USA-Chile Workshop on Nonlinear Analysis,
Electron. J. Diff. Eqns., Conf. 06, 2001, pp. 123–129.
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A remark on infinity harmonic functions *

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Abstract

A real-valued function u is said to be *infinity harmonic* if it solves the nonlinear degenerate elliptic equation $-\sum_{i,j=1}^{n} u_{x_1} u_{x_j} u_{x_i x_j} = 0$ in the viscosity sense. This is equivalent to the requirement that u enjoys comparison with cones, an elementary notion explained below. Perhaps the primary open problem concerning infinity harmonic functions is to determine whether or not they are continuously differentiable. Results in this note reduce the problem of whether or not a function u which enjoys comparison with cones has a derivative at a point x_0 in its domain to determining whether or not maximum points of u relative to spheres centered at x_0 have a limiting direction as the radius shrinks to zero.

1 Introduction

Let $U \subset \mathbb{R}^n$ be an open set and $u : U \to \mathbb{R}$. Then u is said to be *infinity harmonic* in U if it is continuous and satisfies

$$-\Delta_\infty u = -\sum_{i,j=1}^{n} u_{x_1} u_{x_j} u_{x_i x_j} = 0 \quad \text{in } U \tag{1}$$

in the sense of viscosity solutions of nonlinear possibly degenerate elliptic equations. We refer to Crandall, Evans and Gariepy [4] for terminology not defined here; however, the theory of viscosity solutions is not needed to read this paper. Indeed, this note employs only an elementary property which is shown to be equivalent to infinite harmonicity in [4]. One says that u enjoys comparison with cones from above in U if for every open set $V \Subset U$ (i.e., V is an open subset of U whose closure is a compact subset of U), $a, b \in \mathbb{R}$ and $y \in \mathbb{R}^n$ such that

$$u(x) \le a + b|x - y| \quad \text{for } x \in \partial(V \setminus \{y\})$$

one then has $u(x) \le a + b|x - y|$ for $x \in V$. Similarly, one defines comparison with cones from below and then defines "u enjoys comparison with cones" as the conjunction of the two one-sided notions. In [4] it is shown that $-\Delta_\infty u = 0$ in U in the viscosity sense if and only if u enjoys comparison with cones in U.

Mathematics Subject Classifications: 35D10, 35J60, 35J70.
Key words: infnity Laplacian, degenerate elliptic, regularity, fully nonlinear

Published January 8, 2001.
L.C.E. Supported by NSF Grant DMS-942342

We refer to [4] for orientation and references beyond the following remarks. The equation (1) arises as the Euler equation for the problem of minimizing $\|Du\|$ in the L^∞ norm subject to Dirichlet conditions. That is, (1) guarantees that if $V \Subset U$ and $u = v$ on ∂V, then $\|Du\|_{L^\infty(V)} \le \|Dv\|_{L^\infty(V)}$. We call this property (AML) for "absolutely minimizing Lipschitz". This aspect of (1) was introduced and studied by Aronsson [1], [2], [3] in the case of smooth functions. However, the nonsmooth function $u(x_1, x_2) = x_1^{4/3} - x_2^{4/3}$ solves (1) (equivalently, enjoys comparison with cones) in $\mathbb{R}^2$. This function has Hölder continuous first partial derivatives but is not twice differentiable along the lines $x_1 = 0, x_2 = 0$. This example, again due to Aronsson, sets limits on the reliability of results derived for (1) via pde arguments valid for smooth functions. For example, Aronsson showed that classical solutions of (1) in two variables are either constant or have a nowhere vanishing gradient. However, the example is not constant and the gradient vanishes at the origin. Classical solutions of (1) also have the property that $|D(u(X(t))|$ is constant when $dX(t)/dt = Du(X(t))$ (gradient flow). In fact, this property is easily seen to be equivalent to (1) for smooth functions. However $|Du|$ is not constant on gradient flow lines in the example.

R. Jensen [5] used approximation of (1) by the p-Laplace equation to prove the equivalence of (1) in the viscosity sense and the extremal property (AML). He proved as well that the Dirichlet problem for (1) was uniquely solvable. Jensen used approximation by the p-Laplacian in his remarkable analysis. A direct elementary proof that functions enjoying comparison with cones have the property (AML) was given in [4].

It remains a mystery as to whether or not solutions of (1) are everywhere differentiable. Our main results imply that u can only fail to be differentiable via complex phenomena — roughly speaking, simple bad behaviors that do not scale to planes are ruled out. More precisely, we prove that if u enjoys comparison with cones in U, $x_0 \in U$, $r_j \downarrow 0$ and

$$v(x) = \lim_{j\to\infty} \frac{u(r_j x + x_0) - u(x_0)}{r_j} \tag{2}$$

holds locally uniformly in $\mathbb{R}^n$, then v is linear.

As a consequence of the above and arguments in the proof, we establish an equivalent condition to differentiability of u at x_0. Suppose x_0 is not a point at which $Du(x_0)$ exists and $Du(x_0) = 0$. Then u is differentiable at x_0 if and only if the following condition holds: if $|z_r - x_0| = r$ and $u(z_r) = \max_{\{|z-x_0|=r\}} u(z)$ for small $r > 0$, then

$$\lim_{r\downarrow 0} \frac{z_r - x_0}{r} \quad \text{exists.} \tag{3}$$

We reduce the problem of showing u is differentiable at x_0 to verifying (3). However, we have not yet determined if (3) is always satisfied.

2 The Proofs

We employ the notation

$$S_r(x_0) = \{z \in \mathbb{R}^n;\ |z - x_0| = r\}$$

for the sphere of radius r centered at the x_0.

Let u enjoy comparison with cones in the open subset U of $\mathbb{R}^n$. We note that only this simple property is needed in all that follows. The equivalence with the satisfaction of the equation is not needed, nor are devices more conveniently dealt with in terms of the equation (changes of variables, etc.). It follows from Lemma 2.5 of [4] that u is locally Lipschitz continuous. The functions

$$L_r^+(y) = \max_{z \in S_r(y)} \frac{u(z) - u(y)}{r} \quad \text{and} \quad L_r^-(y) = \min_{z \in S_r(y)} \frac{u(z) - u(y)}{r}$$

are defined for $y \in U$ and $r < \operatorname{dist}(y, \partial U)$ as are the limits

$$L^+(y) = \lim_{r \downarrow 0} L_r^+(y) \quad \text{and} \quad L^-(y) = \lim_{r \downarrow 0} L_r^-(y). \tag{4}$$

The latter quantities are well defined by Lemma 2.4 of [4], which establishes that $L_r^+(y)$ (respectively, $L_r^-(y)$) is nonnegative (respectively, nonpositive) and nondecreasing (respectively, nonincreasing) in r. Moreover, by Lemma 2.7 of [4]

$$L^+(y) = -L^-(y). \tag{5}$$

We will show that if $0 \in U$, $r_j \downarrow 0$ and $v \in C(\mathbb{R}^n)$ is the locally uniform limit

$$v(x) = \lim_{j \to \infty} \frac{u(r_j x) - u(0)}{r_j} \tag{6}$$

then v is necessarily linear. Note that as u is locally Lipschitz continuous, any sequence $R_k \downarrow 0$ has a subsequence $r_j \downarrow 0$ for which the limit (6) is defined on $\mathbb{R}^n$ and the convergence is uniform on any bounded subset of $\mathbb{R}^n$.

Define

$$L_0 = L^+(0) = -L^-(0).$$

This number is computed from the original function u. Let v be given by (6). It is obvious that locally uniform limits of functions enjoying comparison with cones likewise enjoy comparison with cones, so v enjoys comparison with cones. Hereafter the quantities

$$L_r^+(y) = \max_{z \in S_r(y)} \frac{v(z) - v(y)}{r} \quad \text{and} \quad L_r^-(y) = \min_{z \in S_r(y)} \frac{v(z) - v(y)}{r} \tag{7}$$

are computed from v. Correspondingly, $L^+(y), L^-(y)$ are now defined from v and not u.

To see that v is linear, we will prove that

$$-L_r^-(y),\ L_r^+(y) \le L_0 \quad \text{for } y \in \mathbb{R}^n,\ 0 < r, \tag{8}$$

and

$$-L^-(0) = L_0 = L^+(0). \tag{9}$$

Let us first show why (8), (9) imply that v is linear. First (8), (9) and $L^+(y) \le L_r^+(y)$ together with Lemma 2.7 (ii) of [4] imply that

$$L_0 \text{ is the least Lipschitz constant for } v. \tag{10}$$

Next, let $z_r^\pm \in S_r(0)$ be such that

$$L_r^+(0) = \frac{v(z_r^+) - v(0)}{r} = \frac{v(z_r^+)}{r}, \quad L_r^-(0) = \frac{v(z_r^-) - v(0)}{r} = \frac{v(z_r^-)}{r}. \tag{11}$$

Since $L_0 \le L_r^+(0)$ and $L_0 \le -L_r^-(0)$ because of (9) and monotonicity, it follows from (8) and (11) that

$$L_r^+(0) = -L_r^-(0) = L_0 = \frac{v(z_r^+) - v(z_r^-)}{2r}. \tag{12}$$

The function

$$g(t) = v(z_r^- + t(z_r^+ - z_r^-)) - v(z_r^-)$$

is Lipschitz with constant $L_0|z_r^+ - z_r^-|$ by (10) and satisfies

$$|g(1) - g(0)| = |g(1)| = |v(z_r^+) - v(z_r^-)| = 2rL_0$$

by (12). Hence $2rL_0 \le L_0|z_r^+ - z_r^-|$ and this implies that

$$z_r^+ = -z_r^- \tag{13}$$

since both points lie in $S_r(0)$. Moreover, $|g(1) - g(0)|$ is then the Lipschitz constant for g, which forces g to be linear for $0 \le t \le 1$. We conclude that

$$v(z_r^- + t(z_r^+ - z_r^-)) - v(z_r^-) = L_0 2rt \quad \text{for } 0 \le t \le 1.$$

The net result of this analysis is that for each $r > 0$ there is a unique point $z_r^+ \in S_r(0)$ such that v is linear on the line segment joining $-z_r^+$ and z_r^+ and achieves its Lipschitz constant on these segments. The uniqueness follows from the result (13). Then all the points z_r^+ lie on a common line and thus there is a line through the origin on which v is linear and attains its Lipschitz constant. This forces v to be linear. Indeed, we may change coordinates to assume that the line through the origin is along the first coordinate axis $(x_1, 0, \cdots, 0)$, $-\infty < x_1 < \infty$. That is, we assume that $v(x_1, 0, \ldots, 0) - L_0 x_1 \equiv 0$. Write (x_1, y) for a general point in $\mathbb{R}^n$; "y" contains the last $n-1$ coordinates. Consider $w(x_1, y) = v(x_1, y)/L_0$, which has 1 as a Lipschitz constant and satisfies $w(x_1, 0) \equiv x_1$. Then we have

$$\begin{aligned} |w(x_1, y) - w(s, 0)|^2 &= |w(x_1, y) - x_1 + x_1 - w(s, 0)|^2 \\ &= |w(x_1, y) - x_1 + x_1 - s|^2 \\ &= |w(x_1, y) - x_1|^2 + 2(x_1 - s)(w(x_1, y) - x_1) + |x_1 - s|^2 \\ &\le |x_1 - s|^2 + |y|^2. \end{aligned} \tag{14}$$

Here the last inequality is due to 1 being a Lipschitz constant for w:

$$|w(x_1,y)-w(s,0)|^2 \le |(x_1,y)-(s,0)|^2.$$

We conclude that $2(x_1-s)(w(x_1,y)-x_1) \le |y|^2$ where s is free. This can only be if $w(x_1,y)-x_1 \equiv 0$.

Let $\langle\cdot,\cdot\rangle$ be the Euclidean inner-product. For later use, let us observe that if $v(x)=\langle p,x\rangle$ then $|p|$ is determined by (10):

$$|p| = L_0 = \lim_{r\downarrow 0}\max_{|z|=r}\frac{u(z)-u(0)}{r}. \tag{15}$$

Remark: The proof just given is essentially a special case of the proof of Lemma 4.2 of [4]. There is is shown that if w has 1 as a Lipschitz constant then $x_1 \to w(x_1,y)-x_1$ is nonincreasing and its limits as $x_1\to\pm\infty$ are independent of y. In our case, when $y=0$ these limits are 0. In view of the monotonicity, we then have $w(x_1,y)-x_1\equiv 0$, again establishing the linearity.

It remains to prove (8), (9). We treat the case of the superscript "+". First fix y and consider $z\in S_r(y)$ such that

$$L_r^+(y) = \frac{v(z)-v(y)}{r} = \lim_{j\to\infty}\frac{u(r_jz)-u(r_jy)}{r_jr}.$$

We have $r_jz \in S_{r_jr}(r_jy)$ and then

$$\frac{u(r_jz)-u(r_jy)}{r_jr} \le L^+_{u,r_jr}(r_jy) \le L^+_{u,R}(r_jy) \tag{16}$$

for $r_jr < R < \operatorname{dist}(r_jy,\partial U)$. Here the notation is reflecting that we are computing quantities associated with u and the final inequality is due to monotonicity. However, $L^+_{u,R}(r_jy)$ is continuous in r_jy so combining (1) and (16) we find

$$L_r^+(y) \le \lim_{j\to\infty} L^+_{u,R}(r_jy) = L^+_{u,R}(0)$$

and in the limit $R\downarrow 0$ (8) follows. Finally, if we take $y=0$ then for $z\in S_r(0)$

$$L_r^+(0) \ge \max_{z\in S_r(0)}\frac{v(z)}{r} = \lim_{j\to\infty}\max_{z\in S_r(0)}\frac{u(r_jz)-u(0)}{rr_j} = \lim_{j\to\infty} L^+_{u,r_jr}(0) = L_0.$$

Thus (9) holds.

Remark: Given a Lipschitz continuous function u such that every limit (2) is linear, the most optimistic among us might hope that u is then necessarily differentiable. However, this is not so. D. Priess provided us with a classical counterexample in $\mathbb{R}$: $u(x) = x\sin(\log(|\log(|x|)|))$. The function u is differentiable except at $x=0$ and it is Lipschitz continuous near $x=0$, which is all that matters here. All subsequential limits of the quotients $u(rx)/r$ as $r\downarrow 0$ are linear, but the limiting slope can be arranged to be any number between -1 and 1.

We have still to establish that (3) characterizes differentiability of u at points x_0 where u does not have a vanishing derivative. Assume $x_0 = 0$ and let $L_0 = L^+(0)$ be as above. A first point is that if $Du(0)$ exists and $Du(0) = 0$, then $L^+(0) = 0$ is immediate. Conversely, if $L^+(0) = 0$ then

$$\frac{|u(x) - u(0)|}{|x|} \le \max\left(L^+_{|x|}(0), -L^-_{|x|}(0)\right) \le \max\left(L^+_r(0), -L^-_r(0)\right)$$

for $0 < |x| < r$ and $L^+_r(0), -L^-_r(0) \to L_0 = 0$ shows that $Du(0) = 0$. Thus $L^+(0) = L_0 > 0$ is assumed throughout what follows.

In general, if u is differentiable at 0 and $p = Du(0) \neq 0$ we have

$$u(x) = u(0) + \langle p, x\rangle + o(|x|) \quad \text{as } x \to 0; \tag{17}$$

If z_r is a maximum point of u on the sphere $|x| = r$ then $u(x) \le u(z_r)$ for $|x| = r$ and so, by (17),

$$\langle p, x\rangle \le \langle p, z_r\rangle + o(r) \quad \text{for } |x| = r$$

and elementary considerations show that this implies $z_r/r \to p/|p|$ as $r \downarrow 0$. For the converse, if $r_j \downarrow 0$ and

$$v(x) = \lim_{j\to\infty} \frac{u(r_j x) - u(0)}{r_j} \tag{18}$$

holds locally uniformly, it was proved above v that is linear. According to (15) and current assumptions v is not constant. Say $v(x) = \langle p, x\rangle$. If we show that p is independent of the sequence $r_j \downarrow 0$ the result follows. Indeed, then a standard compactness argument shows that

$$v(x) = \langle p, x\rangle = \lim_{r\downarrow 0} \frac{u(rx) - u(0)}{r} \text{ holds locally uniformly.}$$

That is, $Du(0)$ exists and $Du(0) = p$.

We already know by (15) that $|p|$ is unique. Let z_r be any maximum point for u on the sphere of radius r centered at the origin and assume that then

$$\lim_{r\downarrow 0} \frac{z_r}{r} = \omega. \tag{19}$$

Of course, we show $p = |p|\omega$ to complete the argument. Now $x = z_{r_j}/r_j$ is a maximum point for the function on the right of (18) relative to the sphere $|x| = 1$. Using (19) to pass to the limit in the relation

$$\frac{u(r_j x) - u(0)}{r_j} \le \frac{u(z_{r_j}) - u(0)}{r_j} = \frac{u\left(r_j \frac{z_{r_j}}{r_j}\right) - u(0)}{r_j} \quad \text{for } |x| = 1$$

we find $v(x) \le v(\omega)$ for $|x| = 1$. As $v(x) = \langle p, x\rangle$ we conclude that $p = |p|\omega$. This completes the argument.

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