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Center manifold and exponentially-bounded solutions of a forced Newtonian system with dissipation *

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Abstract

We study the existence of exponentially-bounded solutions to the following system of second-order ordinary differential equations with dissipation:

$$u'' + cu' + Au + kH(u) = P(t), \quad u \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

where c and k are positive constants, H is a globally Lipschitz function, and P is a bounded and continuous function. A is a $n \times n$ symmetric matrix whose first eigenvalue is equal to zero and the others are positive. Under these conditions, we prove that for some values of c, and k there exist a continuous manifold such that solutions starting in this manifold are exponentially bounded. Our results are applied to the spatial discretization of well-known second-order partial differential equations with Neumann boundary conditions.

1 Introduction

In this note, we study the existence of exponentially bounded solutions of the following system of second-order ordinary differential equations with a damping force and dissipation in $\mathbb{R}^n$:

$$u'' + cu' + Au + kH(u) = P(t), \quad u \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{1.1}$$

where c and k are positive constants, $P \in C_b(\mathbb{R};\mathbb{R}^n)$ (the space of continuous and bounded functions) and $H : \mathbb{R}^n \to \mathbb{R}^n$ is a globally Lipschitz function. i.e., there exists a constant $L > 0$ such that

$$\|H(U_1) - H(U_2)\| \leq L\|U_1 - U_2\|, \quad U_1, U_2 \in \mathbb{R}^n. \tag{1.2}$$

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A is a $n \times n$ symmetric matrix whose first eigenvalue λ_1 is equal to zero and the other eigenvalues are positive

$$0 = \lambda_1 < \lambda_2 < \cdots \lambda_l$$

each one with multiplicity γ_j equal to the dimension of the corresponding eigenspace.

The equation (1.1) has been studied in [5] for the case that the first eigenvalue λ_1 of the matrix A is positive ($\lambda_1 > 0$); under these conditions they prove that for some values of c the equation (1.1) has a bounded solution which is exponentially stable and, if $P(t)$ is almost periodic, this bounded solution is also almost periodic.

The fact that the first eigenvalue λ_1 of the matrix A is equal to zero, in general, does not allow us to prove the existence of bounded solutions of (1.1). However, we prove that for some values of c and k there exist a positive number η depending on c and a continuous manifold $\mathcal{M} = \mathcal{M}(c, k, P(\cdot))$ such that any solution of the system (1.1) starting in $\mathcal{M}$ is exponentially bounded. i.e.,

$$\sup_{t \in \mathbb{R}} e^{-\eta|t|} \left\{ \|u'(t)\|^2 + \|u(t)\|^2 \right\}^{1/2} < \infty.$$

Our method is similar to the one used in [5], we just rewrite the equation (1.1) as a first order system of ordinary differential equations and prove that the linear part of this system has an exponential trichotomy with trivial unstable space. Next, we use the variation constants formula and some ideas from [7] [8] to find a formula for the exponentially bounded solutions of (1.1). From this formula we can prove the existence of such manifold $\mathcal{M} = \mathcal{M}(c, k, P(\cdot))$. These results are applied to the spatial discretization of very well known second order partial differential equations with Neumann boundary conditions:

Example 1.1 The Sine-Gordon equation with Neumann boundary conditions is given by

$$\begin{gathered} U_{tt} + cU_t - dU_{xx} + k \sin U = p(t,x), \quad 0 < x < L, \quad t \in \mathbb{R}, \qquad (1.3) \\ U_x(t,0) = U_x(t,L) = 0, \quad t \in \mathbb{R}, \end{gathered}$$

where c, d and k are positive constants, $p : \mathbb{R} \times [0, L] \to \mathbb{R}$ is continuous and bounded.

The following paragraph was taken from Temam's book([6], pg. 184). With Neumann boundary the Sine-Gordon Equation is physically interesting, the average value of the function U is not expected to remain bounded and actually leads to nontrivial dynamics. From mathematical point of view this case is also interesting.

For each $N \in \mathbb{N}$ the spatial discretization of this equation is given by

$$\begin{gathered} u_i'' + cu_i' + d\delta^{-2}(2u_i - u_{i+1} - u_{i-1}) + k \sin u_i = p_i(t), \quad 1 \le i \le N, \quad t \in \mathbb{R}, \\ u_0 = u_1, \quad u_N = u_{N+1} = 0\,. \qquad (1.4) \end{gathered}$$

This equation can be written in the form of (1.1) with

$$A = d\delta^{-2}\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \tag{1.5}$$

where $\delta = L/(N+1)$. The eigenvalues of this matrix are simple, the first one being zero and the others positive.

Example 1.2 A telegraph equation with Neumman boundary conditions

$$\begin{gathered} U_{tt} + cU_t - dU_{xx} + \arctan U = p(t,x), \quad 0 < x < L, \quad t \in \mathbb{R}, \qquad (1.6) \\ U_x(t,0) = U_x(t,L) = 0, \quad t \in \mathbb{R}, \end{gathered}$$

where c, d are positive constants, $p : \mathbb{R} \times [0,L] \to \mathbb{R}$ is continuous and bounded.

For each $N \in \mathbb{N}$ the spatial discretization of this equation is given by

$$\begin{gathered} u_i'' + cu_i' + d\delta^{-2}(2u_i - u_{i+1} - u_{i-1}) + \arctan u_i = p_i(t) \qquad (1.7) \\ u_0 = u_1, \quad u_N = u_{N+1} = 0\,. \end{gathered}$$

This equation can be written in the form of (1.1) with the the same matrix A as in (1.5).

2 Preliminaries

Most of the ideas presented in this section can be found in [5]. So, we shall prove only the new results. Equation (1.1) can be written as a first order system of ordinary differential equations in the space $W = \mathbb{R}^n \times \mathbb{R}^n$ as follows:

$$w' + \mathcal{A}w + k\mathcal{H}(w) = \mathcal{P}(t), \quad w \in W, \quad t \in \mathbb{R}, \tag{2.1}$$

where $v = u'$ and

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{H}(w) = \begin{pmatrix} 0 \\ H(u) \end{pmatrix}, \quad \mathcal{P}(t) = \begin{pmatrix} 0 \\ P(t) \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} 0 & -I \\ A & cI \end{pmatrix}. \tag{2.2}$$

Now, we are ready to study the linear part of (2.1):

$$w' + \mathcal{A}w = 0, \quad w \in W, \quad t \in \mathbb{R}. \tag{2.3}$$

For the rest of this article, we shall assume that each eigenvalue of the matrix A has multiplicity γ_j equal to the dimension of the corresponding eigenspace and the first one is equal to zero and the others are positive. Therefore, if $0 = \lambda_1 < \lambda_2 < \cdots \lambda_l$ are the eigenvalues of A, we have the following:

a) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvector of A in $\mathbb{R}^n$

b) For all $x \in \mathbb{R}^n$ we have

$$Ax = \sum_{j=1}^{l} \lambda_j \sum_{k=1}^{\gamma_j} \langle x, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{l} \lambda_j E_j x, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$ and

$$E_j x = \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k}. \tag{2.5}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in R^n and $x = \sum_{j=1}^{l} E_j x$, $x \in \mathbb{R}^n$

c) The exponential matrix e^{-At} is given by

$$e^{-At} = \sum_{j=1}^{l} e^{-\lambda_j t} E_j \tag{2.6}$$

Theorem 2.1 *Suppose that $c \neq 2\sqrt{\lambda_j}$, $j = 1, 2, \ldots, l$. Then the exponential matrix $e^{-\mathcal{A}t}$ of the matrix $-\mathcal{A}$ given by (2.2) can be written as follow*

$$e^{-\mathcal{A}t} w = \sum_{j=1}^{l} \left\{ e^{\rho_1(j)t} Q_1(j) w + e^{\rho_2(j)t} Q_2(j) w \right\}, \quad w \in W, \quad t \in \mathbb{R}, \tag{2.7}$$

where

$$\rho(j) = \frac{-c \underline{+} \sqrt{c^2 - 4\lambda_j}}{2}, \quad j = 1, 2, \ldots, l \tag{2.8}$$

and $\{Q_i(j) : i = 1, 2\}_{j=1}^{l}$ is a complete orthogonal system of projections in W.

Corollary 2.2 *The spectrum $\sigma(-\mathcal{A})$ of the matrix $-\mathcal{A}$ is given by*

$$\sigma(-\mathcal{A}) = \{ \frac{-c \underline{+} \sqrt{c^2 - 4\lambda_j}}{2}, \quad j = 1, 2, \ldots, l \}.$$

Corollary 2.3 *Under the hypothesis of Theorem 2.1, there exist two orthogonal projectors $\pi_0, \pi_s : W \to W$ and a constant $M > 0$ such that*

$$\|e^{-\mathcal{A}t} \pi_0\| \leq M, \quad t \in \mathbb{R},$$
$$\|e^{-\mathcal{A}t} \pi_s\| \leq e^{-\beta t}, \quad t \geq 0,$$
$$I_W = \pi_0 + \pi_s, \quad W = W_0 \oplus W_s \,,$$

where $W_0 = \mathrm{Ran}(\pi_0)$, $W_s = \mathrm{Ran}(\pi_s)$, and $\beta = \beta(c) > 0$ is given by

$$-\beta = \max \big\{ -c, \quad \mathrm{Re}(\rho_j) = \mathrm{Re} \big(\frac{-c \pm \sqrt{c^2 - 4\lambda_j}}{2} \big) : j = 2, \ldots, l.\ i = 1, 2. \big\}.$$

Proof Since Theorem 2.1 and Corollary 2.2 are proved in [5], we shall only prove Corollary 2.3. From Corollary 2.2, we have that $\mathrm{Re}(\rho_i(j) < 0, \quad j = 2, \dots, l$. Therefore $\beta > 0$. Since $\lambda_1 = 0$, the formula (2.7) can be written as

$$e^{-\mathcal{A}t}w = Q_1(1)w + e^{-ct}Q_2(1)w + \sum_{j=2}^{l}\left\{e^{\rho_1(j)t}Q_1(j)w + e^{\rho_2(j)t}Q_2(j)w\right\},$$

with $w \in W$ and $t \in \mathbb{R}$. If we put $\pi_0 = Q_1(1)$ and

$$\pi_s = I - \pi_0 = Q_2(1) + \sum_{j=2}^{l}\left\{Q_1(j) + Q_2(j)\right\},$$

we obtain the required projections, and

$$\|e^{-\mathcal{A}t}\pi_0 w\| = \|\pi_0 e^{-\mathcal{A}t}w\| = \|Q_1(1)w\| \le \|Q_1(1)\|\|w\|, \quad t \in \mathbb{R}.$$

Therefore, $\|e^{-\mathcal{A}t}\pi_0\| \le \|Q_1(1)\| = M$. In the same way we get

$$\begin{aligned}
&\|e^{-\mathcal{A}t}\pi_s w\|^2 \\
&= \|\pi_s e^{-\mathcal{A}t}w\|^2 = \|e^{-ct}Q_2(1)w + \sum_{j=2}^{l}\left\{e^{\rho_1(j)t}Q_1(j)w + e^{\rho_2(j)t}Q_2(j)w\right\}\|^2 \\
&= e^{-2ct}\|Q_2(1)w\|^2 + \sum_{j=2}^{l}\left\{e^{2\,\mathrm{Re}\,\rho_1(j)t}\|Q_1(j)w\|^2 + e^{2\,\mathrm{Re}\,\rho_2(j)t}\|Q_2(j)w\|^2\right\} \\
&\le e^{-2\beta t}\left\{\|Q_2(1)w\|^2 + \sum_{j=2}^{l}\|Q_1(j)w\|^2 + \|Q_2(j)w\|^2\right\} \\
&\le e^{-2\beta t}\|w\|^2, \quad t \ge 0.
\end{aligned}$$

Therefore, $\|e^{-\mathcal{A}t}\pi_s\| \le e^{-\beta t}$ for $t \ge 0$. ◇

Corollary 2.4 *For each $\epsilon \in [0, \beta)$ there exists some $M(\epsilon) > 0$ such that*

$$\begin{aligned}
\|e^{-\mathcal{A}t}\pi_0\| &\le M(\epsilon)e^{\epsilon|t|}, \quad t \in \mathbb{R}, \\
\|e^{-\mathcal{A}t}\pi_s\| &\le M(\epsilon)e^{-(\beta-\epsilon)t}, \quad t \ge 0.
\end{aligned}$$

3 Main Result

In this section we shall prove the main Theorem of this paper under the hypothesis of Theorem 2.1 ($c \ne 2\sqrt{\lambda_j}, \quad j = 1, 2, \dots, l$).

The solution of (2.1) passing through the point w_0 at time $t = t_0$ is given by the variation constants formula

$$w(t) = e^{-\mathcal{A}(t-t_0)}w_0 + \int_{t_0}^{t} e^{-\mathcal{A}(t-s)}\left\{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\right\}ds, \quad t \in \mathbb{R}. \tag{3.1}$$

We shall use the following notation: For each $\eta \geq 0$ we denote by Z_η the Banach space

$$Z_\eta = \Big\{ z \in C(\mathbb{R}; W) : \|z\|_\eta = \sup_{t\in\mathbb{R}} e^{-\eta|t|}\|z(t)\| < \infty \Big\}. \tag{3.2}$$

In particular, $Z_0 = C_b(\mathbb{R}, W)$ the space of bounded and continuous functions defined in $\mathbb{R}$ taking values in $W = \mathbb{R}^n \times \mathbb{R}^n$.

Theorem 3.1 *Suppose that H is a bounded function or $H(0) = 0$. Then for some c and k positive there exist $\eta = \eta(c) \in (0, \beta)$ and a continuous manifold $\mathcal{M} = \mathcal{M}(c, k, P)$ such that any solution $u(t)$ of (1.1) with $(u(0), u'(0)) \in \mathcal{M}$ satisfies:*

$$\sup_{t\in\mathbb{R}} e^{-\eta|t|} \left\{ \|u(t)\|^2 + \|u'(t)\|^2 \right\}^{1/2} < \infty, \tag{3.3}$$

where β, W_0 and W_s are as in corollary 2.3. Moreover,

(a) *There exists a globally Lipschitz function $\psi : W_0 \to W_s$ depending on P such that*

$$\mathcal{M} = \{w_0 + \psi(w_0) \quad : w_0 \in W_0\}, \tag{3.4}$$

and if we put $\psi(w) = \psi(w, P)$, there exist $M \geq 1$ and $0 < \Gamma < 1$ such that

$$\|\psi(w_1, P_1) - \psi(w_2, P_2)\| \leq \frac{kLM(1-\Gamma)^{-1}}{\beta - \eta}\|w_1 - w_2\| + \frac{1}{\beta}\|P_1 - P_2\|, \tag{3.5}$$

for $w_1, w_2 \in W_0, \quad P_1, P_2 \in C_b(\mathbb{R}, \mathbb{R}^n)$.

(b) *If H is bounded, then ψ is also bounded.*

(c) *If $P = 0$ and $H(0) = 0$, then $\mathcal{M}$ is unique and invariant under the equation $w' + \mathcal{A}w + k\mathcal{H}(w) = 0$. In this case $\mathcal{M}$ is called center manifold and it is tangent to the space W_0 at $w_0 = 0$.*

Before proving the main theorem, we need some previous results.

Lemma 3.2 *Let $z \in Z_0 = C_b(\mathbb{R}, W)$. Then, z is a solution of (2.1) if and only if there exists some $w_0 \in W_0$ such that*

$$\begin{aligned} z(t) &= e^{-\mathcal{A}t}w_0 + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0 \left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau \\ &\quad + \int_{-\infty}^t e^{-\mathcal{A}(t-\tau)}\pi_s \left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau, \quad t \in \mathbb{R}. \end{aligned} \tag{3.6}$$

Proof Suppose that z is a solution of (2.1). Then, from Corollary 2.3 we get $z(t) = \pi_0 z(t) + \pi_s z(t)$ and from the variation of constants formula (3.1) we obtain

$$\pi_0 z(t) = e^{-\mathcal{A}t}\pi_0 z(0) + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0 \left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau, \quad t \in \mathbb{R}, \tag{3.7}$$

and

$$\pi_s z(t) = e^{-\mathcal{A}(t-t_0)}\pi_s z(t_0) + \int_{t_0}^{t} e^{-\mathcal{A}(t-\tau)}\pi_s\left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau, \quad t \in \mathbb{R}. \tag{3.8}$$

Since $z(t)$ is bounded, there exists $R > 0$ such that $\|z(t)\| \leq R$, for all $t \in \mathbb{R}$. Then , from corollary 2.3 we obtain that

$$\|e^{-\mathcal{A}(t-t_0)}\pi_s z(t_0)\| \leq Re^{-\beta(t-t_0)} \to 0 \quad \text{as} \quad t_0 \to -\infty.$$

Now, if we put

$$l = \sup_{\tau\in\mathbb{R}} \|\mathcal{H}(z(\tau))\| \quad \text{and} \quad L_p = \sup_{\tau\in\mathbb{R}} \|\mathcal{P}(\tau)\|,$$

then

$$\begin{aligned} \|\int_{-\infty}^{t} e^{-\mathcal{A}(t-\tau)}\pi_s\left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau\| &\leq \|\int_{-\infty}^{t} e^{-\beta(t-\tau)}\left\{kl + L_p\right\} d\tau\| \\ &= \frac{kl + L_p}{\beta}. \end{aligned}$$

Hence, passing to the limit in (3.8), as t_0 approaches $-\infty$, we obtain

$$\pi_s z(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-\tau)}\pi_s\left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau, \quad t \in \mathbb{R}. \tag{3.9}$$

Therefore, putting $w_0 = \pi_0 z(0)$ we get (3.6).

Conversely, suppose that z is a solution of (3.6). Then

$$\begin{aligned} z(t) &= e^{-\mathcal{A}t}w_0 + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0\left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau \\ &\quad + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_s\left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau \\ &\quad + \int_{-\infty}^0 e^{-\mathcal{A}(t-\tau)}\pi_s\left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau \\ &= e^{-\mathcal{A}t}\left\{w_0 + \int_{-\infty}^0 e^{\mathcal{A}\tau}\pi_s\left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau\right\} \\ &\quad + \int_0^t e^{-\mathcal{A}(t-\tau)}\left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau \\ &= e^{-\mathcal{A}t}z(0) + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0\left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau, \end{aligned}$$

where

$$z(0) = w_0 + \int_{-\infty}^0 e^{\mathcal{A}\tau}\pi_s\left\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\right\} d\tau. \tag{3.10}$$

This concludes the proof of the lemma. $\diamondsuit$

Lemma 3.3 *Suppose that $H(0) = 0$ and $z \in Z_\eta$ for $\eta \in [0, \beta)$. Then, z is a solution of (2.1) if and only if there exists some $w_0 \in W_0$ such that z satisfies (3.6).*

Proof Suppose that z is a solution of (2.1). Then, in the same way as the proof of lemma 3.2, we consider

$$\pi_0 z(t) = e^{-\mathcal{A}t}\pi_0 z(0) + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0 \{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\} \, d\tau, \quad t \in \mathbb{R}.$$

and

$$\pi_s z(t) = e^{-\mathcal{A}(t-t_0)}\pi_s z(t_0) + \int_{t_0}^t e^{-\mathcal{A}(t-\tau)}\pi_s \{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\} \, d\tau, \quad t \in \mathbb{R}.$$

Since z belongs to Z_η, there exists $R > 0$ such that $\|z(t)\| \le Re^{\eta|t|}$, for all $t \in \mathbb{R}$. Fix some $t \in \mathbb{R}$ and let $t_0 \le \min\{t, 0\}$; then we have

$$\|e^{-\mathcal{A}(t-t_0)}\pi_s z(t_0)\| \le Re^{-\beta(t-t_0)}e^{-\eta t_0} = Re^{-\beta t}e^{(\beta-\eta)t_0} \to 0 \quad \text{as} \quad t_0 \to -\infty.$$

On the other hand, we obtain the following estimate

$$\| \int_{-\infty}^t e^{-\mathcal{A}(t-\tau)}\pi_s \{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\} \, d\tau \| \le e^{\eta|t|} \left\{ \frac{kLR}{\beta+\eta} + \frac{kLR}{\beta-\eta} + \frac{L_p}{\beta} \right\}, \tag{3.11}$$

where L is the Lipschitz constant of H. Hence, putting $w_0 = \pi_0 z(0)$ and passing to the limit when t_0 goes to $-\infty$ we get (3.6).

The converse follows in the same way as the foregoing lemma. ♢

Lemma 3.4 *Suppose that H is bounded and $z \in Z_\eta$ for $\eta \in [0, \beta)$. Then, z is a solution of (2.1) if and only if there exists some $w_0 \in W_0$ such that z satisfies (3.6).*

Now, from (3.6) we only have to prove that the following set

$$\mathcal{M} = \mathcal{M}(c, k, P) = \{z(0) : z \in Z_\eta, \text{ and satisfies (3.6) } \} \tag{3.12}$$

is a continuous manifold for some values of c and $\eta \in (0, \beta(c))$. From (3.10) we get that

$$\mathcal{M} = \{w_0 + \pi_s z(0) \quad : (w_0, z) \in W_0 \times Z_\eta, \quad (w_0, z) \quad \text{satisfying} \quad (3.6)\} \tag{3.13}$$

We shall need the following definition and notations:

Definition

(a) For each $w_0 \in W_0$ we define the function $Sw_0 : \mathbb{R} \to W$ by:

$$(Sw_0)(t) = e^{-\mathcal{A}t}w_0, \quad t \in \mathbb{R};$$

(b) for each function $z : \mathbb{R} \to W$ we define the non-autonomous Nemytski operator $G(z) : \mathbb{R} \to W$ by

$$G(z)(t) = -k\mathcal{H}(z(t)) + \mathcal{P}(t), \quad t \in \mathbb{R};$$

(c) for those functions $z : \mathbb{R} \to W$ for which the integrals make sense we define $Kz : \mathbb{R} \to W$ by

$$Kz(t) = \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0 z(\tau)d\tau + \int_{-\infty}^t e^{-\mathcal{A}(t-\tau)}\pi_s z(\tau)d\tau, \quad t \in \mathbb{R}.$$

With this notation, (3.6) can be written in the following equivalent form in Z_η

$$z = Sw_0 + K \circ G(z). \tag{3.14}$$

The proof of the following lemma is not hard:

Lemma 3.5 *(a) S is a bounded operator from W_0 into Z_η for each $\eta \geq 0$.*

(b) if $\mathcal{H}(0) = 0$ or $\mathcal{H}$ is bounded, then G maps Z_η into itself for $\eta \geq 0$ and

$$\|G(z_1) - G(z_2)\|_\eta \leq kL\|z_1 - z_2\|_\eta, \quad z_1, z_2 \in Z_\eta;$$

(c) for $\eta \in (0, \beta)$ the linear operator K is bounded from Z_η into itself and

$$\|K\|_\eta \leq R(c) = M(\epsilon)\left\{\frac{1}{\eta - \epsilon} + \frac{1}{\beta(c) - \eta}\right\}, \tag{3.15}$$

where L is the Lipschitz constant of H, β is given by corollary 2.3, $0 < \epsilon < \eta < \beta$ and $M(\epsilon)$ is given by corollary 2.4.

Lemma 3.6 *Let $c > 0$, $k > 0$ and $\eta \in (0, \beta)$ such that*

$$\Gamma = \|K\|_\eta kL < 1. \tag{3.16}$$

Then $(I - K \circ G) : Z_\eta \to Z_\eta$ is a homeomorphism with inverse $\Psi : Z_\eta \to Z_\eta$ and the manifold $\mathcal{M} = \mathcal{M}(c, k, P)$ is given by

$$\mathcal{M} = \{w_0 + \pi_s\Psi(Sw_0)(0) \quad : w_0 \in W_0\}. \tag{3.17}$$

Proof It follows from Lemma 3.5 that $K \circ G$ maps Z_η into itself for $\eta \in (0, \beta)$ and is globally Lipschitzian with Lipschitz constant Γ. Then, under the condition (3.16) the map $(I - K \circ G) : Z_\eta \to Z_\eta$ is invertible, with inverse $\Psi : Z_\eta \to Z_\eta$ which is also globally Lipschitzian with Lipschitz constant $(1 - \Gamma)^{-1}$. In particular Ψ is a continuous function. Therefore, the equation (3.14) has a unique solution given by

$$z(t) = (I - K \circ G)^{-1}(Sw_0)(t) = \Psi(Sw_0)(t), \quad t \in \mathbb{R}. \tag{3.18}$$

Hence, from (3.13) we get (3.17). ♢

Proof of Theorem 3.1. If we take for example $\eta = \beta/2$, then $R(c)$ given by (3.15) can be written as follow

$$R(c) = M(\epsilon)\left\{\frac{2}{\beta - 2\epsilon} + \frac{2}{\beta}\right\},$$

with $0 < \epsilon < \frac{\beta}{2}$. Hence, from $\lim_{c\to\infty} R(c) = 0$. So, we can choose c such that

$$\Gamma = \|K\|_\eta kL \leq R(c)kL < 1.$$

Then, using Lemma 3.6 we obtain the first conclusion of the Theorem and defining $\psi : W_0 \to W_s$ by

$$\psi(w_0) = \pi_s \Psi(Sw_0)(0), \quad w_0 \in W_0,$$

we get (3.4). Clearly, the function ψ is globally Lipschitzian.

On the other hand, from (3.10) we have

$$\begin{aligned} z(0) &= w_0 + \pi_s z(0) = w_0 + \psi(w_0) \\ &= w_0 + \int_{-\infty}^{0} e^{\mathcal{A}\tau}\pi_s\left\{-k\mathcal{H}(\Psi(Sw_0)(\tau)) + \mathcal{P}(\tau)\right\} d\tau. \end{aligned}$$

Therefore,

$$\psi(w_0, P) = \int_{-\infty}^{0} e^{\mathcal{A}\tau}\pi_s\left\{-k\mathcal{H}(\Psi(Sw_0)(\tau)) + \mathcal{P}(\tau)\right\} d\tau. \tag{3.19}$$

To complete the proof of part (a), let us consider $w_1, w_2 \in W$, $\quad P_1, P_2 \in C_b(\mathbb{R}, \mathbb{R}^n)$ and

$$\begin{aligned} \psi(w_1, P_1) - \psi(w_2, P_2) &= \int_{-\infty}^{0} -ke^{\mathcal{A}\tau}\pi_s\left\{\mathcal{H}(\Psi(Sw_1)(\tau)) - \mathcal{H}(\Psi(Sw_2)(\tau))\right\} d\tau \\ &\quad + \int_{-\infty}^{0} e^{\mathcal{A}\tau}\pi_s\left\{\mathcal{P}_1(\tau) - \mathcal{P}_2(\tau)\right\} d\tau. \end{aligned}$$

From Corollary 2.3 we get that

$$\begin{aligned} \|Sw_1 - Sw_2\|_\eta &= \sup_{\tau\in\mathbb{R}} e^{-\eta|\tau|}\|e^{\mathcal{A}\tau}(Sw_1 - Sw_2)\| \\ &\leq M\|w_1 - w_2\|, \end{aligned}$$

and from Lemma 3.6 we get that

$$\begin{aligned} \|\Psi(Sw_1) - \Psi(Sw_2)\|_\eta &\leq (1-\Gamma)^{-1}\|Sw_1 - Sw_2\|_\eta \\ &\leq M(1-\Gamma)^{-1}\|w_1 - w_2\|. \end{aligned}$$

Therefore,

$$\begin{aligned} \|\psi(w_1, P_1) - \psi(w_2, P_2)\| &\leq \int_{-\infty}^{0} kLM(1-\Gamma)^{-1}\|w_1 - w_2\| e^{(\beta-\eta)\tau} d\tau \\ &\quad + \int_{-\infty}^{0} \|P_1 - P_2\| e^{\beta\tau} d\tau. \end{aligned}$$

Hence,

$$\|\psi(w_1,P_1)-\psi(w_2,P_2)\| \leq \frac{kLM(1-\Gamma)^{-1}}{\beta-\eta}\|w_1-w_2\| + \frac{1}{\beta}\|P_1-P_2\|.$$

To prove part (b), let us suppose that: $\|H(u)\| \leq l, \quad u \in \mathbb{R}^n$ and $L_p = \sup_{\tau\in\mathbb{R}}\|P(\tau)\|$. Then, from (3.19) we get that

$$\|\psi(w_0)\| \leq \int_{-\infty}^{0} e^{\beta\tau}\{kl+L_P\}d\tau \leq \frac{kl+L_P}{\beta}, \quad w_0 \in W_0.$$

Part (c) follows from Theorem 2.1 of [7]. ♢

Remark The equation (3.6) may not have bounded solutions in $\mathbb{R}$. However, if $H = 0$ and $\mathcal{P}$ satisfies the condition

$$\sup\left\{ \mid \int_0^t \|\pi_0\mathcal{P}(\tau)\|d\tau \mid : \quad t \in \mathbb{R}\right\} < \infty,$$

then for each $w_0 \in W_0$ the equation (3.6) has a bounded solution which is given by

$$z(t) = e^{-\mathcal{A}t}w_0 + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0\mathcal{P}(\tau)d\tau + \int_{-\infty}^t e^{-\mathcal{A}(t-\tau)}\pi_s\mathcal{P}(\tau)d\tau, \quad t \in \mathbb{R}.$$

An open question, is the following: What conditions do we have to impose to the functions H and P to insure the existence of bounded solutions of the equation (3.6)?

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