation results in Section 4.

Lemma 6.1. *Suppose that Ω is of class C^∞, and that the coefficients of $\mathcal{A}(t)$ are in $C_T^\infty(\Omega \times \mathbb{R})$. Moreover, let $\varphi \in \mathcal{D}(\Omega)$ be nonnegative such that*

$$T \int_\Omega \varphi^2 \, dx = 1. \tag{6.2}$$

Finally define $w \in C_T^\infty(\bar\Omega \times \mathbb{R}, \mathbb{R}^N)$ by $w := \varphi b + 2A(\nabla\varphi)^\mathsf{T}$. Then the principal eigenvalue, μ, of (6.1) satisfies the estimate

$$\mu \le \frac{1}{4} \int_0^T \int_\Omega w^\mathsf{T} A^{-1} w + \varphi^2 c_0 \, dx\, dt. \tag{6.3}$$

Proof. Let $u \in C_T^\infty(\bar\Omega \times \mathbb{R})$ denote an eigenfunction of (6.1) to the principal eigenvalue μ. We can choose u such that $u(x,t) > 0$ for all $(x,t) \in \Omega \times [0,T]$. By this choice of u the function $\psi \in C_T^\infty(\Omega \times \mathbb{R})$, given by $\psi(x,t) := -\log u(x,t)$ for all $(x,t) \in \Omega \times [0,T]$, is well defined. As u is an eigenfunction of (6.1) we get that

$$-\frac{\partial}{\partial t}\psi = \frac{1}{u}\frac{\partial}{\partial t}u = \mu - \frac{1}{u}\mathcal{A}(t)u,$$

and thus by definition of ψ and $\mathcal{A}(t)$

$$\begin{aligned}\frac{1}{u}\mathcal{A}(t)u &= -\frac{1}{u}\operatorname{div}\bigl((\nabla u)A\bigr) + \frac{1}{u}(\nabla u)b + c_0\\ &= -\operatorname{div}\Bigl(\frac{1}{u}(\nabla u)A\Bigr) - \frac{1}{u^2}(\nabla u)A(\nabla u)^\mathsf{T} - (\nabla\psi)b + c_0\\ &= \operatorname{div}\bigl((\nabla\psi)A\bigr) - (\nabla\psi)A(\nabla\psi)^\mathsf{T} - (\nabla\psi)b + c_0.\end{aligned}$$

Combining the above two identities we see that

$$\mu = -\frac{\partial}{\partial t}\psi + \operatorname{div}\bigl((\nabla\psi)A\bigr) - (\nabla\psi)A(\nabla\psi)^\mathsf{T} - (\nabla\psi)b + c_0.$$

Next we multiply the above equation by φ^2 and integrate over $\Omega \times (0,T)$. We can do this because φ has compact support in Ω, and u is bounded away from zero on the support of φ. Taking into account our assumption (6.2) we get that

$$\begin{aligned}\mu = -\int_0^T \int_\Omega \varphi^2 \frac{\partial}{\partial t}\psi \, dx\, dt &+ \int_0^T \int_\Omega \varphi^2 \operatorname{div}\bigl((\nabla\psi)A\bigr)\, dx\, dt\\ &- \int_0^T \int_\Omega \varphi^2(\nabla\psi)A(\nabla\psi)^\mathsf{T} + \varphi^2(\nabla\psi)b - \varphi^2 c_0 \, dx\, dt.\end{aligned}$$

As ψ is T-periodic in $t \in \mathbb{R}$ and φ is independent of t the first integral on the right hand side of the above identity is zero. The second integral can be rewritten as

$$\int_0^T \int_\Omega \varphi^2 \operatorname{div}\bigl((\nabla\psi)A\bigr)\,dx\,dt = \int_0^T \int_\Omega \operatorname{div}\bigl(\varphi^2(\nabla\psi)A\bigr)\,dx\,dt - \int_0^T \int_\Omega 2\varphi(\nabla\psi)A(\nabla\varphi)^\mathsf{T}\,dx\,dt.$$

As φ has compact support an application of the divergence theorem shows that the first integral on the right hand side of the above equation is zero, and thus

$$\mu = -\int_0^T \int_\Omega (\nabla\psi)A(\nabla\psi)^\mathsf{T} + 2\varphi(\nabla\psi)A(\nabla\varphi)^\mathsf{T} + (\nabla\psi)b\,dx\,dt + \int_0^T \int_\Omega c_0\,dx\,dt. \tag{6.4}$$

We next estimate the first of the above integrals by a quantity independent of ψ. To do so first note that by the ellipticity condition (2.2) the matrix A is invertible, and hence $v := \varphi(\nabla\psi)^\mathsf{T} + \frac{1}{2}A^{-1}w$ is well defined. Recalling that $w = \varphi b + 2A(\nabla\varphi)^\mathsf{T}$ and that A is symmetric, an elementary calculation shows that

$$v^\mathsf{T}Av = \varphi^2(\nabla\psi)A(\nabla\psi)^\mathsf{T} + \tfrac{1}{4}w^\mathsf{T}A^{-1}w + \varphi^2(\nabla\psi)b + 2(\nabla\varphi)A(\nabla\psi)^\mathsf{T}. \tag{6.5}$$

Clearly $v^\mathsf{T}Av \geq 0$ by the ellipticity assumption (2.2). If we add $\int_0^T\int_\Omega v^\mathsf{T}Av\,dx\,dt$ to the right hand side of (6.4) and take into account (6.5) we immediately arrive at (6.3), concluding the proof of the lemma. □

In the calculations in the above proof it was quite essential that φ does not depend on $x \in \Omega$. This can be relaxed a little bit by looking at a transformed problem.

Lemma 6.2. *Suppose that $\gamma \in C^1(\mathbb{R}, \mathbb{R}^N)$ is T-periodic. Further assume that $\varphi_0 \in \mathcal{D}(\Omega)$ is a nonnegative function satisfying* (3.7)*. Also assume that $\varphi(x,t) := \varphi_0(x - \gamma(t)) \in \Omega \times \mathbb{R}$ for all $(x,t) \in \operatorname{supp}(\varphi_0) \times \mathbb{R}$, and set*

$$w := \varphi(b - d\gamma/dt) + 2A(\nabla\varphi)^\mathsf{T}.$$

Then the principal eigenvalue, μ, of (6.1) *satisfies the estimate* (6.3)*.*

Proof. Define the diffeomorphism $\theta \in C^1(\mathbb{R}^{N+1}, \mathbb{R}^{N+1})$ by $\theta(x,t) := (x - \gamma(t), t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Then the inverse of θ is given by $\theta^{-1}(y,t) = (y + \gamma(t), t)$. Next set $Q_T := \theta(\Omega \times (0,T))$, and define

$$\mathcal{A}_\gamma(t)v := -\operatorname{div}(\nabla v(A \circ \theta^{-1}) + \nabla v(b \circ \theta^{-1} - \dot\gamma) + (c_0 \circ \theta^{-1})v.$$

Suppose now that u is a positive principal eigenfunction to the principal eigenvalue μ of (6.1). Then, using that u is an eigenfunction of (6.1), a simple calculation shows that the function $v := u \circ \theta^{-1}$ satisfies the equation

$$\frac{\partial}{\partial t}v + \mathcal{A}_\gamma v = \mu v$$

in Q_T. By our assumptions we have that $\operatorname{supp}(\varphi_0) \times (0,T) \subset Q_T$. Therefore we can apply Lemma 6.1 to conclude that

$$\mu \leq \int_0^T \int_{\operatorname{supp}(\varphi_0)} (w \circ \theta^{-1})^\mathsf{T}(A^{-1} \circ \theta^{-1})(w \circ \theta^{-1}) + c_0 \circ \theta^{-1}\varphi_0\,dy\,dt.$$

(Note that in the proof of Lemma 6.1 we the did not use the boundary conditions, but only the fact that u is positive in $\Omega \times [0,T]$.) As $\det D\theta = 1$ we can apply the transformation formula for integrals and the definition of φ to get (6.3). □

Next we get rid of the smoothness assumptions on the domain and the coefficients of $\mathcal{A}(t)$. The idea is to use the approximation procedure from Section 5, and then the perturbation results in Section 4.

Proposition 6.3. *Suppose that the assumptions of Lemma 6.2 are satisfied, but that $\Omega \subset \mathbb{R}^N$ is an arbitrary bounded domain, and that the coefficients of $\mathcal{A}(t)$ are only bounded and measurable (Assumption 2.1). Then the assertions of Lemma 6.2 remain true.*

Proof. Suppose that $\mathcal{A}_n(t)$ and Ω_n are as constructed in Section 5. If we define A_n and w_n accordingly we see from Lemma 6.2 that the principal eigenvalue, μ_n, of

$$\begin{aligned} \partial_t u + \mathcal{A}_n(t)u &= \mu u && \text{in } \Omega_n \times [0,T] \\ u(\,\cdot\,,t) &= 0 && \text{on } \partial\Omega_n \times [0,T] \\ u(\cdot,0) &= u(\cdot,T) && \text{in } \Omega_n \end{aligned}$$

satisfies the estimate

$$\mu_n \leq \frac{1}{4}\int_0^T \int_\Omega w_n^\mathsf{T} A_n^{-1} w_n + \varphi^2 c_0^{(n)}\,dx\,dt. \tag{6.6}$$

for all $n \in \mathbb{N}$. As the inversion of a matrix is a smooth operation, and the ellipticity constant of $\mathcal{A}_n$ is uniformly bounded from below we have that $w_n^\mathsf{T} A_n^{-1} w_n$ converges to $w^\mathsf{T} A^{-1} w$ in $L_1(\Omega)$. Applying Proposition 5.3 the estimate (6.3) follows from (6.6) by letting n go to infinity. This completes the proof of the proposition. □

Appendix A. Perturbations of the Initial Value Problem

The purpose of this appendix is to show that the results in [6] hold under our more general notion of domain convergence given in Definition 2.4. The only place we need the explicit notion of domain convergence is in the proof of Theorem 3.1, all subsequent results only use the assertions of that theorem. If these assertions are true for our new notion of domain convergence then all other results from [6] are valid. We consider perturbations of the initial boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t}u + \mathcal{A}(t)u &= f && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{A.1}$$

We next state [6, Theorem 3.1], and then provide the necessary changes in its proof assuming the domains converge in the more general sense given in Definition 2.4.

Theorem A.1. *Suppose that Ω is a bounded open and stable set, and that Ω_n is a sequence of domains with $\Omega_n \to \Omega$ in the sense of Definition 2.4. Moreover, assume that $p > 2N(N+2)^{-1}$, and that $u_{0n} \in L_2(\Omega_n)$ and $f_n \in L_2\big((0,T), L_p(\Omega_n)\big)$ are such that $u_{0n} \rightharpoonup u_0$ weakly in $L_2(\mathbb{R}^N)$ and $f_n \rightharpoonup f$ weakly in $L_2\big((0,T), L_p(\mathbb{R}^N)\big)$. Finally, suppose that u_n is the weak solution of*

$$\begin{aligned} \frac{\partial}{\partial t}u + \mathcal{A}_n(t)u &= f_n && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega_n \times (0,T), \\ u(\cdot,0) &= u_{0n} && \text{in } \Omega_n. \end{aligned} \tag{A.2}$$

Then u_n converges to u strongly in $L_2\big((0,T), L_q(\mathbb{R}^N)\big)$ for all $q \in [1, 2N(N-2)^{-1})$, and weakly in $L_2\big((0,T), W_2^1(\mathbb{R}^N)\big)$. Moreover, u is a weak solution of (A.1).

Proof. It follows in exactly the same way as in the proof of [6, Theorem 3.1] that u_n is bounded in $L_2\big((0,T),W_2^1(\mathbb{R}^N)\big)$, and that it converges to a function u weakly in that space. Recall that we did not assume that Ω_n stays in a common bounded set for all $n\in\mathbb{N}$ (c.f. Remark 2.5). Hence we cannot directly apply [6, Lemma 2.1] to conclude that the convergence of u_n takes place strongly in $L_2\big((0,T),L_q(\mathbb{R}^N)\big)$ for all $q\in[1,2N(N-2)^{-1})$. However, an obvious modification of the proof of that lemma shows that for every bounded subset $B\subset\mathbb{R}^N$ the sequence (u_n) converges to u in $L_2\big((0,T),L_q(B)\big)$ for all $q\in[1,2N(N-2)^{-1})$. As u_n is bounded in $L_2\big((0,T),W_2^1(\mathbb{R}^N)\big)$ it follows from the Sobolev embedding theorem that u_n is bounded in $L_2\big((0,T),L_r(\mathbb{R}^N)\big)$ for all $q\in[1,2N(N-2)^{-1})$. Fix now q,r such that $1\leq q<r<2N(N-2)^{-1}$. Then, by Hölder's inequality we have that

$$\begin{aligned}\|u_n\|_{L_2((0,T),L_q(\mathbb{R}^N\setminus\bar\Omega^{\complement}))} &= \Big(\int_0^T\Big(\int_{\Omega_n\cap\bar\Omega^{\complement}}|u_n(x,t)|^q\,dx\Big)^{\frac{2}{q}}\,dt\Big)^{\frac12}\\ &\leq \big(\operatorname{meas}(\Omega_n\cap\bar\Omega^{\complement})\big)^{\frac1q-\frac1r}\|u_n\|_{L_2((0,T),L_q(\mathbb{R}^N\setminus\bar\Omega^{\complement}))}.\end{aligned}$$

We already saw that the sequence (u_n) is bounded in $L_2((0,T),L_q(\mathbb{R}^N))$. By assumption (see Definition 2.4) $\operatorname{meas}(\Omega_n\cap\Omega^{\complement})$ converges to zero. This shows that $u_n|_{\mathbb{R}^N\setminus\Omega^{\complement}}$ converges to zero in $L_2\big((0,T),L_q(\mathbb{R}^N\setminus\Omega^{\complement})\big)$ for all $q\in[1,2N(N-2)^{-1})$. In particular $u=0$ almost everywhere in $\mathbb{R}^N\setminus\bar\Omega^{\complement}$. This implies that $\operatorname{supp}(u(t))\subset\bar\Omega$ for almost all $t\in(0,T)$. Hence by the stability of the domain (Definition 2.6) it follows that $u(t)\in\mathring{W}_2^1(\Omega)$ for almost all $t\in(0,T)$. Finally note that we already proved that u_n converges to u in $L_2\big((0,T),L_q(B)\big)$ for all $q\in[1,2N(N-2)^{-1})$ and all bounded sets B. Hence, the assertion of the theorem follows. □

Appendix B. Local Uniform Convergence of Convex Functions

Suppose that $f_n\colon\mathbb{R}\to\mathbb{R}$ are convex functions converging pointwise to a function f. Then, clearly f is convex. We want to show that f_n converges locally uniformly to f. The idea is to show that the family (f_n) is bounded and equi-continuous, and then apply the Arzelá-Ascoli theorem.

Proposition B.1. *Let $f_n\colon\mathbb{R}\to\mathbb{R}$ be convex functions converging pointwise to the real valued function f. Then f is convex, and f_n converges to f uniformly on every compact subset of $\mathbb{R}$.*

Proof. It is easy to see that f is convex, so we only prove local uniform convergence. We first show that the family (f_n) is bounded on any compact interval $[a,b]\subset\mathbb{R}$. From the convexity it is clear that $f_n(x)\leq\max\{f_n(a),f_n(b)\}$ for all $x\in[a,b]$. As f_n converges pointwise there exists $M_0>0$ such that $\max\{f_n(a),f_n(b)\}\leq M_0$ for all $n\in\mathbb{N}$. This proves the existence of a uniform upper bound. We now establish a uniform lower bound. Setting $x_0:=(b-a)/2$, the convexity of f_n implies that $2f_n(x_0)\leq f_n(x_0+z)+f_n(x_0-z)$ for all $z\in\mathbb{R}$. Using the upper bound already established we therefore get

$$\inf_{x\in[a,b]}f_n(x)\geq 2f_n(x_0)-\sup_{x\in[a,b]}f_n(x)\geq 2f_n(x_0)-M_0$$

for all $n\in\mathbb{N}$. As $f_n(x_0)$ is bounded this yields a uniform lower bound. Hence, the family (f_n) is bounded on $[a,b]$.

Next we prove the equi-continuity of the family (f_n). Let $I := [\alpha, \beta]$ be a compact interval. Fix $\delta > 0$ and let $x, y \in I$ with $x < y$. By the convexity of f_n

$$\frac{f_n(x) - f_n(\alpha - \delta)}{x - \alpha + \delta} \leq \frac{f_n(y) - f_n(x)}{y - x} \leq \frac{f_n(b + \delta) - f_n(y)}{\beta + \delta - y}$$

for all $n \in \mathbb{N}$. By what we proved already the family (f_n) is bounded in the interval $[\alpha - \delta, \beta + \delta]$ by some $M > 0$. Therefore

$$-2M\delta^{-1} \leq \frac{f_n(y) - f_n(x)}{y - x} \leq 2M\delta^{-1}$$

for all $n \in \mathbb{N}$ and all $x, y \in I$ with $x < y$. Setting $L := 2M\delta^{-1}$ we conclude that $|f_n(x) - f_n(y)| \leq L|x - y|$ for all $n \in \mathbb{N}$ and $x, y \in I$. Hence, the family (f_n) is bounded and equi-continuous, and by the Arzelà-Ascoli theorem (see [17, Section III.3]) it is relatively compact in $C(I)$. Since, by assumption, it converges pointwise, it therefore converges in $C(I)$, i.e. uniformly on I. □

References

[1] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory*, Springer, Berlin, 1996.

[2] A. Beltramo, *Über den Haupteigenwert von periodisch-parabolischen Differentialoperatoren*, Ph.D. thesis, Universität Zürich, 1984.

[3] A. Beltramo and P. Hess, *On the principal eigenvalue of a periodic-parabolic operator*, Comm. Partial Differential Equations **9** (1984), 919–941.

[4] A. Castro and A. C. Lazer, *Results on periodic solutions of parabolic equations suggested by elliptic theory*, Boll. Un. Mat. Ital. B (6) **1** (1982), 1089–1104.

[5] E. N. Dancer, *Some remarks on classical problems and fine properties of Sobolev spaces*, Differential and Integral Equations **9** (1996), 437–446.

[6] D. Daners, *Domain Perturbation for Linear and Nonlinear Parabolic Equations*, J. Differential Equations **129** (1996), 358–402.

[7] ———, *Periodic-parabolic eigenvalue problems with indefinite weight functions*, Archiv der Mathematik **68** (1997), 388–397.

[8] G. B. Folland, *Real Analysis: Modern techniques and their applications*, John Wiley & Sons Inc., New York, 1984.

[9] T. Godoy, E. Lami Dozo, and S. Paczka, *The periodic parabolic eigenvalue problem with L^∞ weight*, Math. Scand. **81** (1997), 20–34.

[10] P. Hess, *On positive solutions of semilinear periodic-parabolic problems*, Infinite-Dimensional Systems (F. Kappel and W. Schappacher, eds.), Lecture Notes in Mathematics, vol. 1076, Springer-Verlag, Berlin, 1994, pp. 101–122.

[11] ———, *Periodic-parabolic Boundary Value Problems and Positivity*, Pitman Research Notes in Mathematics Series, vol. 247, Longman Scientific & Technical, Harlow, Essex, 1991.

[12] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Springer, Berlin, 1976.

[13] A. C. Lazer, *Some remarks on periodic solutions of parabolic differential equations*, Dynamical Systems. II (Proceedings of the Second International Symposium held at the University of Florida, Gainesville, Fla., February 25–27, 1981) (A. R. Bednarek and L. Cesari, eds.), Academic Press, 1982, pp. 227–246.

[14] J. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Springer, Berlin, 1972.

[15] J. López-Gómez, *The maximum principle and the existence of principal eigenvalues for some linear weighted boundary value problems*, J. Differential Equations **127** (1996), 263–294.

[16] N. Trudinger, *Pointwise estimates and quasilinear parabolic equations*, Comm. Pure Appl. Math. **21** (1968), 206–226.

[17] K. Yosida, *Functional Analysis*, 6th ed., Springer, Berlin, 1980.

Daniel Daners

Department of Mathematics, Brigham Young University, Provo, Utah 84602, USA

Current address: School of Mathematics and Statistics, Univ. of Sydney, NSW 2006, Australia

E-mail address: D.Daners@maths.usyd.edu.au