USA-Chile Workshop on Nonlinear Analysis,
Electron. J. Diff. Eqns., Conf. 06, 2001, pp. 225–241
http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu or ejde.math.unt.edu (login: ftp)

Exponential dichotomies for linear systems with impulsive effects *

Raúl Naulin

Abstract

In this paper we give conditions for the existence of a dichotomy for the impulsive equation

$$\mu(t,\varepsilon)x' = A(t)x,\ t \neq t_k,$$
$$x(t_k^+) = C_k x(t_k^-)\,,$$

where $\mu(t,\varepsilon)$ is a positive function such that $\lim \mu(t,\varepsilon) = 0$ in some sense. The results are expressed in terms of the properties of the eigenvalues of matrices $A(t)$, the properties of the eigenvalues of matrices $\{C_k\}$ and the location of the impulsive times $\{t_k\}$ in $[0,\infty)$.

1 Introduction

In this paper we study the dichotomic properties of the impulsive system

$$\begin{aligned} \mu(t,\varepsilon)x'(t) = A(t)x(t), &\quad t \neq t_k,\ J = [0,\infty), \\ x(t_k^+) = C_k x(t_k^-), &\quad k \in \mathbb{N} = \{1,2,3,\dots\}\,, \end{aligned} \tag{1}$$

where $x(t_k^{\pm}) = \lim_{t\to t_k^{\pm}} x(t)$. The function $A(\cdot)$ and the sequence $\{C_k\}$ have properties to be specified later. The function $\mu(t,\varepsilon)$ depends on a parameter ε, in general, belonging to a metric space E. We will assume that $\mu(t,\varepsilon)$, for each fixed ε, is continuous. The cases we are interested in most are $\mu(t,\varepsilon) = \varepsilon > 0$, $\mu(t,\varepsilon) = \mu(t)$, such that $\lim_{t\to\infty} \mu(t) = 0$ and $\mu(t,\varepsilon) = 1$. In what follows, for technical purposes we shall suppose that

$$0 < \mu(t,\varepsilon) \leq 1,\ \forall (t,\varepsilon) \in J \times E. \tag{2}$$

For ordinary differential equations, the singular perturbed case ($\mu(t,\varepsilon) = \varepsilon > 0$) has been intensively studied in [7, 15]; the regular case ($\mu(t,\varepsilon) = 1$) has been

* *Mathematics Subject Classifications:* 34A05, 34E05.
Key words: Impulsive linear systems, singularly perturbed impulsive systems, dichotomies, splitting of impulsive systems.

Published January 8, 2001.
Supported by Proyecto UDO-CI-5-1003-0936/00.

considered in [6]; the general setting of the problem (1), when $\mu(t,\varepsilon) = \mu(t)$, $\lim_{t\to\infty} \mu(t) = 0$ was studied in [13].

The aim of this paper is to give a set of algebraic conditions of existence of a (μ_1, μ_2)-dichotomy [4], meaning by this conditions involving the properties of the functions of eigenvalues of matrices $A(t)$, the eigenvalues of matrices belonging to the sequence $\{C_k\}$, and the location of the impulsive times $\{t_k\}$.

2 Notations and basic hypotheses

In this paper V stands for the field of complex numbers. We will assume that a fixed norm $\|\cdot\|$ on the space V^n is defined. For a matrix $A \in V^{n\times n}$, $\|A\|$ will denote the corresponding functional matrix norm. If m and n are integral numbers, then the set $\{m, m+1, m+2, \ldots, n\}$ will be denoted by $\overline{m,n}$. The symbol $\{t_k\}$ identifies a strictly increasing sequence of positive numbers, satisfying $\lim_{k\to\infty} t_k = \infty$. The solutions of all considered impulsive systems are uniformly continuous on each interval $J_k = (t_{k-1}, t_k]$. Further notations;

- For a bounded function f, we denote $\|f\|_\infty = \sup\{\|f(t)\| : t \in J\}$,
- For an absolutely integrable function f, we denote $\|f\|_1 = \int_0^\infty \|f(t)\| dt$,
- For a bounded sequence $\{C_k\}$, we denote $\|\{C_k\}\|_\infty = \sup\{\|C_k\| : \ k \in \mathbb{N}\}$,
- For a summable sequence $\{C_k\}$, we denote $\|\{C_k\}\|_1 = \sum_{k=1}^\infty \|C_k\|$,
-$C(\{t_k\}) = \{f : J \to V^n : f$ is uniformly continuous on all intervals $J_k\}$,
-$BC(\{t_k\}) = \{f \in C(\{t_k\}) : f$ is bounded$\}$.
- The function $i[s,t)$ will denote the number of impulsive times contained in the interval $[s,t)$ if $t > s$; if $s \le t_k < t_{k+1} < \cdots < t_h < t$, we define

$$\sum_{[s,t)} C_i = C_k + C_{k+2} + \cdots + C_h, \quad \sum_{[t,t)} C_i = 0,$$
$$\prod_{[s,t)} C_i = C_h C_{h-1} \cdots C_k, \quad \prod_{[t,t)} C_i = I\,.$$

We will denote by $X(t) = X(t,\varepsilon)$ the fundamental matrix of the impulsive system (1). By this we mean a function $X : J \to V^{n\times n}$ uniformly continuous, of class C^1 on each interval J_k, such that $X(0^+) = I$ and X satisfies (1). The definition and basic properties of function $X(t,\varepsilon)$, for each fixed ε, are described in [2, 8].

Below, we list the basic hypotheses **H1-H5** we will use.

H1: *The function A is bounded and piecewise uniformly continuous on J with respect to $\{t_k\}$. This last means: For any $\rho > 0$, there exists a number $\delta(\rho) > 0$, such that $\|A(t) - A(s)\| < \rho$, if $|t-s| < \delta$, $t, s \in J_k$ for all $k \in N$.*

H2: *There exist numbers $p \ge 0$ and $q > 1$, such that*

$$|i[s,t) - p(t-s)| \le q, \ s \le t.$$

H3: $\{C_k\}_{k=1}^{\infty}$ *is a bounded sequence of invertible matrices.*

H4: *There exists a positive number* γ*, such that for any* k*, all eigenvalues* μ_k *of the matrix* C_k *satisfy the condition* $\gamma|\mu_k| \geq 1$.

Definition 1 *We shall say that* $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$*, the eigenvalues of matrix* A*, are ordered by real parts (respectively, ordered by norms) iff*

$$Re\lambda_1 \leq Re\lambda_2 \leq \ldots \leq Re\lambda_n, \ (\textit{respectively } |\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_n|).$$

In the sequel, we will assume that $\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)\}$ the eigenvalues of matrix $A(t)$ are ordered by real parts, and $\{\mu_1(k), \mu_2(k), \ldots, \mu_n(k)\}$ the eigenvalues of matrix C_k are ordered by norms.

We will consider the following piecewise constant function

$$u_m : J \to \mathbb{R}, \ u_m(t) = \frac{\ln|\mu_m(k)|}{t_k - t_{k-1}}, \ \text{ if } t \in J_k. \tag{3}$$

In order to alleviate the writing, let us denote for $m \in \overline{1, n-1}$

$$\alpha_m(t,\varepsilon) = \frac{\mathrm{Re}(\lambda_m(t) - \lambda_{m+1}(t))}{\mu(t,\varepsilon)} + u_m(t) - u_{m+1}(t).$$

The following hypothesis is a slight modification of a condition of splitting used in [9].

H5: *There exists a positive constant* M *such that the function*

$$\begin{aligned} U_m(t,\varepsilon) &= \int_0^t \frac{1}{\mu(s,\varepsilon)} \exp\left\{\int_s^t \alpha_m(\tau,\varepsilon)d\tau\right\} ds, \\ &+ \int_t^{+\infty} \frac{1}{\mu(s,\varepsilon)} \exp\left\{\int_t^s \alpha_m(\tau,\varepsilon)\tau\right\} ds \end{aligned}$$

satisfies

$$\|U_m(t,\varepsilon)\| \leq M, \ \forall (t,\varepsilon) \in [0,\infty) \times E.$$

3 The quasidiagonalization method

We will assume that, for some positive number r, the families of matrices $\{A(t) : t \in J\}$ and $\{C_k : \ k \in \mathbb{N}\}$ are contained in the set

$$\mathcal{M}(r) = \{F \in V^{n\times n} : \|F\| \leq r\}.$$

For each matrix $F \in \mathcal{M}(r)$ and $\sigma > 0$, by Theorem 1.6 in [1], we may choose a nonsingular matrix S such that

$$S^{-1}FS = \Lambda(F) + R(F,\sigma), \ \|R(F,\sigma)\| \leq \sigma/2, \tag{4}$$

where $\Lambda(F)$ denotes the diagonal matrix of eigenvalues of matrix F, ordered by real parts. Let us consider the ball $B[F,\rho] = \{G \in V^{n\times n} : \|f - G\| \le \rho\}$. For any $G \in B[F,\rho]$ we have

$$S^{-1}GS = \operatorname{Re}\Lambda(F) + i \operatorname{Im}\Lambda(F) + S^{-1}(G-F)S + R(F,\sigma),\ i^2 = -1,$$

where

$$\Lambda(F) = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\},\ \operatorname{Re}\Lambda(F) = \operatorname{diag}\{\operatorname{Re}\lambda_1, \operatorname{Re}\lambda_2, \dots, \operatorname{Re}\lambda_n\}.$$

¿From this decomposition we obtain

$$S^{-1}GS = \operatorname{Re}\Lambda(G) + i \operatorname{Im}\Lambda(F) + T(F,\rho) + R(F,\sigma),$$

where

$$T(F,\rho) = (\Lambda(F) - \Lambda(G)) + S^{-1}(G-F)S.$$

¿From Hurwitz's theorem (see [5], page 148), the function $\mathcal{L} : V^{n\times n} \to V^{n\times n}$ defined by $\mathcal{L}(F) =$ Re $\Lambda(F)$ is continuous. This assertion implies, for a fixed number $\sigma > 0$ and a matrix $F \in \mathcal{M}(r)$ the existence of a nonsingular matrix S and a $\rho > 0$, such that if $G \in B[F,\rho]$, then

$$S^{-1}GS = \operatorname{Re}\Lambda(G) + i \operatorname{Im}\Lambda(F) + \Gamma(F,\sigma),\ \Gamma(F,\sigma) := T(F,\rho) + R(F,\sigma),$$

and $\|\Gamma(F,\sigma)\| \le \sigma$. Since $\mathcal{M}(r)$ is compact, then given a $\sigma > 0$, there exist a covering $\mathcal{F} = \{B[F_j,\rho_j]\}_{j=1}^m$ of $\mathcal{M}(r)$, and nonsingular matrices $\mathcal{S} = \{S_1, S_2, \dots, S_m\}$ having the following property: For a fixed $G \in \mathcal{M}(r)$ there exists an index $j \in \{1, 2, \dots, m\}$, such that $G \in B[F_j,\rho_j]$ and

$$S_j^{-1}GS_j = \operatorname{Re}\Lambda(G) + i \operatorname{Im}\Lambda(F_j) + \Gamma_j(\sigma),\ \|\Gamma_j(\sigma)\| \le \sigma. \tag{5}$$

Let $\rho > 0$ be a Lebesgue number of the covering $\mathcal{F}$. According to **H1**, there exists a $\delta > 0$, non depending on k, such that for $t, s \in J_k$, $|t - s| \le \delta$ we have $\|A(t) - A(s)\| < \rho$. Let us define

$$n(k,\delta) = \inf\{j \in \mathbb{N} : \frac{t_k - t_{k-1}}{j} \le \delta\},$$

and the partition of the interval J_k:

$$\mathcal{P}_k = \{t_0^k, t_1^k, \dots, t_{n(k)}^k\},\ t_0^k = t_{k-1},\ t_{n(k)}^k = t_k,$$

defined by

$$|t_{i-1}^k - t_i^k| = \delta_k,\ i \in \overline{1, n(k)},\ \delta_k := \frac{t_k - t_{k-1}}{n(k,\delta)}.$$

We emphasize that $n(k,\delta) = 1$ iff $t_k - t_{k-1} \le \delta$. This and **H2** yield

$$n(k,\delta) \le L(p,\delta)(t_k - t_{k-1}),\ L(p,\delta) := \max\{\frac{p}{q-1}, \frac{2}{\delta}\}. \tag{6}$$

According to the decomposition (5), we may assign to the interval $(t^k_{i-1}, t^k_i]$ a nonsingular matrix $S_{k,i} \in \mathcal{S}$ and $F_{k,i} \in \{F_j\}_{j=1}^m$, such that

$$S_{k,i}^{-1} A(t) S_{k,i} = Re\Lambda(t) + iIm\Lambda(F_{k,i}) + \Gamma_{k,i}(\sigma),\ t \in (t^k_{i-1}, t^k_i], \tag{7}$$

where we have abbreviated $\Lambda(t) = \Lambda(A(t))$ and

$$\|\Gamma_{k,i}(\sigma)\| \leq \sigma. \tag{8}$$

Regarding the sequence $\{C_k\}_{k=1}^{\infty}$, we will accomplish a similar procedure. Let us consider a matrix $D \in \mathcal{M}(r)$ and $\sigma > 0$. For some nonsingular matrix T we will have, instead of (4), the decomposition

$$T^{-1} D T = N(D) + R(D, \sigma),\ \|R(D,\sigma)\| < \sigma, \tag{9}$$

where the matrix $N(D)$ is defined by means of the eigenvalues D:

$$N(D) = \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_n\},\ |\mu_1| \leq |\mu_2| \leq \ldots \leq |\mu_n|.$$

We may write (9) in the form

$$T^{-1} D T = |N(D)| e^{i\,\mathrm{Arg}(D)} + R(D, \sigma),$$

where

$$\mathrm{Arg}(D) = \mathrm{diag}\{\arg(\mu_1), \arg(\mu_2), \ldots, \arg(\mu_n)\}$$

and

$$|N(D)| = \mathrm{diag}\{|\mu_1|, |\mu_2|, \ldots, |\mu_n\}|.$$

For a matrix $C \in B[D, \rho]$, $\rho > 0$, we write

$$\begin{aligned} T^{-1} C T &= |N(C)| e^{i\,Arg(D)} + (|N(C)| - |N(D)|) e^{i\,Arg(D)} \\ &+ T^{-1}(C - D) T + R(D, \sigma),\ \|R(D, \rho)\| \leq \sigma. \end{aligned}$$

The Hurwitz's theorem implies that the function $\mathcal{N} : V^{n\times n} \to V^{n\times n}$ defined by $\mathcal{N}(C) = |C|$ is continuous. Since $\mathcal{M}(r)$ is compact, then for a given $\sigma > 0$, there exists a covering $\mathcal{D} = \{B[D_i, \rho_i]\}_{i=1}^{\tilde{m}}$ of $\mathcal{M}(r)$, and a set of nonsingular matrices $\mathcal{T} = \{T_1, T_2, \ldots, T_{\tilde{m}}\}$, such that for each C_k there exists a $T_k \in \mathcal{T}$ and $D_k \in \{D_i\}_{i=1}^{\tilde{m}}$ such that

$$T_k^{-1} C_k T_k = |N(C_k)| e^{i\,Arg(D_k)} + \tilde{\Gamma}_k(\sigma),\ \|\tilde{\Gamma}_k(\sigma)\| \leq \sigma. \tag{10}$$

4 A change of variables

Let $g : [0,1] \to [0,1]$ be a strictly increasing function, $g \in C^{\infty}$, such that $g(0) = g'(0) = g'(1) = 0$, $g(1) = 1$. For an ordered pair (Q, R) of invertible matrices we define

$$\theta : [a,b] \to V^{n\times n},\ \theta(t) = Q \exp\left\{ g\left(\frac{t-a}{b-a}\right) Ln(Q^{-1}R) \right\}.$$

The path θ is of class C^∞. Moreover $\theta(t)$ is a nonsingular matrix for each t, and $\theta(a) = Q$, $\theta(b) = R$, $\theta'(a) = 0$, $\theta'(b) = 0$. In the sequel, we shall say that the path θ splices the ordered pair of matrices (Q, R) on the interval $[a, b]$. In order to perform a change of variable of system (1), we splice matrices $(S_{k,i}, S_{k,i+1})$, $i \in \overline{1, n(k) - 1}$ on an interval $[t_i^k - \nu_k(\varepsilon)\delta_{k,i}/2, t_i^k + \nu_k(\varepsilon)\delta_{k,i}/2]$, where $\nu_k(\varepsilon) = \inf\{\ \mu(t, \varepsilon) : t \in J_k\}$, and $\delta_{k,i}$ are small numbers satisfying $\nu_k(\varepsilon)\delta_{k,i} < \delta_k$ and another condition we will specify in the forthcoming definition of number ν (see (13). Let us define the path

$$\theta_{k,i} : [t_i^k - \nu_k(\varepsilon)\delta_{k,i}/2,\ t_i^k + \nu_k(\varepsilon)\delta_{k,i}/2] \to V^{n\times n}$$

splicing the matrices $(S_{k,i}, S_{k,i+1})$ in the following way

$$\theta_{k,i}(t) = S_{k,i}\ \exp\ \left\{ g\left(\frac{t - t_i^k + \nu_k(\varepsilon)\delta_{k,i}}{\mu_k(\varepsilon)\delta_{k,i}} \right) Ln(S_{k,i}^{-1} S_{k,i+1}) \right\}.$$

For the constant

$$K_1(\sigma) = \max\left\{ \left(\|S_k\| + \|Ln(S_k^{-1}S_i)\|\right)\ \exp\ \left\{\|Ln(S_k^{-1}S_i)\|\right\} : 1 \le k, i \le m \right\}$$

we have the estimates

$$\|\theta_{k,i}(t)\|_\infty \le K_1(\sigma),\ \|\theta'_{k,i}(t)\|_\infty \le \frac{K_1(\sigma)}{\nu_k(\varepsilon)\delta_{k,i}}. \tag{11}$$

The matrix T_{k+1} assigned to the impulsive time $t_0^{k+1} = t_{k+1} = t_{n(k)}^k$ and the matrix $S_{k+1,1}$ are spliced on the interval $[t_0^{k+1}, t_0^{k+1} + \mu_{k+1}(\varepsilon)\delta_{k+1,0}/2]$ by a path we denote by $\theta_{k+1,0}$. The matrices $(S_{k,n(k)}, T_{k+1})$ are spliced on the interval $[t_{n(k)}^k - \nu_k(\varepsilon)\delta_{k,n(k)}/2,\ t_{n(k)}^k]$ by a path we denote by $\theta_{k,n(k)}$. We emphasize that $\theta_{k+1,0}(t_k) = T_{k_1} = \theta_{k,n(k)}(t_k)$. A special mention deserves the time $t = 0$ which is not considered as an impulsive time. We will attach to the time $t = 0$ the matrix $S_{1,1}$. For these splicing paths are valid similar estimates to (11), with a modified constant for which we maintain the notation $K_1(\sigma)$.

Let us define the intervals

$$\begin{aligned} I_k &= [t_0^{k+1} - \nu_k(\varepsilon)\delta_{k,0}/2, t_0^{k+1} + \nu_{k+1}(\varepsilon)\delta_{k+1,0}/2], k = 1, 2, \dots, \\ I_{k,i} &= (t_i^k - \nu_k(\varepsilon)\delta_{k,i}/2, t_i^k + \nu_k(\varepsilon)\delta_{k,i}/2),\ i \in \overline{1, n(k) - 1}, \end{aligned} \tag{12}$$

and the number

$$\nu = \sum_{k=1}^{\infty} \sum_{i=1}^{n(k)} \delta_{k,i}. \tag{13}$$

The choice of the numbers $\delta_{k,i}$ is at our disposal. Therefore, ν can be made as small as necessary. Let us consider the C^∞ function

$$S(t) = \begin{cases} \theta_{k+1,0}(t),\ t \in [t_0^{k+1}, t_0^{k+1} + \nu_{k+1}(\varepsilon)\delta_{k,0}/2], & k = 0, 1, \dots \\ S_{k,i},\ t \in [t_i^k + \nu_k(\varepsilon)\delta_{k,i}/2, t_{i+1}^k - \nu_k(\varepsilon)\delta_{k,i+1}/2], & i \in \overline{0, n(k) - 1}, \\ \theta_{k,i}(t),\ t \in [t_i^k - \nu_k(\varepsilon)\delta_{k,i}/2, t_i^k + \nu_k(\varepsilon)\delta_{k,i}/2], & i \in \overline{1, n(k) - 1}, \\ \theta_{k,n(k)}(t),\ t \in [t_{n(k)-1}^k - \nu_k(\varepsilon)\delta_{k,n(k)}/2, t_{n(k)}^k], & k = 1, 2, \dots \end{cases}$$

From this definition $S'(t) = 0$ except on the intervals I_k and $I_{k,i}$. Since $S(t_k) = T_k$, the change of variable $x = S(t)y$ reduces System (1) to the form

$$\begin{aligned} \mu(t,\varepsilon)y'(t) &= \left(S^{-1}(t)A(t)S(t) - \mu(t,\varepsilon)S^{-1}(t)S'(t)\right) y(t), t \neq t_k, \\ y(t_k^+) &= \left(|N(C_k)|e^{i\ \mathrm{Arg}(D_k)} + \tilde{\Gamma}_k(\sigma)\right) y(t_k), k \in \mathbb{N}, \end{aligned} \tag{14}$$

where $\|\tilde{\Gamma}_k(\sigma)\| \leq \sigma$. Thus, this change of variable yields a notable simplification of the discrete component of (1). Let us define the left continuous function $L : J \to V^{n\times n}$ by

$$L(0) = S_{1,1}, \quad L(t) = S_{k,i}, \quad t \in (t_{i-1}^k, t_i^k],\ i \in \overline{1, n(k)}.$$

From $S^{-1}(t)A(t)S(t) = L^{-1}(t)A(t)L(t) + F(t,\sigma)$, where

$$F(t,\sigma) = S^{-1}(t)A(t)S(t) - L^{-1}(t)A(t)L(t), \tag{15}$$

we may write System (14) in the form

$$rcl \quad \mu(t,\varepsilon)y'(t) = \left(L^{-1}(t)A(t)L(t) + F(t,\sigma) - \mu(t,\varepsilon)S^{-1}(t)S'(t)\right) y(t),\ t \neq t_k,$$
$$y(t_k^+) = \left(N_k e^{i\,\mathrm{Arg}(D_k)} + \tilde{\Gamma}_k(\sigma)\right) y(t_k), \quad k \in \mathbb{N}\,.$$

From (7) and the definition of the piecewise constant functions

$$G(t) = Im\Lambda(F_{k,i}),\ t \in (t_{i-1}^k, t_i^k], \quad \Gamma(t,\sigma) = \Gamma_{k,i}(\sigma),\ t \in (t_{i-1}^k, t_i^k], \tag{16}$$

we can write the last system in the form

$$\begin{aligned} \mu(t,\varepsilon)y'(t) = \Big(&\operatorname{Re}\Lambda(t) + iG(t) + \Gamma(t,\sigma) + F(t,\sigma) \\ &-\mu(t,\varepsilon)S^{-1}(t)S'(t)\Big)y(t),\ t \neq t_k, \\ y(t_k^+) = \Big(&N_k e^{i\,\mathrm{Arg}(D_k)} + \tilde{\Gamma}_k(\sigma)\Big)\, y(t_k), \quad k \in \mathbb{N}\,. \end{aligned} \tag{17}$$

Lemma 1

$$\|\Gamma(t,\sigma)\|_\infty \leq \sigma, \quad \|\{\tilde{\Gamma}_k(\sigma)\}\|_\infty \leq \sigma, \tag{18}$$

$$\|\mu(.,\varepsilon)^{-1}F(.,\sigma)\|_1 \leq K_2(\sigma)\nu, \tag{19}$$

$$\int_s^t \|S^{-1}(\tau)S'(\tau)\|d\tau \leq K_3(\sigma)L(\delta,p)(t-s),\ t \geq s. \tag{20}$$

Proof. The first estimate of (18) follows from the definition of function $\Gamma(t,\sigma)$ given by (16) and (8), and the second follows from (10). From definition (15), there exists a constant $K_2(\sigma)$ depending only on σ such that

$$\|F(\cdot,\sigma)\|_\infty \leq K_2(\sigma).$$

Moreover, from (15) we observe that $F(\cdot,\sigma)$ vanishes outside of the intervals $I_{k,i}$ and I_k. Therefore, from the definitions (12)-(13) we obtain

$$\int_0^\infty |\frac{F(t,\sigma)}{\mu(t,\varepsilon)}|dt = K_2(\sigma)\Big(\sum_{i,k}\int_{I_{k,i}}\frac{1}{\mu_k(\varepsilon)}dt + \sum_k \int_{I_k}\frac{1}{\mu_k(\varepsilon)}dt\Big) \le K_2(\sigma)\nu\,.$$

In order to obtain (20) we observe that $S^{-1}(t)S'(t)$ vanishes outside of the intervals $I_{k,i}$ and I_k. Moreover, there exists a constant $K_3(\sigma)$ depending only on σ, such that on each interval $[t^k_{i-1}, t^k_i]$ we have

$$\int_{t^k_{i-1}}^{t^k_i} \|S^{-1}(\tau)S'(\tau)\|d\tau \le K_3(\sigma).$$

From this estimate and (6), it follows

$$\int_s^t \|S^{-1}(\tau)S'(\tau)\|d\tau \le K_3(\sigma)L(p,\delta)(t-s).$$

In what follows we unify the notations of the constants $K_i(\sigma), i = 1,2,3$ in a simple constant $K(\sigma)$.

5 Splitting and dichotomies

We are interested in the proof of existence of a dichotomy for the System (17). In this task we will follow the way indicated by Coppel in [6]: First we split System (17) in two systems of lower dimensions and after this, the Gronwall inequality for piecewise continuous functions [3] will give the required result. Following the ideas of paper [11], we write System (17) in the form:

$$\begin{aligned} \mu(t,\varepsilon)y'(t) &= \Big(\operatorname{Re}\Lambda(t) + iG(t) + \Gamma(t,\sigma) + F(t,\sigma) \\ &\qquad -\mu(t,\varepsilon)S^{-1}(t)S'(t)\Big)y(t), \quad t \ne t_k, \\ \Delta y(t_k) &= \Big(B_k + \hat\Gamma_k(\sigma)\Big)\, y(t_k^+), \quad k \in \mathbb{N}\,, \end{aligned} \tag{21}$$

where $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$, $B_k = I - N_k^{-1}e^{-i\ \operatorname{Arg}(D_k)}$, and

$$\hat\Gamma_k(\sigma) = N_k^{-1}e^{-i\operatorname{Arg}(D_k)}\Gamma_k(\sigma)\left(N_k e^{i\operatorname{Arg}(D_k)} + \Gamma_k(\sigma)\right)^{-1}.$$

From hypotheses **H3-H4** and (18) we obtain, for a small σ, the estimate

$$|\hat\Gamma_k(\sigma)| \le \frac{\sigma\gamma^2}{1-\gamma\sigma}. \tag{22}$$

On the other hand, the fundamental matrix of system

$$\begin{aligned} \mu(t,\varepsilon)w'(t) &= (\operatorname{Re}\Lambda(t) + iG(t))\, w(t), \quad t \ne t_k, \\ \Delta w(t_k) &= B_k w(t_k^+), \quad k \in \mathbb{N}\,, \end{aligned}$$

coincides with the fundamental matrix $Z(t,\varepsilon) = Z(t)$ of the diagonal system

$$\begin{aligned} \mu(t,\varepsilon)z'(t) &= (\operatorname{Re}\Lambda(t) + iG(t))\, z(t), \quad t \neq t_k, \\ z(t_k^+) &= |N(C_k)|e^{i\,\operatorname{Arg}(D_k)} z(t_k), \quad k \in \mathbb{N}\,, \end{aligned} \tag{23}$$

which is equal to $Z(t) := \Phi(t)\Psi(t)$, where

$$\Psi(t) = \exp\left\{\int_0^t \frac{\operatorname{Re}\Lambda(\tau) + iG(\tau)}{\mu(\tau,\varepsilon)} d\tau\right\}, \quad \Phi(t) = \prod_{[0,t)} |N(C_k)|e^{i\,\operatorname{Arg}(D_k)}.$$

For the projection matrix $P = \operatorname{diag}\{\overbrace{1,1,\ldots,1}^{m},0,\ldots,0\}$, the function Φ satisfies the following estimates:

$$\|\Phi(t)P\| \leq \exp\Big\{\sum_{[0,t)} \ln|\mu_m(k)|\Big\}.$$

From definition (3), we may write

$$\begin{aligned} \|\Phi(t)P\| &\leq L\exp\Big\{\int_0^t u_m(\tau)d\tau\Big\}, \\ \|\Phi^{-1}(t)(I-P)\| &\leq L\exp\Big\{\int_t^0 u_{m+1}(\tau)d\tau\Big\}, \end{aligned}$$

where L is a constant depending on the condition **H3** only. Since $\Phi(t)$ and $\Psi(t)$ commute with P, then for $t \geq s$ we obtain the following estimates

$$\begin{aligned} \|Z(t)PZ^{-1}(s)\| &\leq L_1\exp\Big\{\int_s^t \big(\frac{\operatorname{Re}\lambda_m}{\mu(\cdot,\varepsilon)} + u_m\big)(\tau)d\tau\Big\}, \\ \|Z(s)(I-P)Z^{-1}(t)\| &\leq L_1\exp\Big\{\int_t^s \big(\frac{\operatorname{Re}\lambda_{m+1}}{\mu(\cdot,\varepsilon)} + u_{m+1}\big)(\tau)d\tau\Big\}, \end{aligned} \tag{24}$$

where L_1 is a constant independent of σ and ϵ. In the sequel $W(t,s)$ will denote the matrix: $W(t,s) = Z(t)Z^{-1}(s)$. ¿From (24), for $t \geq s$, we have

$$\|W(t,s)P\|\|W(s,t)(I-P)\| \leq L_1^2\exp\Big\{\int_s^t \alpha_m(\tau,\varepsilon)d\tau\Big\}. \tag{25}$$

For a given matrix C, we write $\{C\}_1 = PCP + (I-P)C(I-P)$.

Definition 2 *By a splitting of System (21), we mean the existence of a function $T : J \to V^{n\times n}$ with the following properties:*

T1: *T is continuously differentiable on each interval J_k,*

T2: *For each impulsive time t_k, there exists the right hand side limit $T(t_k^+)$,*

T3: *$T(t)$ is invertible for each $t \in J_k$. $T(t_k^+)$ are invertible for all k,*

T4: *The functions T and T^{-1} are bounded,*

T5: *The change of variables $y(t) = T(t)z(t)$ reduces System (21) to*

$$\begin{aligned} \mu(t,\varepsilon)z'(t) &= \Big(\operatorname{Re}\Lambda(t) + iG(t) + \{(\Gamma(t,\sigma) + F(t,\sigma))\,T(t)\}_1 \\ &\quad -\mu(t,\varepsilon)\{S^{-1}(t)S'(t)T(t)\}_1\Big)z(t), \quad t \neq t_k, \\ \Delta z(t_k) &= \Big(B_k + \{\hat{\Gamma}_k(\sigma)\}_1\Big)\, z(t_k^+), \quad k \in \mathbb{N}\,. \end{aligned} \tag{26}$$

For ordinary differential equations, problem **T1-T5** was solved in [6]. For difference equations, it was solved in [14]. The problem of splitting for impulsive equations is treated in [11]. None of the cited works study the splitting of system (21), where the unbounded coefficient $\{S^{-1}(t)S'(t)\}_1$ appears.

Following the general setting of [6, 14, 11], we will seek a function T in the form $T(t) = I + H(t)$, where $H \in BC(\{t_k\})$, $\|H\|_\infty \leq 1/2$, such that T satisfies conditions **T1-T5**. In the following we use the notations

$$H_k = H(t_k),\ H_k^+ = H(t_k^+).$$

Let us consider the following operators: The operator of continuous splitting

$$\begin{aligned} \mathcal{O}(H)(t) \;\;=\;\; & \int_{t_0}^{t} \frac{1}{\mu(s,\varepsilon)} W(t,s)P(I - H(s))(\Gamma(s,\sigma) \\ & +F(s,\sigma))(I + H(s))(I - P)W(s,t)ds \\ & -\int_{t}^{\infty} \frac{1}{\mu(s,\varepsilon)} W(t,s)(I - P)(I - H(s))(\Gamma(s,\sigma) \\ & +F(s,\sigma))(I + H(s))PW(s,t)\,ds\,; \end{aligned}$$

the operator of discrete splitting

$$\begin{aligned} \mathcal{D}(H)(t) \;\;=\;\; & \sum_{[t_0,t)} W(t,t_k)P(I - H_k)\tilde{\Gamma}_k(\sigma)(I + H_k^+)(I - P)W(t_k^+,t) \\ & -\sum_{[t,\infty)} W(t,t_k)(I - P)(I - H_k)\tilde{\Gamma}_k(\sigma)(I + H_k^+)PW(t_k^+,t\,); \end{aligned}$$

and the operator of impulsive splitting

$$\begin{aligned} & \mathcal{S}(H)(t) \\ & \;\;=\;\; -\int_{t_0}^{t} W(t,s)P(I - H(s))(S^{-1}(s)S'(s)(I + H(s))(I - P)W(s,t)ds \\ & \qquad +\int_{t}^{\infty} W(t,s)(I - P)(I - H(s))S^{-1}(s)S(s)(I + H(s))PW(s,t)ds\,. \end{aligned}$$

Lemma 2 *Uniformly with respect to $t_0 \in J$, for some constant L_2 non depending on σ nor on ε, we have the following estimates*

$$\|\mathcal{O}(H)\|_\infty \leq L_2(\sigma + K(\sigma)\nu), \tag{27}$$

and

$$\|\mathcal{D}(H)(t)\|_\infty \leq L_2\sigma. \tag{28}$$

Proof. From condition **H5** and (25) we have the estimate

$$\begin{aligned}\|\mathcal{O}(H)(t)\| &= \int_{t_0}^{t} \frac{9L_1^2}{4\mu(s,\varepsilon)} \exp\Big\{\int_s^t \alpha_m(\tau,\varepsilon)d\tau\Big\} \left(\|\Gamma(s,\sigma)\| + \|F(s,\sigma)\|\right) ds \\ &\quad + \int_t^\infty \frac{9L_1^2}{4\mu(s,\varepsilon)} \exp\Big\{\int_t^s \alpha_m(\tau,\varepsilon)d\tau\Big\} \left(\|\Gamma(s,\sigma)\| + \|F(s,\sigma)\|\right) ds \\ &\leq \frac{9L_1^2}{4}\left(\sigma\|U_m(\cdot,\varepsilon)\|_\infty + \int_{t_0}^\infty \frac{\|F(s,\sigma)\|}{\mu(s,\varepsilon)} ds\right).\end{aligned}$$

Now the estimate (26) follows from (18) and **H5**, for some constant L_2.

For a fixed $t > 0$, let us consider the impulsive times divided as follows:

$$t_1 < t_2 < \ldots < t_k < t \leq t_{k+1} < t_{k+2} < \ldots$$

From (17) and (24) we can write the estimate

$$\begin{aligned}&\|\mathcal{D}(H)(t)\| \\ &\leq \frac{9L_1^2\sigma}{4}\sum_{i=1}^{k} \exp\Big\{\int_{t_i}^t \alpha_m(\tau,\varepsilon)d\tau\Big\} + \frac{9L_1^2\sigma}{4}\sum_{i=k+1}^{\infty} \exp\Big\{\int_t^{t_i} \alpha_m(\tau,\varepsilon)d\tau\Big\} \\ &\leq \frac{9L_1^2\sigma}{4}\Big(2 + \sum_{i=1}^{k-1} \exp\Big\{\int_{t_i}^t \alpha_m(\tau,\varepsilon)d\tau\Big\} + \sum_{i=k+2}^{\infty} \exp\Big\{\int_t^{t_i} \alpha_m(\tau,\varepsilon)d\tau\Big\}\Big) \\ &\leq \frac{9L_1^2\sigma}{4}\Big(2 + \sum_{i=1}^{k-1} \frac{1}{t_i - t_{i-1}}\int_{t_{i-1}}^{t_i} \exp\Big\{\int_s^t \alpha_m(\tau,\varepsilon)d\tau\Big\}ds \\ &\quad + \sum_{i=k+2}^{\infty} \frac{1}{t_{i+1} - t_i}\int_{t_i}^{t_{i+1}} \exp\Big\{\int_t^s \alpha_m(\tau,\varepsilon)d\tau\Big\}ds\Big)\end{aligned}$$

From (2) and **H2** we obtain

$$\begin{aligned}\|\mathcal{D}(H)(t)\| &\leq \frac{9L_1^2\sigma p}{4(K-1)}\Big(2 + \frac{p}{4(q-1)}\int_0^t \frac{1}{\mu(s,\varepsilon)} \exp\Big\{\int_s^t \alpha_m(\tau,\varepsilon)d\tau\Big\}ds \\ &\quad + \frac{p}{4(q-1)}\int_t^\infty \frac{1}{\mu(s,\varepsilon)} \exp\Big\{\int_t^s \alpha_m(\tau,\varepsilon)d\tau\Big\}\Big).\end{aligned}$$

From this estimate it follows (28) for some constant L_2. ♢

The estimate of operator $\mathcal{S}$ is more complicated. From (25) we obtain

$$\|\mathcal{S}(H)(t)\| \leq I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_{t_0}^{t} \exp\Big\{ \int_s^t \alpha_m(\tau,\varepsilon)d\tau \Big\} \|S^{-1}(s)S'(s)\| ds \,,$$
$$I_2(t) = \int_{t}^{\infty} \exp\Big\{ \int_t^s \alpha_m(\tau,\varepsilon)d\tau \Big\} \|S^{-1}(s)S'(s)\| ds.$$

We can write I_1 in the form

$$I_1(t) = \int_{t_0}^{t} \exp\Big\{ \int_s^t \alpha_m(\tau,\varepsilon)d\tau \Big\} \frac{d}{ds} \int_t^s \|S^{-1}(\xi)S'(\xi)\| d\xi ds \,.$$

Integration by parts gives

$$\begin{aligned} I_1(t) \quad = \quad & \exp\Big\{ \int_{t_0}^t \alpha_m(\tau,\varepsilon)d\tau \Big\} \int_{t_0}^t \|S^{-1}S'\|(u)du \\ & - \int_{t_0}^t \alpha_m(s,\varepsilon) \exp\Big\{ \int_s^t \alpha_m(\tau,\varepsilon)d\tau \Big\} \int_s^t \|S^{-1}S'\|(u)du \,. \end{aligned}$$

Taking into account the estimate (20) we obtain

$$\begin{aligned} I_1(t) \quad \leq \quad & K(\sigma)L(\delta,p) \exp\Big\{ \int_{t_0}^t \alpha_m(\tau,\varepsilon)d\tau \Big\} (t-t_0) \\ & - K(\sigma)L(\delta,p) \int_{t_0}^t \alpha_m(s,\varepsilon) \exp\Big\{ \int_s^t \alpha_m(\tau,\varepsilon)d\tau \Big\} (t-s) ds \,. \end{aligned}$$

Once again, integrating by parts the last integral, from the right hand side of this inequality we obtain

$$I_1(t) \leq K(\sigma)L(\delta,p) \int_{t_0}^t \exp\Big\{ \int_s^t \alpha_m(\tau,\varepsilon)d\tau \Big\} ds \,. \tag{29}$$

By similar tokens

$$I_2(t) \leq K(\sigma)L(\delta,p) \int_t^\infty \exp\Big\{ \int_t^s \alpha_m(\tau,\varepsilon)d\tau \Big\} ds \,. \tag{30}$$

Using (2) and the hypothesis **H5** we obtain the estimate

$$I_i(t) \leq MK(\sigma)L(\delta,p) \|\mu(\cdot,\varepsilon)\|_\infty, \quad i=1,2 \,.$$

Thus, for a given $\alpha > 0$, if $\|\mu(\cdot,\varepsilon)\|_\infty$ is small enough, we will have

$$\|\mathcal{S}(H)(t)\| \leq \alpha \,. \tag{31}$$

Theorem 1 *The conditions* **H1-H5** *imply, for a small values of the norm* $\{\mu(\cdot,\varepsilon)\}$*, the existence of a function* $T : [t_0,\infty) \to V^{n\times n}$ *satisfying* **T1-T5**. *Moreover* $\|T\| \le \frac{3}{2}$, $\|T^{-1}\| \le 2$.

Proof. According to Lemma 4 and Lemma 5, the operator $\mathcal{T} = \mathcal{O} + \mathcal{D} + \mathcal{S}$, for small values of σ, ν and α (see (31), satisfies

$$\mathcal{T} : \{H \in BC(\{t_k\}) : \|H\|_\infty \le 1/2\} \to \{H \in BC(\{t_k\}) : \|H\|_\infty \le 1/2\}.$$

Also, for small values of σ, ν and α this operator is a contraction. This and further details of this theory are well known for exponential dichotomies. The corresponding result for the dichotomy (24) are similar [6, 14, 12]. ♢

Once we have split (17), we write System (26) in the form

$$\begin{aligned} \mu(t,\varepsilon)z'(t) = \Big(&\operatorname{Re}\Lambda(t) + iG(t) + \{(\Gamma(t,\sigma) + F(t,\sigma))\,T(t)\}_1 \\ &-\mu(t,\varepsilon)\{S^{-1}(t)S'(t)T(t)\}_1\Big) z(t), \quad t \ne t_k, \\ z(t_k^+) = \Big(&N_k e^{i\operatorname{Arg}(D_k)} + \{G_k(\sigma)\}_1\Big)\, z(t_k), \quad k \in \mathbb{N}, \end{aligned} \tag{32}$$

where

$$G_k(\sigma) = \Big(I - N_k e^{i\operatorname{Arg}(D_k)}\{\hat\Gamma_k(\sigma)\}_1\Big)^{-1} N_k e^{i\operatorname{Arg}(D_k)} - N_k e^{i\operatorname{Arg}(D_k)}.$$

From (22) we obtain

$$\|G_k(\sigma)\| \le L_3\sigma, \quad L_3 = 2\|\{C_k\}\|_\infty, \quad \text{if } 0 < 2\sigma < \|\{C_k\}\|_\infty^{-1}. \tag{33}$$

The right hand side equation of (32) commute with projection P. Therefore, (32) may be written as two systems of dimensions m and $n-m$,

$$\begin{aligned} \mu(t,\varepsilon)z_j'(t) = \Big(&\operatorname{Re}\Lambda_j(t) + iG_j(t) + \Gamma_j(t,\sigma) + F_j(t,\sigma) \\ &+\mu(t,\varepsilon)V_j(t)\Big) z_i(t), \quad t \ne t_k, \end{aligned} \tag{34}$$

$$z_j(t_k^+) = \Big(N_{k,j} e^{i\operatorname{Arg}(D_{k,j})} + G_{k,j}(\sigma)\Big)\, z_j(t_k), \quad k \in \mathbb{N}, \tag{35}$$

where $j = 1, 2$. The matrices $\Lambda_1(t)$, $\Lambda_2(t)$ are defined by

$$\Lambda_1(t) = \{\lambda_1(t), \lambda_2(t), \dots, \lambda_m(t)\},\ \Lambda_2(t) = \{\lambda_{m+1}(t), \lambda_{m+2}(t), \dots, \lambda_n(t)\},$$

and similarly the diagonal matrices $G_j(t)$, $N_{k,j}$ and $D_{k,j}$ are defined. The matrices $G_{k,j}(\sigma)$ satisfy estimate (33). $\Gamma_j(t,\sigma)$ has the estimate (18), where instead of σ it is necessary to write 3σ, $F_j(t,\sigma)$ has the estimate (19) and

$$\|\int_s^t V_j(\tau)d\tau\| \le 3\|\int_s^t S^{-1}(\tau)S'(\tau)d\tau\| \le 3L(\delta,p)K(\sigma)(t-s), \quad t \ge s.$$

The Gronwall inequality for piecewise continuous functions [3] gives the following estimates for $Z_i(t)$, the fundamental matrices of systems (34), $j = 1, 2$:

$$\|Z_1(t)Z_1^{-1}(s)\| \le L \exp\Big\{ \int_s^t \mu_1(\tau, \varepsilon) d\tau \Big\}, \quad s \le t,$$
$$\|Z_2(t)Z_2^{-1}(s)\| \le L \exp\Big\{ \int_s^t \mu_2(\tau, \varepsilon) d\tau \Big\}, \quad t \le s,$$

where L is a constant non depending on ε neither on σ, and

$$\mu_1(t, \varepsilon) = \frac{Re(\lambda_m(t))}{\mu(t, \varepsilon)} + u_m(t) + L_4\sigma + 3L(\delta, p)K(\sigma),$$
$$\mu_2(t, \varepsilon) = \frac{Re(\lambda_{m+1}(t))}{\mu(t, \varepsilon)} + u_{m+1}(t) + L_4\sigma + 3L(\delta, p)K(\sigma),$$

with a constant $L_4 = 3 + L_3$. Since the decoupled system (34) is kinetically similar to System (1), we obtain for this system the following

Theorem 2 *If the hypotheses* **H1-H5** *are fulfilled, then for a small value of* $\|\mu(\cdot, \varepsilon)\|$ *the System (1) has the following* (μ_1, μ_2)*-dichotomy:*

$$\|X(t, \varepsilon)PX^{-1}(s, \varepsilon)\| \le L \exp\Big\{ \int_s^t \mu_1(\tau, \varepsilon) d\tau \Big\}, \quad s \le t, \tag{36}$$
$$\|X(t, \varepsilon)PX^{-1}(s, \varepsilon)\| \le L \exp\Big\{ \int_s^t \mu_2(\tau, \varepsilon) d\tau \Big\}, \quad t \le s\,,$$

where L *is a constant independent of* ε *and* σ*.*

6 Dichotomies for linear differential systems

In this section we present some applications of formulas (36).

The case $\|\mu(\cdot, \varepsilon)\|_\infty \le \varepsilon$

Theorem 3 *Under conditions* **H1-H5***, if* $\|\mu(\cdot, \varepsilon)\| \le \varepsilon$, $\varepsilon \in (0, \infty)$*, then there exists a positive number* ε_0 *such that for each* $\varepsilon \in (0, \varepsilon_0)$*, the impulsive system (1) has the dichotomy (36).*

In the particular case $\mu(t, \varepsilon) = \varepsilon$, we obtain the system

$$\begin{aligned} \varepsilon x'(t) &= A(t)x(t), \quad t \ne t_k, \quad J = [0, \infty), \\ x(t_k^+) &= C_k x(t_k^-), \quad k \in \mathbb{N} = \{1, 2, 3, \ldots\}, \end{aligned} \tag{37}$$

and the dichotomy (36) has the form

$$\begin{aligned} \mu_1(t, \varepsilon) &= \frac{Re(\lambda_m(t)) + \varepsilon u_m(t) + L_4\varepsilon\sigma + 3\varepsilon L(\delta, p)K(\sigma)}{\varepsilon}, \\ \mu_2(t, \varepsilon) &= \frac{Re(\lambda_{m+1}(t)) + \varepsilon u_{m+1}(t) + L_4\varepsilon\sigma + 3\varepsilon L(\delta, p)K(\sigma)}{\varepsilon}. \end{aligned}$$

Considering in (37) $C_k = I$ for $k \in N$, we obtain that the solutions of this sytems coincide with the solutions of the ordinary system with a small and a positive parameter at the derivative

$$\varepsilon y'(t) = A(t)y(t). \tag{38}$$

Denoting by $Y(t,\varepsilon)$ the fundamental matrix of System (38), from (36) we obtain the dichotomy

$$\|Y(t,\varepsilon)PY^{-1}(s,\varepsilon)\| \leq K \exp\big\{\int_s^t \mu_1(\tau,\varepsilon)d\tau\big\}, \quad s \leq t,$$
$$\|Y(t,\varepsilon)(I-P)Y^{-1}(s,\varepsilon)\| \leq K \exp\big\{-\int_t^s \mu_2(\tau,\varepsilon)\big\}, \quad t \leq s,$$

where

$$\mu_1(t,\varepsilon) = \frac{Re(\lambda_m(t)) + L_4\varepsilon\sigma + \varepsilon L(\delta,0)K(\sigma)}{\varepsilon},$$
$$\mu_2(t,\varepsilon) = \frac{Re(\lambda_{m+1}(t)) + L_4\varepsilon\sigma + 3\varepsilon L(\delta,0)K(\sigma)}{\varepsilon}.$$

If $Re(\lambda_m(t)) \leq -\alpha < 0$ and $Re(\lambda_m(t)) \geq \beta > 0$, for all values of t, for a small ε_0, we obtain for (38) the dichotomy

$$\|Y(t,\varepsilon)PY^{-1}(s,\varepsilon)\| \leq L \exp\left\{-\frac{\alpha}{2\varepsilon}(t-s)\right\}, \quad s \leq t,$$
$$\|Y(t,\varepsilon)(I-P)Y^{-1}(s,\varepsilon)\| \leq L \exp\big\{\frac{\beta}{2\varepsilon}(t-s)\big\}, \quad t \leq s,$$

for $\varepsilon \in (0,\varepsilon_0]$ and L is independent of ε. This dichotomy was obtained by Chang [7] for almost periodic systems and by Mitropolskii-Lykova [9] for a system (38) which function $A(t)$ is uniformly continuous on J.

The case $\mu(t,\varepsilon) = \mu(t) \to 0$, if $t \to \infty$

In this case the condition $\lim_{t\to\infty} \mu(t) = 0$ allows to obtain a small value of $|\mu(t,\varepsilon)|$ if we consider $t \in [t_0,\infty)$. All the reasoning leading to Theorem 2 can be acomplished on the interval $[t_0,\infty)$ instead of $[0,\infty)$.

Theorem 4 *If we assume valid* **H1-H5***, where $U(t,\varepsilon)$ is defined with*

$$\alpha_m(t,\varepsilon) = \frac{\lambda_m(t) - \lambda_{m+1}(t)}{\mu(t)} + u_m(t) - u_{m+1}(t),$$

(therefore $U(t,\varepsilon)$ does not depend on ε), then the impulsive system

$$\mu(t)x'(t) = A(t)x(t), \quad t \neq t_k,\ J = [0,\infty)$$
$$x(t_k^+) = C_k x(t_k^-), \quad k \in \mathbb{N} = \{1,2,3,\ldots\},$$

has the dichotomy

$$\|X(t)PX^{-1}(s)\| \leq K \exp\left\{\int_s^t \mu_1(\tau)d\tau\right\}, \quad s \leq t,$$
$$\|X(t)(I-P)X^{-1}(s)\| \leq K \exp\left\{\int_t^s \mu_2(\tau)\right\}, \quad t \leq s\,,$$

where

$$\mu_1(t) = \frac{Re(\lambda_m(t)) + L_4\mu(t)\sigma + \mu(t)L(\delta,0)K(\sigma)}{\mu(t)},$$
$$\mu_2(t) = \frac{Re(\lambda_{m+1}(t)) + L_4\sigma\mu(t) + 3\mu(t)L(\delta,0)K(\sigma)}{\mu(t)}.$$

As an application of the above formula let us consider the ordinary system

$$\mu(t)x'(t) = A(t)x(t), \ \lim_{t\to\infty}\mu(t) = 0. \tag{39}$$

Theorem 5 *If $A(\cdot)$ satisfies* **H1** *and the function $U_m(t)$ defined in* **H5** *with*

$$\alpha_m(t,\varepsilon) = \frac{\lambda_m(t) - \lambda_{m+1}(t)}{\mu(t)},$$

is bounded, then system (39) has the dichotomy (36), where

$$\mu_1(t) = \frac{Re(\lambda_m(t)) + 3\sigma\mu(t) + \mu(t)L(\delta,0)K(\sigma)}{\mu(t)},$$
$$\mu_2(t) = \frac{Re(\lambda_{m+1}(t)) - 3\sigma\mu(t) - 3\mu(t)L(\delta,0)K(\sigma)}{\mu(t)}.$$

The above theorem gives conditions of existence of a (μ_1,μ_2)- dichotomy for (39) with an unbounded function $\mu(t)^{-1}A(t)$. These systems have been studied in [13].

References

[1] Bellman, R.: *Stability Theory of Differential Equations*, Dover Publications, New York (1953).

[2] Bainov, D.D.; Simeonov, P.S.: *Systems with impulse effect (Stability Theory and Applications)*, Ellis Horwood & John Wiley, New York (1989).

[3] Bainov, D.D.; Simeonov P.S.: *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht (1992).

[4] Simeonov, P.S.; Bainov, D.D.: Criteria for dichotomies of linear impulsive differential equations, *Tamkang J. Math.*, 25, 2, 101-112 (1994).

[5] Conway, J.B.: *Functions of One Complex Variables*, Springer, Student Edition, New York (1973).

[6] Coppel, W.A.: *Dichotomies in Stability Theory*, Lecture Notes in Mathematics, 629, Springer Verlag (1978).

[7] Chang, K.W.: Almost periodic solutions of singularly perturbed systems of differential equations, *J. Diff. Eqns.* 4, 300-307 (1968).

[8] Lakshmikantham, V.; Bainov, D.D.; Simeonov, P.S.: *Theory of Impulsive Differential Equations*, World Scientific (1989).

[9] Lizana, M.; Muldowney, J.S.: Existence and roughness of (μ_1, μ_2)-dichotomies, preprint (1997).

[10] Mitropolskii, Y.A.; Likova, O.B.: *Integral Manifolds in Nonlinear Mechanich*, (Russian) Nauka, Moscow (1973).

[11] R. Naulin, R.; Pinto, M.: Splitting of linear differential systems with impulsive effect, to appear in *Rocky Mountain J. Math.* (1999).

[12] R. Naulin, R.; Pinto, M.: Quasi-diagonalization of linear impulsive systems and applications, *J. Math. Anal. Appl.,* Vol. 208, 281-297 (1997).

[13] R. Naulin, R.; Pinto, M.: Dichotomies for differential systems with unbounded coefficients", *Dyn. Syst. Appl.* , 3, 333-348 (1994).

[14] Papashinopoulos, G.: Some roughness results concerning reducibility for linear difference equations, *Internat. J. Math.* & *Math. Sci.* , Vol. 11, N^o 4, 793-804 (1988).

[15] Smith, D.R.: *Singular Perturbation Theory*, Cambridge University Press (1985).

Raúl Naulin
Departamento de Matemáticas, Universidad de Oriente
Apartado 245, Cumaná 6101-A, Venezuela
e-mail: rnaulin@cumana.sucre.udo.edu.ve