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Behaviour near zero and near infinity of solutions to elliptic equalities and inequalities *

Marie-Françoise Bidaut-Véron

Abstract

Here we consider elliptic equations and inequalities involving quasilinear operators in divergence form and nonlinear lower order terms:

$$-\operatorname{div}(\mathcal{A}(x,u,\nabla u)) \geq |x|^\sigma u^Q \quad (Q>0, \sigma\in\mathbb{R}),$$

in dimension $N \geq 3$. We study the asymptotic behaviour of the solutions and give *a priori* estimate and non-existence results.

1 Introduction

Here we study the existence and the asymptotic behaviour near zero and near infinity of nonnegative solutions to elliptic problems involving quasilinear operators in divergence form. We study equalities of the form

$$-\operatorname{div}[\mathcal{A}(x,u,\nabla u)] = |x|^\sigma u^Q, \tag{1.1}$$

and more generally inequalities of the form

$$-\operatorname{div}[\mathcal{A}(x,u,\nabla u)] \geq |x|^\sigma u^Q, \tag{1.2}$$

where $Q, \sigma \in \mathbb{R}$, $Q > 0$, in an open set Ω of $\mathbb{R}^N$ $(N \geq 3)$. A great part of the results extends to systems of the form

$$\begin{aligned} -\operatorname{div}[\mathcal{A}(x,u,\nabla u)] &= |x|^a u^S v^R, \\ -\operatorname{div}[\mathcal{B}(x,v,\nabla v)] &= |x|^b u^Q v^T, \end{aligned} \tag{1.3}$$

where $Q, R, S, T \geq 0$, and to systems of inequalities; see for example [6, 9].

Let $B_r = \{|x| < r\}$ with $r > 0$. Let Ω be either $\mathbb{R}^N$ or $\mathbb{R}^N \setminus \{0\}$, or an exterior or interior domain

$$\Omega_e = \left\{x\in\mathbb{R}^N \mid |x|>1\right\}, \quad \Omega_i = \left\{x\in\mathbb{R}^N \mid 0<|x|<1\right\} = B_1\setminus\{0\}$$

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or the half-space $\mathbb{R}^{N+} = \{x \in \mathbb{R}^N \mid x_N > 0\}$, or

$$\Omega_e^+ = \Omega_e \cap \mathbb{R}^{N+}, \quad \Omega_i^+ = \Omega_i \cap \mathbb{R}^{N+}.$$

Our aim is to point out many results on this subject and to show some short proofs to some results. We cannot present a complete survey, because it would be too long, we rather give references that seem to be significant.

2 The Laplacian case

We begin by the model case of the Laplace operator, with the equation

$$-\Delta u = |x|^\sigma u^Q, \tag{2.1}$$

or the inequality

$$-\Delta u \geq |x|^\sigma u^Q, \tag{2.2}$$

where $\sigma \in \mathbb{R}$, $Q > 0, Q \neq 1$. By solution of (2.1) or (2.2), we mean any nonnegative function $u \in C^0(\Omega) \cap W^{1,1}_{\mathrm{loc}}(\overline{\Omega})$ with $\Delta u \in L^1_{\mathrm{loc}}(\overline{\Omega})$, solution in the sense of $\mathcal{D}'(\ \Omega)$. We set

$$Q_\sigma = (N+\sigma)/(N-2).$$

Recall that the equation admits a particular solution of the form

$$u^* = C^*|x|^{-(2+\sigma)/(Q-1)}, \tag{2.3}$$

for some $C^* > 0$ if and only if $Q > Q_\sigma > 1$, or $Q < Q_\sigma < 1$. First remark that the problem in Ω_i or Ω_e are equivalent to solve, and in the same way in Ω_i^+ or Ω_e^+, from the Kelvin transform: setting

$$u_0(x) = |x|^{2-N} u(y), \quad y = x/|x|^2$$

then (2.2) is equivalent to

$$-\Delta u_0 \geq |y|^{\sigma_0} u_0^Q, \quad \sigma_0 = (N-2)Q - (N+2+\sigma) \tag{2.4}$$

Now let us recall the Brézis-Lions theorem in Ω_i in its simplest form:

Theorem 2.1 ([13]) *Let $w \in L^1_{\mathrm{loc}}(\Omega_i)$ be any nonnegative superharmonic function, such that $\Delta w \in L^1_{\mathrm{loc}}(\Omega_i)$. Then $f = \Delta w_{/\Omega_i} \in L^1_{\mathrm{loc}}(B_1)$, $w \in M^{N/(N-2)}_{\mathrm{loc}}(B_1)$, $|\nabla w| \in M^{N/(N-1)}_{\mathrm{loc}}(B_1)$ and there exists $\lambda \geq 0$ such that*

$$-\Delta w = -\Delta w_{/\Omega_i} + \lambda \delta_0 \quad \textit{in } \mathcal{D}'(B_1).$$

Then one gets a first nonexistence result concerning inequality (2.2), given in [4]. Up to some changes of variable, in the radial case of the equation, it comes from the study of Fowler [16, 17], of the equation

$$-y" = r^{\theta} y^{Q},$$

$\theta \in \mathbb{R}$. He made a complete description of the solutions, with the restriction $Q, \theta \in \mathbb{N}$ because the phase plane techniques for ODE's were not known; see also [2]. This result was extended to the inequality in the radial case with more general operators by Ni and Serrin [25]. The result was also found again in the case $\sigma = -2$ by [12].

Theorem 2.2 *Assume $Q > 1$.*
i) There exists a nontrivial solution of (2.2) in Ω_i if and only if $\sigma > -2$.
ii) There exists a nontrivial solution of (2.2) in Ω_e if and only if $Q > Q_\sigma$.
iii) There exists a nontrivial solution of (2.2) in $\mathbb{R}^N$ or $\mathbb{R}^N \setminus \{0\}$ if and only if $Q > Q_\sigma$ and $\sigma > -2$.

Proof. i) and ii) For the part "if", the particular solution (2.3) is a solution in $\mathbb{R}^N \setminus \{0\}$, hence in Ω_i and Ω_e. For the part "only if", *the problem reduces to the radial one.* By Kelvin transform we reduce to the case of Ω_i. Suppose there exists a nontrivial solution u of (2.2). Let

$$\overline{u}(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u(r, \theta)\, d\theta \tag{2.5}$$

be the mean value of u on the sphere of center 0 and radius r. Then $\overline{u}$ also satisfies (2.2), from the Jensen inequality, that is

$$-(r^{N-1}\overline{u}_r)_r \geq r^{N-1+\sigma}\overline{u}^Q,$$

and $\overline{u} > 0$. Then either $\lim_{r\to 0} r^{N-1}\overline{u}_r \in (0, +\infty]$; then $\lim_{r\to 0} \overline{u} = C > 0$ and we reach a contradiction. Or $\overline{u}_r \leq 0$ near 0. By integration we get

$$r^{N-1}\overline{u}_r + \overline{u}^Q \int_0^r t^{N-1+\sigma} dt \leq 0$$

hence $\sigma + N > 0$ and

$$\overline{u}^{-Q}\overline{u}_r + r^{\sigma+1}/(N+\sigma) \leq 0\,.$$

Integrating again it implies that $\sigma > -2$, and we have the estimate near 0:

$$\overline{u} \leq Cr^{-(2+\sigma)/(Q-1)}. \tag{2.6}$$

iii) The part "only if" is obvious. For the part "if", when $Q > Q_\sigma$ and $\sigma > -2$, the function $u(x) = c(1 + |x|^{2+\sigma})^{-1/(Q-1)}$ is a solution of (2.2) in $\mathbb{R}^N$, hence in $\mathbb{R}^N \setminus \{0\}$ if c is small enough. This example can be found in [22] when $\sigma = 0$. $\diamondsuit$

Now we consider the case $Q < 1$. The following was proved by [28] for the equation, and extended in [10] and [6].

Theorem 2.3 *Assume $Q < 1$.*
i) There exists a nontrivial solution of (2.2) in Ω_i if and only if $Q < Q_\sigma$.
ii) There exists a nontrivial solution of (2.2) in Ω_e if and only if $\sigma < -2$.

Proof. Assume there is a nontrivial solution in Ω_i. Then $u > 0$, and we can define $w = 1/u$. It is subharmonic and satisfies

$$-\Delta w + |x|^\sigma w^m \leq 0$$

with $m = 2 - Q > 1$. Then from Osserman's estimate (see [6]),

$$w \leq C \begin{cases} |x|^{-(2+\sigma)/(m-1)} & \text{if } \sigma \neq -2 \\ |\ln |x||^{-1/(m-1)} & \text{if } \sigma = -2 \end{cases}$$

in $\frac{1}{2}\Omega_i$. That means

$$u \geq C \begin{cases} |x|^{(2+\sigma)/(1-Q)} & \text{if } \sigma \neq -2, \\ |\ln |x||^{1/(1-Q)} & \text{if } \sigma = -2. \end{cases} \tag{2.7}$$

But $|x|^\sigma u^Q \in L^1_{\text{loc}}(B(0,1))$, hence in any case $Q < Q_\sigma$. For the part "if", see [28]. $\diamondsuit$

Remark 2.4 Here also *the problem could be reduced to the radial one.* Indeed we have the following property, which proves that $\overline{u}$ satisfies an inequality of the same form as (2.2).

Lemma 2.5 ([7]) *Let $w \in C^2(\Omega_i)$ be any nonnegative superharmonic function. Then there exists a constant $C(N) > 0$ such that for any $x \in \frac{1}{2}\Omega_i$,*

$$w(x) \geq C(N)\overline{w}(|x|). \tag{2.8}$$

Remark 2.6 In particular, the exterior problem

$$-\Delta u \geq u^Q$$

in Ω_e has no solution except 0 for any $0 < Q < N/(N-2)$, $Q \neq 1$.

Remark 2.7 The non existence results are very linked to the estimates of $\overline{u}$. In case of Ω_i we have for any solution of (2.2), from (2.6) (2.7) and the superharmonicity,

$$\overline{u} \leq C \min(r^{-(2+\sigma)/(Q-1)}, r^{2-N}) \quad \text{in } \frac{1}{2}\Omega_i \quad \text{if } Q > 1,$$

$$C_1 r^{(2+\sigma)/(1-Q)} \leq \overline{u} \leq C_2 r^{2-N} \quad \text{in } \frac{1}{2}\Omega_i \quad \text{if } Q < 1.$$

In the case of Ω_e, it follows that

$$\overline{u} \leq C \min(r^{-(2+\sigma)/(Q-1)}, 1) \quad \text{in } 2\Omega_e \quad \text{if } Q > 1,$$

$$C_1 r^{(2+\sigma)/(1-Q)} \leq \overline{u} \leq C_2 \quad \text{in } 2\Omega_e \quad \text{if } Q < 1.$$

Now let us come to the case of the equation. In the radial case, we have a well-known nonexistence result in whole $\mathbb{R}^N$.

Lemma 2.8 *There exists a nontrivial radial solution of (2.1) in $\mathbb{R}^N$ (that means a radial ground state) if and only if*

$$Q \geq Q_\sigma^* = \frac{N+2+2\sigma}{N-2} > 1 \tag{2.9}$$

Proof. Assume (2.9). First one constructs a local solution near 0 such that $u(0) = 1$ and $u_r(0) = 0$. By concavity it extends to a solution of the equation

$$-\Delta u = |x|^\sigma |u|^{Q-1} u$$

in $[0, +\infty)$. Now suppose that $u(r_0) = 0$ for some $r_0 > 0$. The change of variable (first used by Fowler)

$$u(r) = r^{-\gamma} U(t) \quad \gamma = \frac{2+\sigma}{Q-1}, \quad t = -\ln r,$$

reduces the equation to an autonomous one:

$$U_{tt} - AU_t - BU + |U|^{Q-1} U = 0$$

with $A = N-2-2\gamma > 0$ and $B = \gamma((N-2-\gamma) > 0$. Then the energy function

$$E = \frac{U_t^2}{2} - B\frac{U^2}{2} + \frac{|U|^{Q+1}}{Q+1}$$

is nondecreasing, since $E_t = AU_t^2$, with $\lim_{t\to+\infty} E(t) = 0$, and $E(-\ln r_0) \geq 0$. Then $E(t) = E_t(t) = 0$ for $t \geq -\ln r_0$, hence U is constant, and we reach a contradiction. Reciprocally suppose there exists a ground state. Then first $\sigma > -2$. Suppose $Q < Q_\sigma^*$. Then E is nonincreasing, hence nonnegative, and bounded. Then $\lim_{t\to-\infty} E(t) = L > 0$ and $\lim_{t\to-\infty} U_t(t) = 0$, since U_{tt} is bounded and $\int_{-\infty}^0 U_t^2 < +\infty$. Then $\lim_{t\to-\infty} U(t) = \ell = (B(Q+1)/2)^{1/(Q-1)}$. By linearisation $U(t) \equiv \ell$, hence a contradiction holds. $\diamondsuit$

Remark 2.9 The existence in $\mathbb{R}^N \setminus \{0\}$ is obviously different: there exists a nontrivial radial positive solution of (2.1) in $\mathbb{R}^N \setminus \{0\}$ if and only if $Q > Q_\sigma > 1$. Indeed the particular solution (2.3) exists in that range.

Now let us come to the nonradial case. Here the results are not complete: they require that

$$Q \leq Q_0^* = \frac{N+2}{N-2},$$

where the well-known Q_0^* is the limit value of Q for the compacity of the Sobolev injection from L^{Q+1} into $W^{1,2}$. Or they require additional assumptions on the behaviour at infinity, see [32]. They require difficult techniques, either linked to the Bernstein method of a priori estimates of $|\nabla u|^2$, or to the moving plane method of Alexandroff. The pionneer works are due to Gidas, Spruck and Caffarelli [18], [14].

Theorem 2.10 ([18]) *i) Assume that $1 < Q < Q_0^*$. Then any solution in Ω_i (resp. Ω_e) satisfies*

$$u(x) \leq C\,|x|^{-(2+\sigma)/(Q-1)} \quad \text{in } \frac{1}{2}\Omega_i \text{ (resp.in } 2\Omega_e) \tag{2.10}$$

where C does not depend on u.
ii) Assume that $1 < Q_\sigma < Q < Q_0^*$. If $Q < Q_\sigma^*$, then any solution u in $\mathbb{R}^N \backslash \{0\}$ is singular at 0. In particular there is no nontrivial nonnegative solution in $\mathbb{R}^N$. If $Q > Q_\sigma^*$, then either $u = u^*$ or u is a solution in $\mathbb{R}^N$ (ground state).

Remark 2.11 The result was extended to the case $Q = Q_0^*$ in [14]. When $Q > Q_0^*$ the result is not known. In the case $Q = (N+1)/(N-3)$, $\sigma = 0$, it is shown in [8] that(2.10) cannot hold with a constant independant on u.

Now let us give a few results concerning the case of the half-space. Concerning the inequality (2.2), the usual proofs of nonexistence lie on the use of the first eigenvalue $\lambda_1 = N-1$ of the Beltrami operator on the half sphere $(S^{N-1})^+$ with Dirichlet conditions on $\partial(S^{N-1})^+$, and the corresponding positive normalized eigenfunction ϕ_1, and extend to cones and systems. We refer for example to [2] and [11] . In case of the half space, we have the following theorem.

Theorem 2.12 *Assume that $N \geq 2$, and $Q > 1$.*
i)If $Q \leq (N+1+\sigma)/(N-1)$, the problem (2.2) in Ω_e^+, with $u \in C^1(\overline{\Omega_e^+})$, has only the solution $u \equiv 0$.
ii) If $Q + \sigma + 1 \leq 0$, the problem in Ω_i^+, with $u \in C^1(\overline{\Omega_i^+} \backslash \{0\})$ has only the solution $u \equiv 0$.

Proof. We follow the method of [15] given in the case $u \in C^1(\overline{\mathbb{R}^N} \backslash \{0\})$. They still show that the problem can be reduced to a radial one, by considering the mean value function

$$u_\sharp(r) = \frac{1}{|(S^{N-1})^+|} \int_{(S^{N-1})^+} u(r,\theta)\phi_1 \, d\theta. \tag{2.11}$$

Namely function $u_\sharp$ satisfies the inequality

$$-r^{-N}((r^{N+1}(r^{-1}u_\sharp)_r)_r) = -\Delta u_\sharp + (N-1)\frac{u_\sharp}{r^2} \geq r^\sigma u_\sharp^Q.$$

By Kelvin transform we are reduced to the case of Ω_i^+. Let $v = r^{-1}u_\sharp$. Then

$$-(r^{N+1}v_r)_r \geq r^{N+Q+\sigma}v^Q,$$

and $\overline{u} > 0$. Then either $\lim_{r\to 0} r^{N+1}v_r \in (0, +\infty]$; then $\lim_{r\to 0} v = C > 0$ and we reach a contradiction. Or $v_r \leq 0$ near 0. By integration we get

$$r^{N+1}v_r + v^Q \int_0^r t^{N+Q+\sigma} dt \leq 0$$

hence $N+Q+\sigma>0$ and

$$v^{-Q}v_r + r^{\sigma+Q}/(N+Q+\sigma) \leq 0.$$

Integrating again it implies that $\sigma+Q+1>0$, and we have the estimate near 0 :

$$u_\sharp \leq Cr^{-(2+\sigma)/(Q-1)}.$$

$\diamondsuit$

In the case of the equation (2.1), Gidas and Spruck have obtained a better result:

Theorem 2.13 ([19]) *Assume that $Q<(N+2)/(N-2)$. Then equation (2.1) with σ =0 has no nontrivial solution in $\mathbb{R}^{N+}$.*

3 The p-Laplacian case

Now we consider the case of the p-Laplace operator ($p>1$):

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |x|^\sigma u^Q \tag{3.1}$$

and the inequality

$$-\Delta_p u \geq |x|^\sigma u^Q. \tag{3.2}$$

In the radial case, the first estimates concerning (3.2) are due to Guedda and Véron [20], where they give the behaviour in Ω_i and some global properties; and the first nonexistence results are given in [24], [25]. Then the non-radial case was studied in [4], where one can also find a complete description of the radial case.

Here one cannot use any Kelvin transform, so that the behaviour at infinity cannot reduced to the behaviour near 0. Also one cannot use the mean value of u since the problem is not linear. But many of the results can be extended. The equation has a particular solution

$$u^*(x) = C^*|x|^{-\Gamma}, \quad \Gamma = \frac{p+\sigma}{Q-p+1}, \quad C^*>0, \tag{3.3}$$

if and only if $Q>Q_{\sigma,p}>p-1$, or $Q<Q_{\sigma,p}<p-1$, where

$$Q_{\sigma,p} = (N+\sigma)(p-1)/(N-p). \tag{3.4}$$

First theorem 2.2 extends to the p-Laplacian. This was proved in [4] for equation (3.1) in Ω_e without mentioning the critical case $Q=Q_{\sigma,p}$, but the proof extends to the general case and we reproduce it here. The idea is the following: if (3.2) has a solution u in Ω_i or Ω_e, then we can construct a *radial* solution of (3.1) which is less than u. So that *we still are reduced to the radial case*, and with an equation.

Theorem 3.1 *Assume $Q>p-1$.*
i) There exists a nontrivial solution of (3.2) in Ω_i if and only if $\sigma>-p$.
ii) There exists a nontrivial solution of (3.2) in Ω_e if and only if $Q>Q_{\sigma,p}$.

Proof. Let us prove for example ii). Suppose that $Q \leq Q_{\sigma,p}$ and that (3.2) has a nontrivial solution u. Then $u > 0$ from the strong maximum principle. Let $m = \min_{|x|=2} u(x)$. Let $n \in \mathbb{N}^*$ be fixed, such that $n > 2$. By minimization we construct a sequence $(u_{n,k})_{k\in\mathbb{N}}$ of radial nonnegative functions with $u_{n,0} \equiv 0$ and

$$\begin{aligned} -\Delta_p u_{n,k} &= |x|^\sigma u_{n,k-1} \quad \text{for } 2 < |x| < n, \\ u_{n,k} &= m \quad \text{for } |x| = 2, \\ u_{n,k} &= 0 \quad \text{for } |x| = n\,. \end{aligned}$$

Then $0 < u_{n,k} \leq u_{n,k+1} \leq u$ for $2 < |x| < n$. And $\left(r^{N-1}\left|(u_{n,k})_r\right|^{p-2}(u_{n,k})_r\right)_{k\in\mathbb{N}}$ is equi-continuous on $[2,n]$. Thus it converges in $C^1([2,n])$ to a radial fuction u_n such that $u_{n,k} \leq u_n \leq u$ and

$$\begin{aligned} -\Delta_p u_n &= |x|^\sigma u_n \quad \text{for } 2 < |x| < n\,, \\ u_n &= m \quad \text{for } |x| = 2, \\ u_n &= 0 \quad \text{for } |x| = n. \end{aligned}$$

By extraction of a diagonal sequence, there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ converging in $C^1_{\mathrm{loc}}([2,+\infty])$ to a radial solution w of equation (3.1) in $2\Omega_e$, nontrivial, since $w = m$ for $|x| = 2$. But the radial equation has no solution when $Q \leq Q_{\sigma,p}$, by an argument analogous to the one of theorem 2.2 . ◇

The theorem 2.3 extends immediately to the case of the p-Laplacian, by the same proof, since Osserman's estimate extends.

Theorem 3.2 *Assume $Q < p-1$.*
i) There exists a nontrivial solution of (3.2) in Ω_i if and only if $Q < Q_{\sigma,p}$.
ii) There exists a nontrivial solution of (3.2) in Ω_e if and only if $\sigma < -p$.

Now let us come to upper estimates. First the Brézis-Lions lemma extends in the following form, where for simplicity we supposed $p < N$.

Theorem 3.3 ([4]) *Let $1 < p < N$. Let $w \in C(\Omega_i)$ be any nonnegative super-p-harmonic function, such that $\Delta_p w \in L^1_{\mathrm{loc}}(\Omega_i)$. Then $f = -\Delta_p w_{/\Omega_i} \in L^1_{\mathrm{loc}}(B_1)$, $w^{p-1} \in M^{N/(N-p)}_{\mathrm{loc}}(B_1)$, $|\nabla w|^{p-1} \in M^{N/(N-1)}_{\mathrm{loc}}(B_1)$ and there exists $\lambda \geq 0$ such that*

$$-\Delta_p w = -\Delta_p w_{/\Omega_i} + \lambda\delta_0 \quad \textit{in } \mathcal{D}'(B_1). \tag{3.5}$$

Proof. It is divided in four steps.
i) Function f is in $L^1(B_1)$. In order to obtain estimates on f, the idea is to multiply the inequality by a function $P(u)\varphi$, with φ with compact support in B_1, and P *is decreasing in* u, in order to obtain some coercivity. One takes $P(u) = (n+1-u)^+$, with $n \in \mathbb{N}$.

ii) Function w is in $L^k_{\text{loc}}(B_1)$ for any $1 \leq k < N/(N-p)$ and it satisfies the integral estimate for $\rho \leq 1/2$:

$$\int_{B\rho} w^k \leq C\rho^{N-(N-p)k/(p-1)} \tag{3.6}$$

Here we use a test function introduced by Serrin [29] and capacity methods in order to estimate $\min_{|x|=\rho} w(x)$, and then the weak Harnack inequality.

iii) Function $|\nabla w|^{p-1}$ is $L^k_{\text{loc}}(B_1)$ for any $1 \leq k < N/(N-1)$ and also satisfies an integral inequality.

iv) The Marcinkiewicz estimates and (3.5) hold. Here we use ideas of P. Bénilan. $\diamondsuit$

This showed that we can obtain some integral estimates on w, even for a nonlinear problem, replacing the estimates of the mean value for the Laplacian. Indeed defining for any nonnegative g on Ω and any $\omega \subset \Omega$

$$\fint_\omega g = \frac{1}{|\omega|}\int_\omega g$$

then (3.6) can be written

$$\Big(\fint_{B(0,\rho)} w^k\Big)^{1/k} \leq C\rho^{-(N-p)/(p-1)},$$

which extends the classical estimate $\overline{u}(r) \leq Cr^{2-N}$ in case of the Laplacian. This was a motivation to extend also the estimate $\overline{u}(r) \leq Cr^{-(2+\sigma)/(Q-1)}$ of the problem (2.2) to the problem (3.2) and more general operators. One gets the following, where $\mathcal{C}_{\rho_1,\rho_2} = \{\rho_1 < |x| < \rho_2\}$.

Theorem 3.4 ([9]) *Assume that $N \geq p > 1$. Let u be a nonnegative solution of (3.2) in Ω_i (resp. Ω_e).*
i) If $Q > p-1$, then for small ρ (resp. for large ρ)

$$\Big(\fint_{\mathcal{C}_{\rho/2,\rho}} u^Q\Big)^{1/Q} \leq C\rho^{-\Gamma}. \tag{3.7}$$

ii) If $Q < p-1$, either $u \equiv 0$, or

$$u(x) \geq C\,|x|^{-\Gamma} \quad \textit{in } \frac{1}{2}\Omega_i \quad (\textit{resp. in } 2\Omega_e). \tag{3.8}$$

Proof. We just give the proof of i). Let u be a nontrivial solution of (3.2) , hence $u > 0$. Let $1-p < \alpha < 0$. By computation the function $u_\alpha = u^{1+\alpha/(p-1)}$ is also superharmonic and satisfies

$$-\Delta_p u_\alpha \geq C(\alpha)\left(|x|^\sigma u^{Q+\alpha} + u^{\alpha-1}\,|\nabla u|^p\right)$$

for some $C(\alpha) > 0$. Then we multiply by a test function $\varphi = \xi^\lambda$ with λ large enough, and $\xi \in \mathcal{D}(\Omega)$ with values in $[0,1]$, such that $\xi = 1$ for $\rho/2 \le |x| \le \rho$ and $|\nabla\xi| \le C/\rho$. We get (with other constants $C = C(\alpha,\lambda)$)

$$\begin{aligned}\int_{\Omega_i} |x|^\sigma u^{Q+\alpha}\xi^\lambda + \int_{\Omega_i} u^{\alpha-1} |\nabla u|^p \xi^\lambda &\le C\int_{\Omega_i} |\nabla u_\alpha|^{p-1} \xi^{\lambda-1} |\nabla\xi| \\ &\le C\int_{\Omega_i} u^\alpha |\nabla u|^{p-1} \xi^{\lambda-1} |\nabla\xi|\end{aligned}$$

and setting $\theta = Q/(p-1+\alpha) > 1$ we get from the Hölder inequality

$$\begin{aligned}&\int_{\Omega_i} |x|^\sigma u^{Q+\alpha}\xi^\lambda + \int_{\Omega_i} u^{\alpha-1} |\nabla u|^p \xi^\lambda \\ &\quad \le C\left(\int_{\Omega_i} u^Q\xi^\lambda\right)^{1/\theta}\left(\int_{\Omega_i} \xi^{\lambda-p\theta'} |\nabla\xi|^{p\theta'}\right)^{1/\theta'} \qquad (3.9)\end{aligned}$$

Now we take ξ^λ as test function directly in (3.2) and get by using the same α

$$\begin{aligned}\int_{\Omega_i} |x|^\sigma u^Q\xi^\lambda &\le \lambda\int_{\Omega_i} |\nabla u|^{p-1} \xi^{\lambda-1} |\nabla\xi| \\ &\le \lambda\int_{\Omega_i} u^{(\alpha-1)/p'} |\nabla u|^{p-1} u^{(1-\alpha)/p'} \xi^{\lambda-1} |\nabla\xi| \\ &\le C\left(\int_{\Omega_i} u^{\alpha-1} |\nabla u|^p \xi^\lambda\right)^{1/p'}\left(\int_{\Omega_i} u^{(1-\alpha)(p-1)}\xi^{\lambda-p} |\nabla\xi|^p\right)^{1/p}\end{aligned}$$

And from (3.9), choosing α small enough such that $\tau = Q/(1-\alpha)(p-1) > 1$,

$$\begin{aligned}\int_{\Omega_i} u^Q\xi^\lambda &\le C\rho^{-\sigma}\left(\int_{\Omega_i} u^Q\xi^\lambda\right)^{1/\theta p'+1/\tau p} \times \\ &\quad \left(\int_\Omega \xi^{\lambda-\theta' p} |\nabla\xi|^{\theta' p}\right)^{1/\theta' p'}\left(\int_\Omega \xi^{\lambda-\tau' p} |\nabla\xi|^{\tau' p}\right)^{1/\tau' p}. \qquad (3.10)\end{aligned}$$

And $1/\theta p' + 1/\tau p = (p-1)/Q = 1 - (1/\theta' p' + 1/\tau' p)$, hence (3.7) follows. $\diamondsuit$

Remark 3.5 In the case $Q > p-1$, Theorem 3.1 can be found again in a longer way by using these upper estimates. Indeed following the technique of comparison of theorem 3.1, one can prove lower estimates. Consider the radial elementary p-harmonic functions in $\mathbb{R}^N \setminus \{0\}$, that means functions

$$\Phi_{1,p}(r) \equiv 1, \quad \Phi_{2,p}(r) = \begin{cases} r^{(p-N)/(p-1)} & \text{if } N > p, \\ \ln r & \text{if } N = p\,. \end{cases}$$

Then any super-p-harmonic function u in Ω_i (resp. Ω_e) satisfies

$$u \ge C\Phi_{1,p} \quad \text{in } \frac{1}{2}\Omega_i \quad (\text{resp.} \quad u \ge C\Phi_{2,p} \quad \text{in } 2\Omega_e)\,;$$

see [9].

Above all, the integral estimates can give *punctual estimates* in the case of the equation (3.1), in the subcritical case. The following is proved in [4] when $\sigma = 0$, and in [9] in the general case.

Theorem 3.6 *Assume that $N \geq p > 1$. Let u be a nonnegative solution of (3.1) in Ω_i. Assume that*

$$0 < Q < Q_{0,p} = N(p-1)/(N-p).$$

Then u satisfies the Harnack inequality. Consequently, if $Q > p-1$,

$$u(x) \leq C \min(|x|^{-\Gamma}, |x|^{(p-N)/(p-1)}) \quad \text{in } \frac{1}{2}\Omega_i; \tag{3.11}$$

if $Q < p-1$, then

$$u(x) \leq C\,|x|^{(p-N)/(p-1)}) \quad \text{in } \frac{1}{2}\Omega_i.$$

Proof. First suppose $Q > p-1$. We write the equation under the form

$$-\Delta_p u = h\, u^{p-1}, \quad h = |x|^{\sigma} u^{Q-p+1}.$$

If $\sigma = 0$, we remark that $u^Q \in L^1(B_{1/2})$ from the Brézis-Lions theorem. Hence $h^s \in L^1(B_{1/2})$ for $s = Q/(Q-p+1) > N/p$, since $Q < Q_0$. Then we can apply Serrin's results of [29], and conclude. In the general case $\sigma \in \mathbb{R}$, we use the estimate (3.7)

$$\int_{\mathcal{C}_{\rho/2,\rho}} h^s = \int_{\mathcal{C}_{\rho/2,\rho}} |x|^{\sigma s} u^Q \leq \rho^{\sigma s} \int_{\mathcal{C}_{\rho/2,\rho}} u^Q \leq C\rho^{N+\sigma s-\Gamma Q} = C\rho^{N-ps}. \tag{3.12}$$

This implies the Harnack inequality, and (3.11) follows. Now suppose $Q \leq p-1$. We observe that $h(x) \leq C\,|x|^{-p}$ near 0, from (3.8) if $Q < p-1$. Then h satisfies (3.12) for any $s > 1$, and the Harnack inequality still holds. ♢

As in the case $p = 2$, the question of the estimates is harder in the case $Q > Q_{0,p}$. Serrin and Zou have announced in January 2000 the following beautiful result, which extends the one of [18] and of [4]:

Theorem 3.7 ([31]) *Assume that $1 < Q < Q^*_{0,p} = (N(p-1)+p)/(N-p)$. Then any solution of (3.1) with $\sigma = 0$ in Ω_i satisfies*

$$u(x) \leq C\,|x|^{-p/(Q+1-p)} \quad \text{in } \frac{1}{2}\Omega_i; \tag{3.13}$$

where C does not depend on u, and u satisfies the Harnack inequality. Moreover there is no nontrivial nonnegative solution in $\mathbb{R}^N$.

At last we consider the case of an halfspace. First following the ideas of the proof of theorem 3.4, we get upper estimates when $Q > p-1$:

Theorem 3.8 *Assume that $N \geq p > 1$, $Q > p-1$. Let u be a nonnegative solution of (3.2) in Ω_i^+ (resp. Ω_e^+). Let $K_a = \left\{x \in \mathbb{R}^{N+} \middle| \; x_N \geq a\,|x|\right\}$ for any $a > 0$. Then for small ρ (resp. for large ρ)*

$$\left(\oint_{K_a \cap \mathcal{C}_{\rho/2,\rho}} u^Q\right)^{1/Q} \leq C\rho^{-\Gamma}. \tag{3.14}$$

Here also we can find lower estimates by comparison to the p-harmonic functions which vanish on the set $x_N = 0$. In the case $p = 2$, they are given by $x \longmapsto x_N$ and $x \longmapsto x_N / |x|^N$. In the general case, they are given by

$$\Psi_{1,p}(x) = x_N, \quad \Psi_{2,p}(x) = \frac{\varpi(x/|x|)}{|x|^{\beta_{p,N}}}, \tag{3.15}$$

for some unique $\beta_{p,N} > 0$ and $\varpi \in C^1(S^{N-1})$, $\varpi > 0$, with maximum value 1, from [21]. The exact value of $\beta_{p,N}$ is unknown if $p \neq 2$, except in the case $N = 2$. We prove that any super-p-harmonic function u in $C^1(\overline{\Omega_i^+} \setminus \{0\})$ (resp. $C^1(\overline{\Omega_e^+})$) satisfies

$$u \geq C\Psi_{1,p} \quad \text{in } \frac{1}{2}\Omega_i \quad (\text{resp.} \quad u \geq C\Psi_{2,p} \quad \text{in } 2\Omega_e).$$

So that we deduce a new nonexistence result:

Theorem 3.9 ([9]) *Assume that $N \geq p > 1$, and $Q > p-1$.*
i) If $Q < q_{\sigma,p}$, where

$$q_{\sigma,p} = p - 1 + (p+\sigma)/\beta_{p,N},$$

the problem (3.2) in Ω_e^+, with $u \in C^1(\overline{\Omega_e^+})$, has only the solution $u \equiv 0$.
ii) If $Q + \sigma + 1 < 0$, the problem in Ω_i^+, with $u \in C^1(\overline{\Omega_i^+} \setminus \{0\})$ has only the solution $u \equiv 0$.

4 More general operators

Some of the above results are still valid for problems of the form

$$-\operatorname{div}\left[\mathcal{A}(x, u, \nabla u)\right] = |x|^\sigma u^Q, \tag{4.1}$$

or

$$-\operatorname{div}\left[\mathcal{A}(x, u, \nabla u)\right] \geq |x|^\sigma u^Q, \tag{4.2}$$

where $\mathcal{A} : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function, satisfying suitable assumptions. The radial case has been studied by many authors, among them we refer to [27, 15].

We shall say that $\mathcal{A}$ is strongly p-coercive if

$$\begin{aligned} |\mathcal{A}(x,u,\eta)| &\leq K_1 |\eta|^{p-1}, \\ \mathcal{A}(x,u,\eta)\eta &\geq K_2 |\eta|^p . \end{aligned} \tag{4.3}$$

for some $K_1, K_2 > 0$, and for all $(x,u,\eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$. Up to some variants, the condition (4.3) is a classical frame for the study of quasilinear operators, see [29]. It implies the weak Harnack inequality, and hence the strong maximum principle.

We shall say that $\mathcal{A}$ is weakly p-coercive if

$$\mathcal{A}(x,u,\eta).\eta \geq K |\mathcal{A}(x,u,\eta)|^{p'} \tag{4.4}$$

for some $K > 0$, and for all $(x,u,\eta) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^N$. This condition (4.4) is clearly weaker than (4.3), and does not imply the Harnack inequality. It is satisfied in particular by the mean curvature operator $u \mapsto -\operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2})$ with $p = 2$.

Operators with a weak coercivity

For a general weakly p-coercive operator, first we can extend Theorem 3.4.

Theorem 4.1 ([9]) *Assume that $N \geq p > 1$, and $\mathcal{A}$ is weakly p-coercive. Let u be a nonnegative solution of (4.2) in Ω_i (resp. Ω_e). If $Q > p-1$, then (3.7) holds for small ρ (resp. for large ρ). If $Q < p-1$, for any $\ell > p-1-Q$, then*

$$\Big(\oint_{\mathcal{C}_{\rho/2,\rho}} u^\ell \Big)^{1/\ell} \geq C\rho^{-\Gamma}. \tag{4.5}$$

Proof. It is an extension of the one of theorem 3.4: we multiply the inequality by $u^\alpha \varphi$, where $1-p < \alpha < 0$, and φ is a test function, in order to get coercivity, then directly by φ. $\diamondsuit$

Then one can give nonexistence results in whole $\mathbb{R}^N$:

Theorem 4.2 ([9]) *Assume that $N \geq p > 1$, $Q > p-1$, and $\mathcal{A}$ is weakly p-coercive. If $Q \leq Q_{\sigma,p}$, there exists no nontrivial solution of (4.2) in $\mathbb{R}^N$.*

Proof. From the a priori estimate of theorem 4.1, one deduces

$$\int_{B_\rho} |x|^\sigma u^Q \leq C\rho^\theta$$

with $\theta = (N-p)(Q-Q_\sigma)/(Q-p+1) \leq 0$. If $\theta < 0$, then as $\rho \to +\infty$, we deduce that $\int_{\mathbb{R}^N} |x|^\sigma u^Q = 0$, hence $u \equiv 0$. If $\theta = 0$, then $|x|^\sigma u^Q \in L^1(\mathbb{R}^N)$, hence $\lim \int_{\mathcal{C}_{2^n,2^{n+1}}} |x|^\sigma u^Q = 0$. And we show that

$$\int_{B_{2^n}} |x|^\sigma u^Q \leq C\Big(\int_{\mathcal{C}_{2^n,2^{n+1}}} |x|^\sigma u^Q \Big)^{(p-1)/Q}$$

hence again $u \equiv 0$. $\diamondsuit$

For some weakly p-coercive operators which only depend on the gradient of u, we can also extend the nonexistence results in Ω_i and Ω_e.

Theorem 4.3 ([9]) *Assume that $\mathcal{A}(x,u,\eta) = A(|\eta|)\eta$, with $t \mapsto A(t)t$ nondecreasing and*

$$\begin{aligned} A(t) &\leq Mt^{p-2}, \quad \text{for } t > 0, \\ A(t) &\geq M^{-1}t^{p-2} \quad \text{for small } t > 0, \end{aligned} \tag{4.6}$$

for some $M > 0$. If $\sigma \leq -p$, there exists no nontrivial solution of (4.2) in Ω_i. If $Q \leq Q_{\sigma,p}$, there exists no nontrivial solution of (4.2) in Ω_e.

The result applies in particular to the mean curvature operator with $p = 2$.

Operators with a strong coercivity

For a general strongly p-coercive operator, one can give nonexistence results in $\mathbb{R}^N \setminus \{0\}$. The method is a combination of the two techniques of multiplication, either by $u^\alpha (\alpha < 0)$ or by $(k-u)^+ (k > 0)$.

Theorem 4.4 ([9]) *Assume that $N \geq p > 1$, $Q > p-1$, and$\mathcal{A}$ is weakly p-coercive. If $Q < Q_{\sigma,p}$, there exists no nontrivial solution of (4.2) in $\mathbb{R}^N \setminus \{0\}$.*

Moreover Theorems 3.3 and 3.6 extend completely, see [5] (with more general assumptions on $\mathcal{A}$) and [9]. This problem with $Q \geq Q_{0,p}$ for such operators is open.

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Marie-Françoise Bidaut-Véron
Laboratoire de Mathématiques et Physique Théorique
CNRS UMR 6063
Faculté des Sciences et Techniques
Parc de Grandmont
F 37200 Tours France
e-mail: veronmf@balzac.univ-tours.fr