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Behavior of positive radial solutions of a quasilinear equation with a weighted Laplacian *

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Abstract

We obtain a classification result for positive radially symmetric solutions of the semilinear equation

$$-\operatorname{div}(\tilde{a}(|x|)\nabla u) = \tilde{b}(|x|)|u|^{\delta-1}u,$$

on a punctured ball. The weight functions $\tilde{a}$ and $\tilde{b}$ are C^1 on the punctured ball, are positive and measurable almost everywhere, and satisfy certain growth conditions near zero.

1 Introduction

In this work we study the behavior of positive solutions to

$$-\operatorname{div}(\tilde{a}(|x|)\nabla u) = \tilde{b}(|x|)|u|^{\delta-1}u, \quad x \in B^*_{r_0}(0), \quad r_0 > 0, \tag{1.1}$$

near an isolated singularity at the origin. Here $\delta > 1$, $B^*_{r_0}(0)$ is the punctured ball $B_{r_0}(0)\setminus\{0\}$, and $\tilde{a}, \tilde{b}$ are weight functions which are positive and measurable. Many authors have dealt with the non weighted case, i.e., with positive solutions to the equation

$$-\operatorname{div}(\nabla u) = |u|^{\delta-1}u, \text{in } \Omega \subseteq \mathbb{R}^N, \tag{1.2}$$

where $\delta > 1$, $2 \le N$, and $\Omega \subseteq \mathbb{R}^N$ or $\Omega \subseteq \mathbb{R}^N \setminus \{0\}$ is a smooth domain, bounded or unbounded.

When $N > 2$, appear two critical values: $\delta = \frac{N}{N-2}$, and $\delta = \frac{N+2}{N-2}$. The first results for this case were obtained by Emden, and then Fowler [5, 6, 7], where existence results are given as well as a complete classification of the global solutions in $\mathbb{R}^N$ or $\mathbb{R}^N \setminus \{0\}$, in the radial situation.

Lions [12] studied the nonradial case for the behavior near 0 when $\delta < N/N-2$, Avilés [1] did so for $\delta = N/(N-2)$. Then Gidas and Spruck [9], studied the problem for $\delta < (N+2)/(N-2)$ and established local and global results.

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A generalization of the Sobolev exponent $(N+2)/(N-2)$ to a corresponding one for the problem with weights considered in (1.1) has been recently done in [11], where a more general situation of the p-laplacian operator is treated.

Summarizing the above mentioned papers on the behavior of positive solutions to (1.1), the classification result for radial solutions as δ crosses the value $\frac{N}{N-2}$ which we call *Serrin's number*, is as follows: if u is a positive unbounded (near 0) radially symmetric solution to (1.2) defined in a neighborhood of the origin and $2 < N$, then

$$u(r) \approx \begin{cases} r^{2-N} & \text{when } \delta < N/(N-2) \\ r^{2-N}|\log r|^{\frac{N-2}{2}} & \text{when } \delta = \frac{N}{N-2} \\ r^{\frac{-2}{\delta-1}} & \text{when } \frac{N}{N-2} < \delta < \frac{N+2}{N-2}. \end{cases}$$

We are concerned here with the generalization of some of those results for the general equation (1.1).

The radial version of this problem is

$$-(a(r)u')' = b(r)|u|^{\delta-1}u, \quad r \in (0, r_0), \tag{1.3}$$

where $|x| = r$ and now $a(r) := r^{N-1}\tilde{a}(r)$ and $b(r) := r^{N-1}\tilde{b}(r)$ are positive functions satisfying some regularity and growth conditions near 0. Although it is not necessary at all steps, we will assume that

(H1) $a, b \in C^1((0,1), \mathbb{R}^+)$

(H2) $b \in L^1(0,1)$,

(H3) $1/a \notin L^1(0,1)$,

where $\mathbb{R}^+ = (0, \infty)$. From (H1) and (H2), we have that

$$B(r) := \int_0^r b(t)dt, \quad h(r) := \int_r^1 \frac{1}{a}(t)dt \tag{1.4}$$

are well defined, and from (H3),

$$\lim_{r\to 0^+} h(r) = +\infty, \tag{1.5}$$

i.e., h is unbounded near 0.

We note that h is a solution to

$$-(a(r)u')' = 0, \quad r \in (0,1),$$

for this reason we call h a *fundamental solution* to the weighted Laplacian.

By a solution to (1.3) we understand an absolutely continuous function u in the open interval $(0, r_0)$ such that $a(r)u'$ is also absolutely continuous in the open interval $(0, r_0)$. Also, we will say that u is singular if

$$\limsup_{r\to 0^+} u(r) = +\infty.$$

Remark 1.1 Since we are interested in characterizing the behavior of *positive* solutions near an isolated singularity at $r = 0$, we shall see that neither (H2) nor (H3) are really restrictions to our problem. Indeed, if u is a positive singular solution to (1.3), then $-a(r)u'$ is a monotone increasing function and as such, it has a limit ℓ as $r \to 0$. It can be easily checked that if this limit is negative, then the solution is bounded near 0, and thus, we must have that $\ell \geq 0$. Therefore, we obtain that the solution u is decreasing, i.e., $u'(r) < 0$ for $r \in (0, r_0)$ and for $0 < s < r < r_0$,

$$a(r)|u'(r)| \geq a(r)|u'(r)| - a(s)|u'(s)| \geq \int_s^r b(t)u^\delta(t)dt \geq u^\delta(r)\int_s^r b(t)\,dt$$

implying that (H2) must hold. Furthermore,

$$a(r)|u'| \leq \ell, \quad r \in (0, r_0),$$

which yields

$$u(r) \leq u(r_0) \leq \ell(h(r) - h(r_0)),$$

implying that $1/a \notin L^1(0,1)$.

Theorem 1.1 *Let $\delta \in \mathbb{R}$ be such that $\delta > 1$, and let the weight functions a, b satisfy (H1), (H2), (H3), and*

(H4) *There exists a finite number $\tilde{\delta} > 1$, such that $\int_0 b(r)(h(r))^{\tilde{\delta}}\,dr < \infty$.*

Let u be a positive singular solution to (1.3). Then, there exists a positive extended real number $\mathcal{S}$ (which we call the generalized Serrin's number) such that

(i) If $1 < \delta < \mathcal{S}$, then

$$\lim_{r\to 0^+} \frac{u(r)}{h(r)} > 0.$$

Also, if $\delta > \mathcal{S}$, then u cannot be of fundamental type, i.e.,

$$\lim_{r\to 0^+} \frac{u(r)}{h(r)} = 0.$$

Assume next that $\mathcal{S} < \infty$.

(ii) If $\int_0 b(r)(h(r))^{\mathcal{S}}dr < \infty$, then u is also of fundamental type when $\delta = \mathcal{S}$.

Assume further that there exist positive constants c_1 and c_2 such that

$$c_1 \leq B(r)(h(r))^{\mathcal{S}} \leq c_2 \quad \text{for all } r \in (0, r_0), \tag{1.6}$$

and the mapping

$$r \mapsto \frac{b(r)(h(r))^{\mathcal{S}+1}}{|h'(r)|} \quad \text{is monotone in } (0, r_0). \tag{1.7}$$

(iii) Then for $\mathcal{S} < \delta$ and $\delta \neq 2\mathcal{S} - 1$, it holds that

$$\lim_{r\to 0^+} \frac{u(r)}{h(r)^{\frac{\mathcal{S}-1}{\delta-1}}} > 0.$$

Remark 1.2 We recall that in the non weighted case, i.e., the case when $a(r) = b(r) = r^{N-1}$, $N > 2$, we have

$$h(r) = \int_r^1 s^{1-N} ds = r^{2-N}\Big(\frac{1-r^{N-2}}{N-2}\Big), \quad B(r) = \frac{r^N}{N}, \quad \text{and } \mathcal{S} = \frac{N}{N-2}.$$

Also, in this case

$$B(r)(h(r))^{\mathcal{S}} = \frac{1}{N}\Big(\frac{1-r^{N-2}}{N-2}\Big)^{\frac{N}{N-2}}, \quad \frac{b(r)(h(r))^{\mathcal{S}+1}}{|h'(r)|} = \Big(\frac{1-r^{N-2}}{N-2}\Big)^{\frac{N}{N-2}+1}.$$

Thus our assumptions (1.6) and (1.7) are satisfied in that case. In fact, it can be easily shown that these assumptions always hold when the weights are powers near the origin.

Remark 1.3 We note here, that as a first striking difference with the non weighted case, the solutions can behave like the fundamental solution at the critical number $\mathcal{S}$, see example 1 in section 5.

Remark 1.4 As it is stated in the theorem, the number $\mathcal{S}$ can be infinity. This of course happens in the non weighted case when $N = 2$. Nevertheless, in this more general case, it can happen in different situations, see example 2 in section 5.

To prove parts (i) and (ii) of the theorem, as in [10], we think of the critical number $\mathcal{S}$ as the limiting value of δ so that a singular solution behaves like the fundamental solution. (We can show that thanks to assumption (H_4), there exists at least one value of δ with that property). Then, we make an appropriate change of variable, (which corresponds to the one used in [8] in the non weighted case) to study the behavior when it is not of the fundamental type.

The organization of this paper is as follows. In section 2 we prove some preliminary results concerning a priori bounds for the positive solutions to (1.3), some of them can also be found in [4], where the authors establish nonexistence results for an equation containing a more general non-homogeneous operator. In section 3, we find the critical number $\mathcal{S}$ and we prove parts (i) and (ii) of Theorem 1.1. Then in section 4 we prove part (iii). We do this by following the idea in [2] and a regularity result proved in [10], see also [11]. Finally in section 5 we give some examples to illustrate the main differences with respect to the non weighted case.

2 Preliminary Results

We start this section by proving some basic facts concerning positive solutions to (1.3). As we pointed out in the introduction, if u is a positive singular solution to (1.3), then $u'(r) < 0$ for $r \in (0, r_0)$, $\lim_{r\to 0} a(r)|u'(r)| = \ell$ exists and $\ell \geq 0$. Therefore, by L'Hospital's rule, also $\lim_{r\to 0} u(r)/h(r)$ exists (and it is equal to

ℓ). Moreover, we will prove next that u/h is in fact monotone increasing in some right neighborhood of zero (see also [4]).

Lemma 2.1 *Let the weight functions a, b satisfy assumptions (H1), (H2), and (H3), and let u be a positive singular solution to (1.3) such that*

$$\lim_{r\to 0} a(r)|u'(r)| = 0.$$

Then, there exists $r_* \in (0, r_0)$ such that u/h is monotone increasing on $(0, r_*)$.

Proof. The result follows easily by making the change of variable

$$s = \frac{1}{h(r)}, \quad v(s) := su(r).$$

We observe that v turns out to be concave with $v(0) = 0$, and thus, since it is a positive function, it has to be increasing near zero. □

Next we find an a-priori estimate for the growth of u near zero. We have.

Lemma 2.2 *Let the weight functions a, b satisfy assumptions (H1), (H2), and (H3), and let u be a positive singular solution to (1.3) such that $\lim_{r\to 0} a(r)|u'(r)| = 0$. Then*

$$u^{\delta-1}(r) \le (B(r))^{-1}(h(r))^{-1} \quad \text{for all } r \in (0, r_*), \tag{2.1}$$

where r_* is given in Lemma 2.1.

Proof. Let u be a positive singular solution to (1.3). By Lemma 2.1, we have that $|u'| \le |h'|(u/h)$ on $(0, r_*)$, and thus, using that u is decreasing on $(0, r_0)$, we find that

$$a(r)|h'|\frac{u(r)}{h(r)} \ge a(r)|u'| = \int_0^r b(t)u^\delta(t)dt \ge B(r)u^\delta(r),$$

and the result follows by observing that $a(r)|h'| \equiv 1$. □

Remark 2.1 Note that in the non weighted case this last lemma establishes the well known result

$$u(r) \le C\ r^{\frac{-2}{\delta-1}} \quad \text{for small } r > 0.$$

We finish this section by recalling a regularity result from [11].

Lemma 2.3 *Assume that the weight functions a, b satisfy assumptions (H1)-(H4) and let $1 < \delta$ be such that (1.6) holds. Moreover, assume that there exists a nonnegative function $\nu \in C^1(0, r_0)$ such that $a\nu^\delta \in L^1(0, r_0)$ and*

$$-a(r)\nu'(r) \ge K \int_0^r b(t)(\nu(t))^\delta dt, \quad r \in (0, \overline{r}_0), \tag{2.2}$$

for some positive constant K, and some $\overline{r}_0 \in (0, r_0)$. Let u be a positive solution to the equation in (1.3) which is defined on a right neighborhood of 0 and satisfies

$$\lim_{r\to 0}\frac{u(r)}{\nu(r)} = 0. \tag{2.3}$$

Then, u is a bounded solution.

Proof. The proof is rather technical and it consists in proving that there exist an interval $(0, r_*)$, a positive constant C, and a sequence $\{\epsilon_n\}$ tending to 0 as $n \to \infty$, such that

$$u(r) \le \epsilon_n \nu(r) + C \quad \text{for all } r \in (0, r_*),$$

from where the result follows by letting $n \to \infty$. Since it is lengthy and a similar version can be found also in [10], where the non weighted case, but for a non-homogeneous operator is treated, we omit it.

3 Definition of $\mathcal{S}$ and proof of Theorem 1.1

We first observe that a necessary condition for a positive singular solution to (1.3) to behave like the fundamental solution u is that

$$\int_0 b(t)(h(t))^\delta dt < \infty. \tag{3.1}$$

Indeed, this comes from the fact that

$$a(r)|u'(r)| \ge \int_0^r b(t)u^\delta(t)dt,$$

and thus, if $u(r) \ge Ch(r)$ for r small enough, then (3.1) follows.

Let us set

$$\mathcal{W} := \{\delta > 1 \;:\; \int_0 b(t)(h(t))^\delta dt < \infty\}.$$

Thanks to hypothesis (H_4), we have that $\mathcal{W} \ne \emptyset$, and thus we may define

$$\mathcal{S} := \sup \mathcal{W}. \tag{3.2}$$

Proof of Theorem 1.1 parts (i)-(ii).

Let $\mathcal{S}$ be defined as in (3.2). Since $\lim_{r\to 0}\frac{u(r)}{h(r)}$ exists, we only have to prove that if u is a positive singular solution to (1.3), then this limit cannot be 0. We will argue by contradiction and thus assume that $a(r)|u'(r)| \to 0$ as $r \to 0$. Then, by lemma 2.1 there is $r_* \in (0, r_0)$ such that u/h is increasing on $(0, r_*)$. Let us first prove (i), i.e., assume $\delta \in (1, \mathcal{S})$. Then by the definition of $\mathcal{S}$, it holds that $\int_0 b(t)h^\delta(t)dt < \infty$ and thus, given $\varepsilon > 0$, there exists $r_1 \in (0, r_*)$ such that

$$\int_0^r b(t)h^\delta(t)dt < \varepsilon \quad \text{for all } r \in (0, r_1), \tag{3.3}$$

and since we are assuming that $u/h \to 0$ as $r \to 0$, we may assume that r_1 is small enough to have

$$\frac{u(r_1)}{h(r_1)} < \varepsilon^{1-\delta}. \tag{3.4}$$

Then, by the monotonicity of u/h and (3.3) we have that

$$a(r)|u'(r)| = \int_0^r b(t)h^\delta(t)\Big(\frac{u(t)}{h(t)}\Big)^\delta dt < \Big(\frac{u(r)}{h(r)}\Big)^\delta \varepsilon,$$

implying that

$$|u'(r)|u^{-\delta}(r) \leq \varepsilon|h'(r)|h^{-\delta}(r).$$

By integrating this inequality over (r, r_1), we find that

$$u^{1-\delta}(r_1) - u^{1-\delta}(r) < \varepsilon h^{1-\delta}(r_1), \quad r \in (0, r_1),$$

which is equivalent to

$$u^{1-\delta}(r_1) - \varepsilon h^{1-\delta}(r_1) < u^{1-\delta}(r), \quad r \in (0, r_1).$$

Hence, from (3.4) we deduce that

$$u^{\delta-1}(r) < (u^{1-\delta}(r_1) - \varepsilon h^{1-\delta}(r_1))^{-1},$$

contradicting the unboundedness of u near 0. Hence, we must have that

$$\lim_{r\to 0^+} \frac{u(r)}{h(r)} > 0.$$

Next we observe that as we mentioned at the beginning of this section, a necessary condition for u to behave like the fundamental solution near 0 is that $\int_0 b(t)h^\delta(t)dt < \infty$ and thus $\delta \leq \mathcal{S}$, i.e., if $\delta > \mathcal{S}$, then $\lim_{r\to 0}(u/h)(r) = 0$.

To prove (ii), we note that the assumption is equivalent to saying $\mathcal{S} \in \mathcal{W}$, and thus the result follows in the same way as above.

4 Proof of Theorem 1.1 part (iii)

In this section we treat the case $\delta > \mathcal{S}$. We use a similar argument to the one used in [2]. We do this by considering the following change of variable

$$t = \mu \log(h(r)), \quad v(t) = \frac{u(r)}{(h(r))^{\frac{\mathcal{S}-1}{\delta-1}}},$$

where μ could be any positive constant but will be chosen later to compare with the non weighted case.

Proof of (iii). ¿From (2.1) in lemma 2.2, and the assumption $c_1 \leq B(r)h^{\mathcal{S}}(r)$ for all r sufficiently small in (1.6), we have that

$$u(r) \leq Ch^{\frac{\mathcal{S}-1}{\delta-1}}(r) \quad \text{for } r \text{ small enough,}$$

and hence v is bounded by an absolute constant not depending on u.

Also, it can be readily verified that

$$\dot{v} + \frac{\theta}{\mu}v = \frac{u'(r)h^{(1-\theta)}(r)}{\mu h'(r)} > 0, \quad \text{for all } r \in (0, r_0), \tag{4.1}$$

where $\dot{} = \frac{d}{dt}$, $' = \frac{d}{dr}$, and $\theta := \frac{\mathcal{S}-1}{\delta-1}$. We conclude then, by using again the monotonicity of u/h, i.e., that $\frac{|u'(r)|}{|h'(r)|} \leq \frac{u(r)}{h(r)}$ for r small, that

$$|\dot{v}(t)| \leq \frac{\theta+1}{\mu}v(t)$$

and thus $|\dot{v}|$ is also bounded by an absolute constant. Finally, by differentiating (4.1) with respect to r and using the equation in (P_r), we see that (1.3) transforms into

$$\ddot{v} + (\frac{2\theta}{\mu} - \frac{1}{\mu})\dot{v} - \frac{(1-\theta)\theta}{\mu^2}v = -\frac{b(r)(h(r))^{\mathcal{S}+1}}{\mu^2|h'(r)|}v^{\delta}, \quad t \geq t_0.$$

To simplify our writing we will set $q = \frac{\theta}{\mu}$ and re-write this equation as

$$\ddot{v} + (2q - \frac{1}{\mu})\dot{v} - (\frac{1}{\mu} - q)qv = -\frac{b(r)(h(r))^{\mathcal{S}+1}}{\mu^2|h'(r)|}v^{\delta}, \quad t \geq t_0. \tag{4.2}$$

By multiplying the equation in (P_∞) by $\dot{v}$, we find that

$$\begin{aligned}&\frac{d}{dt}\frac{\dot{v}^2}{2} + (2q - \frac{1}{\mu})\dot{v}^2 - (\frac{1}{\mu} - q)q\frac{d}{dt}\frac{v^2}{2} \\ &\quad = \quad -\frac{d}{dt}\frac{b(r)(h(r))^{\mathcal{S}+1}}{\mu^2|h'(r)|}\frac{v^{\delta+1}}{\delta+1} + \frac{v^{\delta+1}}{\delta+1}\frac{d}{dt}\frac{b(r)(h(r))^{\mathcal{S}+1}}{\mu^2|h'(r)|},\end{aligned}$$

or equivalently,

$$\begin{aligned}(2q - \frac{1}{\mu})\dot{v}^2 \quad = \quad & \frac{d}{dt}\Big((\frac{1}{\mu} - q)q\frac{v^2}{2} - \frac{\dot{v}^2}{2} - \frac{b(r)(h(r))^{\mathcal{S}+1}}{\mu^2|h'(r)|}\frac{v^{\delta+1}}{\delta+1} \\ & + \int_{t_0}^{t}\frac{v^{\delta+1}}{\delta+1}\frac{d}{ds}\frac{b(r)(h(r))^{\mathcal{S}+1}}{\mu^2|h'(r)|}ds\Big).\end{aligned} \tag{4.3}$$

Note that

$$C := 2q - \frac{1}{\mu} = \frac{1}{\mu}\Big(\frac{2\mathcal{S} - (\delta+1)}{\delta-1}\Big) \neq 0 \tag{4.4}$$

by assumption. We will prove next that $\frac{bh^{\mathcal{S}+1}}{|h'|}$ is bounded. Indeed, since by assumption (1.7) this function is monotone, its limit as $r \to 0$ exists. Let $\{r_n\}$ be a sequence of positive numbers such that $r_n \to 0$ as $n \to \infty$ and such that

$$\liminf_{r\to 0} \frac{b(r)h(r)}{B(r)|h'(r)|} = \lim_{n\to\infty} \frac{b(r_n)h(r_n)}{B(r_n)|h'(r_n)|}. \tag{4.5}$$

Then, by using assumption (1.6), (4.5), and by L'Hospital's rule, we have that

$$\begin{aligned}
\lim_{r\to 0} \frac{b(r)h^{\mathcal{S}+1}(r)}{|h'(r)|} &= \lim_{n\to\infty} \frac{b(r_n)h^{\mathcal{S}+1}(r_n)}{|h'(r_n)|} \\
&= \lim_{n\to\infty} \frac{b(r_n)h(r_n)}{B(r_n)|h'(r_n)|} B(r_n)h^{\mathcal{S}}(r_n) \\
&\le c_2 \lim_{n\to\infty} \frac{b(r_n)h(r_n)}{B(r_n)|h'(r_n)|} \\
&= c_2 \liminf_{r\to 0} \frac{b(r)h(r)}{B(r)|h'(r)|} \\
&\le c_2 \liminf_{r\to 0} \frac{|\log(B(r))|}{\log(h(r))}.
\end{aligned}$$

We claim that it must be that

$$\liminf_{r\to 0} \frac{|\log(B(r))|}{\log(h(r))} < \infty. \tag{4.6}$$

Indeed, assume on the contrary that this lim inf is equal to ∞. Then, given any $M > 0$, and in particular $M > \mathcal{S}$, there is $r_* > 0$ such that $|\log B(r)| > \log h^M(r)$ for all $r \in (0, r^*)$, hence $B(r)h^M(r) \le 1$ for all $r \in (0, r^*)$. But from the left hand side inequality in (1.6), we conclude that

$$c_1(h(r))^{M-\mathcal{S}} \le 1 \quad \text{for all } r \in (0, r^*),$$

contradicting (H3). Thus we find that there is a positive constant K such that

$$\frac{b(r)h^{\mathcal{S}+1}(r)}{|h'(r)|} \le K < \infty \quad \text{for } r \text{ small enough}, \tag{4.7}$$

proving our assertion and implying in particular that

$$(\frac{1}{\mu} - q)q\frac{v^2}{2} - \frac{\dot{v}^2}{2} - \frac{b(r)(h(r))^{\mathcal{S}+1}}{\mu^2|h'(r)|}\frac{v^{\delta+1}}{\delta+1} \tag{4.8}$$

is bounded.

Next we observe that also from (4.3),

$$\begin{aligned}
F(t) \quad := \quad & (\frac{1}{\mu} - q)q\frac{v^2}{2} - \frac{\dot{v}^2}{2} - \frac{b(r)(h(r))^{\mathcal{S}+1}}{\mu^2|h'(r)|}\frac{v^{\delta+1}}{\delta+1} \\
& + \int_{t_0}^{t} \frac{v^{\delta+1}}{\delta+1}\frac{d}{ds}\frac{b(r)(h(r))^{\mathcal{S}+1}}{\mu^2|h'(r)|}\, ds
\end{aligned} \tag{4.9}$$

is monotone (increasing or decreasing according to whether C is negative or positive) and thus it has a limit as $t \to \infty$. We will prove next that this limit is finite. Clearly, from (4.8), we only have to establish the convergence of the integral

$$\int_{t_0}^{t} \frac{v^{\delta+1}}{\delta+1} \frac{d}{ds} \frac{b(r)(h(r))^{\mathcal{S}+1}}{\mu^2 |h'(r)|} ds.$$

But this follows directly from (1.7), the boundedness of v, and the monotonicity of the change of variables $r = r(t)$, hence we conclude that

$$|\dot{v}|^2 \in L^1(t_0, \infty). \tag{4.10}$$

Finally, we will prove that

$$\lim_{t\to\infty} \dot{v}(t) = 0. \tag{4.11}$$

From (4.2) and the boundedness of v and $\dot{v}$, we have that $|\ddot{v}|$ is bounded and thus (4.11) easily follows from (4.10).

We conclude then from the existence of the limit of F defined in (4.9) that $\lim_{t\to\infty} v(t)$ exists. It only remains to prove that this limit cannot be zero. To this end, we will prove that due to the assumption $\delta > \mathcal{S}$, the function

$$\nu(r) = h^{\frac{\mathcal{S}-1}{\delta-1}}(r)$$

satisfies (2.2), and thus by Lemma 2.3, if $\lim_{t\to\infty} v(t) = 0$, then u is bounded, a contradiction. Indeed,

$$a(r)|\nu'(r)| = \theta h^{\theta-1},$$

where as before, $\theta = \frac{\mathcal{S}-1}{\delta-1}$, and thus

$$(a(r)|\nu'(r)|)' = -\theta(\theta-1)h^{\theta-1}\frac{|h'(r)|}{h(r)}.$$

Hence, from L'Hospital's rule and using that since $\delta > \mathcal{S}$, it is $\theta - 1 < 0$, we have that

$$\liminf_{r\to0} \frac{a(r)|\nu'(r)|}{\int_0^r b(t)(\nu(t))^\delta dt} \ge \theta(1-\theta) \liminf_{r\to0} \frac{|h'(r)|}{b(r)h^{\mathcal{S}+1}},$$

and the result follows from (4.7). □

We end this section with a partial result concerning the case $\delta = \mathcal{S}$. This case, as well as the subcritical and supercritical case for the p-Laplace operator is treated in detail in a forthcoming paper, see [3].

Proposition 4.1 *Let a, b satisfy assumptions (H1)-(H_4), and assume that (1.6) holds. Let u be a positive singular solution to (1.3) with $\delta = \mathcal{S}$, and suppose that $\int_0 b(t)(h(t))^{\mathcal{S}} dt = \infty$. Then, there is $\bar{r}_0 > 0$ and $C > 0$ such that*

$$u(r) \le Ch(r)(\log(h(r)))^{-1/(\mathcal{S}-1)} \quad \textit{for all } r \in (0, \bar{r}_0).$$

Proof. Since the convergence of the integral $\int_0 b(t)(h(t))^{\mathcal{S}}dt$ is a necessary condition to have $\lim_{r\to 0} u(r)/h(r) > 0$, we have that in this case $\lim_{r\to 0} a(r)|u'(r)| = 0$. Hence, by Lemma 2.1 we have that u/h is monotone increasing near 0, that is,

$$\frac{|u'(r)|}{|h'(r)|} \le \frac{u(r)}{h(r)} \quad \text{for } r \text{ sufficiently small.}$$

Hence, from the left hand side inequality in (1.6), we have

$$\begin{aligned}(a(r)|u'(r)|)' &\ge b(r)\Big(\frac{u(r)}{h(r)}\Big)^{\mathcal{S}}(h(r))^{\mathcal{S}}\\ &\ge b(r)\Big(\frac{|u'(r)|}{|h'(r)|}\Big)^{\mathcal{S}}(h(r))^{\mathcal{S}}\\ &\ge c_0\frac{b(r)}{B(r)}\Big(\frac{|u'(r)|}{|h'(r)|}\Big)^{\mathcal{S}},\end{aligned}$$

and thus, using that $a(r)|u'(r)| = \frac{|u'(r)|}{|h'(r)|}$, we find that

$$\Big(\frac{|u'(r)|}{|h'(r)|}\Big)' \ge c_0\frac{b(r)}{B(r)}\Big(\frac{|u'(r)|}{|h'(r)|}\Big)^{\mathcal{S}}$$

implying that

$$\Big(\frac{|u'(r)|}{|h'(r)|}\Big)^{-\mathcal{S}}\Big(\frac{|u'(r)|}{|h'(r)|}\Big)' \ge c_1\frac{b(r)}{B(r)}.$$

Integrating this last inequality over (r, r_*), with r_* sufficiently small we conclude that

$$\Big(\frac{|u'(r)|}{|h'(r)|}\Big)^{\mathcal{S}-1} \le \bar{c}_0|\log B(r)|^{-1} \quad \text{for all } r\in(0,r_*).$$

Hence, for all $r\in(0,r_*)$, it holds that

$$u(r) \le u(r_*) + \bar{c}_0\int_r^{r_*}|h'(t)||\log B(t)|^{\frac{-1}{\mathcal{S}-1}}dt \le Kh(r)|\log B(r)|^{\frac{-1}{\mathcal{S}-1}}$$

for some positive constant K. The result follows now by using the right hand side inequality in (1.6).

5 Examples

We finish this paper by giving some examples to illustrate our results as well as the main differences with the non-weighted case.

Example 1. We consider first a case for which $\mathcal{S} < \infty$ and it is such that when $\delta = \mathcal{S}$, any singular solution behaves like the fundamental solution. Let

$$a(r) = r^{N-1}, \quad N > 2, \quad B(r) = r^{\theta(N-2)}(\log(r^{-1}))^{-2} \quad \text{near zero,} \quad \theta > 1.$$

Then it can be easily verified that $h(r) = Cr^{2-N}$ near zero. Also, if $\delta < \mathcal{S}$, we have that

$$\int_0^r b(t)(h(t))^\delta dt \geq B(r)(h(r))^\delta = C^\delta r^{(\theta-\delta)(N-2)}(\log(r^{-1}))^{-2},$$

and thus it follows that $\mathcal{S} \leq \theta$. Next, by integrating by parts we find that

$$\begin{aligned}\int_s^r b(t)(h(t))^\theta dt &\leq B(r)(h(r))^\theta + \theta \int_s^t B(t)(h(t))^{\theta-1}|h'(t)|dt \\ &\leq B(r)(h(r))^\theta + C_1\theta \int_s^r \left(\log(t^{-1})\right)^{-2}\frac{1}{t}dt \\ &\leq B(r)(h(r))^\theta + C_1\theta(\log(r^{-1}) - \log(s^{-1})),\end{aligned}$$

where C_1 is some positive constant. Hence, $\mathcal{S} = \theta$ and $\theta \in \mathcal{W}$, and the claim follows from Theorem 1.1(i)-(ii), that is, any radially symmetric singular solution to

$$-\Delta u = |x|^{\theta(N-2)-N} \log|x|^{-1}(\theta(N-2)\log|x|^{-1} + 2)|u|^{\delta-1}u, \quad x \in B^*_{r_0}(0),$$

behaves like $|x|^{2-N}$ near zero for $1 < \delta \leq \theta$ and satisfies

$$\lim_{|x|\to 0} |x|^{N-2}u(x) = 0 \quad \text{for } \delta > \theta.$$

Next we give an example for which any positive singular solution is of the fundamental type.

Example 2. Let $a(r) = r^{N-2}$, $N \geq 2$, and set $b(r) = r^{-\theta-1}e^{-1/r^\theta}$, $\theta > 0$. When $N > 2$, the fundamental solution is $h(r) = Cr^{2-N}$, and clearly the integral

$$\int_0 r^{-\theta-1-L(N-2)}e^{-1/r^\theta}dr < \infty$$

for any $L > 0$. Thus $\mathcal{S} = \infty$.

In the case that $N = 2$, the fundamental solution is $h(r) = \log(r^{-1})$, and we also have that

$$\int_0 r^{-\theta-1}e^{-1/r^\theta}(\log(r^{-1}))^L dr \quad \text{converges for any } \theta > 0.$$

Hence in this case we also have $\mathcal{S} = \infty$.

We conclude that any positive radially symmetric solution to

$$-\Delta u = |x|^{-\theta-2-N}\exp(|x|^{-\theta})|u|^{\delta-1}u, \quad x \in B^*_{r_0}(0),$$

behaves like the fundamental solution, for any $N \geq 2$.

Finally, we give a general example to which all our results apply.

Example 3. Let $m(r)$ be any continuous monotone function satisfying

$$m_0 \leq m(r) \leq m_1 \quad \text{for all } r \in [0,1],$$

let $a \in C^1(0,1)$ be a positive function such that $1/a \notin L^1(0,1)$, and set

$$b(r) := m(r)\frac{(h(r))^{-L-1}}{a(r)}.$$

Then a, b satisfy all the assumptions in Theorem 1.1 and it can be easily shown that $\mathcal{S} = L$. Indeed, let $1 < \delta < L$. Then

$$\int_s^r b(t)(h(t))^\delta dt = \int_s^r m(t)(h(r))^{\delta - L - 1}|h'(t)|dt,$$

implying that

$$\int_s^r b(t)(h(t))^\delta dt \leq m_1 \frac{(h(r))^{\delta - L}}{L - \delta} < \infty.$$

Also, it can be easily verified that $\int_0 b(t)(h(t))^L dt = \infty$, and thus $\mathcal{S} = L$. Next, by the mean value theorem it holds that

$$B(r) = m(\xi)\frac{(h(r))^{-L}}{L}, \quad \text{for some } \xi \in (0,r),$$

and thus

$$m_0/L \leq B(r)(h(r))^L \leq m_1/L.$$

Hence we conclude that $\mathcal{S} = L$ and

(i) If $1 < \delta < L$, any positive singular solution to

$$-(a(r)u')' = b(r)|u|^{\delta-1}u, \quad r \in (0, r_0) \tag{5.1}$$

behaves like the fundamental solution h near zero.

(ii) If $\delta > L$, $\delta \neq 2L - 1$, then any positive singular solution to (5.1) behaves like $(h(r))^{\frac{L-1}{\delta-1}}$.

(iii) If $\delta = L$, then

$$\lim_{r\to 0}\frac{u(r)}{h(r)} = 0, \quad \text{and} \quad u(r) \leq Ch(r)(\log(h(r)))^{-1/(\mathcal{S}-1)}$$

for all r sufficiently small.

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