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OSCILLATION OF THIRD ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. We consider third order functional differential equations with discrete and continuous delay. We then develop several theorems related to the oscillatory behavior of these differential equations.

1. Introduction

Our goal in this paper is to study functional differential equations of the form

$$(b(t)(a(t)x'(t))')' + \sum_{i=1}^{m} q_i(t)f(x(\sigma_i(t))) = h(t), \tag{1.1}$$

where $a, b, h \in C([t_0, \infty), \mathbb{R}), a(t), b(t) > 0, \quad f : \mathbb{R} \to \mathbb{R}$ continuous, $\sigma_i(t) \to \infty$, as $t \to \infty, \quad i = 1, 2, \dots, m$, and

$$(b(t)(a(t)x'(t))')' + \int_c^d q(t, \xi)f(x(\sigma(t, \xi)))d\xi = 0, \tag{1.2}$$

where $a, b \in C([t_0, \infty), \mathbb{R})$, $f \in C(\mathbb{R}, \mathbb{R})$. The oscillations of solutions of third order equations were studied by Rao and Dahiya [8], Tantawy [9], Waltman [10] and Zafer and Dahiya [11]. The results in this paper for equation (1.1) are more general comparing to Zafer and Dahiya [11]. The results for equation (1.2) are essentially new.

As is customary, a solution of equations (1.1) and (1.2) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. The solution of equations (1.1) and (1.2) is called almost oscillatory if it is oscillatory or $\lim_{t\to\infty} x^{(i)}(t) = 0$, $i = 0, 1, 2$.

2. Main Results

Oscillatory behavior of third order differential equations with discrete delay. Assume that $xf(x) > 0$, $x \neq 0$, $q_i(t) \geq 0$ is not identically zero in any half

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line of the form (τ, ∞) for some $\tau \geq 0$, $i = 1, 2, \dots, m$ and $\sigma_i(t) < t$, $\sigma_i'(t) > 0$, i=1,2,...,m, $b'(t) \geq 0$, and

$$\int^{\infty} \frac{dt}{b(t)} = \infty, \quad \int^{\infty} \frac{dt}{a(t)} = \infty. \tag{2.1}$$

Theorem 2.1. *Let $f(x) = x$ and $h(t) = 0$. Suppose that there exist a differentiable function $p \in C([t_0, \infty), \mathbb{R})$, $p(t) > 0$ such that*

$$\int^{\infty} \Big[q(t)p(t) - \frac{b(t)(p'(t))^2}{\sum_{i=1}^{m} \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))}\sigma_i'(t)4p(t)}\Big] dt = \infty, \tag{2.2}$$

where $q(t) = min\{q_1(t), q_2(t), \dots, q_m(t)\}$, for every $T \geq 0$, and that

$$\int_{\sigma(t)}^{t} \Big[\int_{\sigma(t)}^{r} \frac{1}{a(u)} du \Big(\int_{u}^{r} \frac{1}{b(v)} dv\Big)\Big] \sum_{i=1}^{m} q_i(r) dr > 1, \tag{2.3}$$

where $\sigma(t) = max\{\sigma_1(t), \sigma_2(t), \dots, \sigma_m(t)\}$. Then the equation (1.1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of (1.1). Assume $x(t)$ is eventually positive. Since $\sigma_i(t) \to \infty$ as $t \to \infty$ for $i = 1, 2, \dots, m$, there exist a $t_1 \geq t_0$ such that $x(t) > 0$ and $x(\sigma_i(t)) > 0$ for $t \geq t_1$. From (1.1), we have

$$(b(t)(a(t)x'(t))')' = -\sum_{i=1}^{m} q_i(t)x(\sigma_i(t)). \tag{2.4}$$

Since $q_i(t)$ is not negative and $x(\sigma_i(t)) > 0$ is positive for $t \geq t_1$, the right-hand side becomes non-positive. Therefore, we have

$$(b(t)(a(t)x'(t))')' \leq 0$$

for $t \geq t_1$. Thus, $x(t)$, $x'(t)$, $(a(t)x'(t))'$ are monotone and eventually one-signed. Now we want to show that there is a $t_2 \geq t_1$ such that for $t \geq t_2$

$$(a(t)x'(t))' > 0. \tag{2.5}$$

Suppose this is not true, then $(a(t)x'(t))' \leq 0$. Since $q_i(t)$, $i = 1, 2, \dots, m$ are not identically zero and $b(t) > 0$, it is clear that there is $t_3 \geq t_2$ such that $b(t_3)(a(t_3)x'(t_3))' < 0$. Then, for $t > t_3$ we have

$$b(t)(a(t)x'(t))' \leq b(t_3)(a(t_3)x'(t_3))' < 0. \tag{2.6}$$

Dividing (2.6) by $b(t)$ and then integrating between t_3 and t, we obtain

$$a(t)x'(t) - a(t_3)x'(t_3) < b(t_3)(a(t_3)x(t_3)')' \int_{t_3}^{t} \frac{1}{b(s)} ds. \tag{2.7}$$

Letting $t \to \infty$ in (2.7), and because of (2.1) we see that $a(t)x'(t) \to -\infty$ as $t \to \infty$. Thus there is a $t_4 \geq t_3$ such that $a(t_4)x'(t_4) < 0$. Using $(a(t)x'(t))' \leq 0$, we have for $t \geq t_4$

$$a(t)x'(t) \leq a(t_4)x'(t_4). \tag{2.8}$$

If we divide (2.8) by $a(t)$ and integrate from t_4 to t with $t \to \infty$, the right-hand side becomes negative. Thus, we have $x(t) \to -\infty$. But this is a contradiction $x(t)$ being eventually positive and therefore it proves that (2.5) holds. Now we consider two cases.

Suppose $x'(t)$ is eventually positive, say $x'(t) > 0$ for $t \geq t_2$. Define the function $z(t)$ by

$$z(t) = \frac{b(t)(a(t)x'(t))'}{\sum_{i=1}^m x(\sigma_i(t))} p(t).$$

It is obvious that $z(t) > 0$ for $t \geq t_2$ and $z'(t)$ is

$$z'(t) = -\frac{\sum_{i=1}^m q_i(t)x(\sigma_i(t))}{\sum_{i=1}^m x(\sigma_i(t))} p(t) + \frac{p'(t)}{p(t)} z(t) - \frac{\sum_{i=1}^m x'(\sigma_i(t))\sigma_i'(t)}{\sum_{i=1}^m x(\sigma_i(t))} z(t).$$

Then

$$z'(t) \leq -q(t)p(t) + \frac{p'(t)}{p(t)} z(t) - \frac{\sum_{i=1}^m x'(\sigma_i(t))\sigma_i'(t)}{\sum_{i=1}^m x(\sigma_i(t))} z(t), \tag{2.9}$$

where $q(t) = min\{q_1(t), q_2(t), \dots, q_m(t)\}$. On the other hand, using $(b(t)(a(t)x^{'}(t))^{'})^{'} \leq 0$, $b'(t) \geq 0$ and (2.5), we can find that

$$(a(t)x'(t))'' \leq 0. \tag{2.10}$$

Using (2.10) and the equality

$$a(t)x'(t) = a(T)x'(T) + \int_T^t (a(s)x'(s))'ds, \tag{2.11}$$

we have

$$a(t)x'(t) \geq (t - T)(a(t)x'(t))' \tag{2.12}$$

for $T \geq t_2$. Since $(a(t)x'(t))'$ is non-increasing, we obtain

$$a(\sigma_i(t))x'(\sigma_i(t)) \geq (\sigma_i(t) - T)(a(t)x'(t))' \quad for \quad i = 1, 2, \dots, m. \tag{2.13}$$

Multiplying both sides of (2.13) by

$$\frac{\sigma_i'(t)}{a(\sigma_i(t))}$$

and taking the summation from 1 to m, we have

$$\sum_{i=1}^m \sigma_i'(t)x'(\sigma_i(t)) \geq \sum_{i=1}^m \frac{(\sigma_i(t) - T)}{a(\sigma_i(t))} \sigma_i'(t)(a(t)x'(t))'. \tag{2.14}$$

Then, using (2.14) in (2.9), it follows that

$$z'(t) \leq -q(t)p(t) + \frac{p'(t)}{p(t)} z(t) - \frac{\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))} \sigma_i'(t)}{b(t)p(t)} z^2(t),$$

and completing the square will leads to

$$z'(t) \leq -q(t)p(t) + \frac{b(t)(p')^2(t)}{\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))} \sigma_i'(t)4p(t)}. \tag{2.15}$$

Integrating (2.15) between T and t and letting $t \to \infty$, we see that $\lim_{t\to\infty} z(t) = -\infty$. This contradicts $z(t)$ being eventually positive.

If $x'(t)$ is eventually negative. We integrate (1.1) from t to ∞ and since

$$b(t)(a(t)x'(t))' > 0,$$

we have

$$-b(t)(a(t)x'(t))' + \int_t^\infty \sum_{i=1}^m q_i(r)x(\sigma_i(r))dr \leq 0. \tag{2.16}$$

Now integrating (2.16) from t to ∞ after dividing by $b(t)$ and using $a(t)x'(t) < 0$, will lead to

$$a(t)x'(t) + \int_t^\infty \Big(\int_t^r \frac{1}{b(u)}du\Big) \sum_{i=1}^m q_i(r)x(\sigma_i(r))dr \le 0. \tag{2.17}$$

Dividing (2.17) by $a(t)$ and integrating again from t to ∞ gives

$$\int_t^\infty \Big[\int_t^r \frac{1}{a(u)}du\Big(\int_u^r \frac{1}{b(v)}dv\Big)\Big] \sum_{i=1}^m q_i(r)x(\sigma_i(r))dr \le x(t). \tag{2.18}$$

Replacing t by $\sigma(t)$ in (2.18), where $\sigma(t) = \max\{\sigma_1(t), \sigma_2(t), \dots, \sigma_m(t)\}$, will give

$$\int_{\sigma(t)}^t \Big[\int_{\sigma(t)}^r \frac{1}{a(u)}du\Big(\int_u^r \frac{1}{b(v)}dv\Big)\Big] \sum_{i=1}^m q_i(r)x(\sigma_i(r))dr \le x(\sigma(t)). \tag{2.19}$$

Using the fact that $\sigma_i(t) < t$ and $x(t)$ is decreasing in (2.19), we obtain

$$\int_{\sigma(t)}^t \Big[\int_{\sigma(t)}^r \frac{1}{a(u)}du\Big(\int_u^r \frac{1}{b(v)}dv\Big)\Big] \sum_{i=1}^m q_i(r)dr \le 1.$$

This is a contradiction to (2.3). Therefore, the proof is complete

Example 2.2. Consider the following functional differential equation

$$(e^{-t}x')'' + \sum_{i=1}^2 (2i-1)e^{-t}x(t-(i+1)\pi) = 0.$$

Now $a(t) = e^{-t}$, $b(t) = 1$, $q_1(t) = e^{-t}$, $q_2(t) = 3e^{-t}$ $\sigma_1(t) = t - 2\pi$, $\sigma_2(t) = t - 3\pi$, $p(t) = e^t$.

We can easily see that the conditions of Theorem 2.1 are satisfied. It is easy to verify that $x(t) = cost$ is a solution of this problem.

Theorem 2.3. *Let $f'(x) \ge \lambda$ for some $\lambda > 0$, and $h(t) = 0$. Suppose that there exist a differentiable function $p \in C([t_0, \infty), \mathbb{R})$, $p(t) > 0$ such that*

$$\int^\infty \Big[q(t)p(t) - \frac{b(t)(p'(t))^2}{\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))}\sigma_i'(t)4\lambda p(t)}\Big]dt = \infty, \tag{2.20}$$

where $q(t) = min\{q_1(t), q_2(t), \dots, q_m(t)\}$, for every $T \ge 0$, and that

$$\limsup_{t\to\infty} \int_{\sigma(t)}^t \Big[\int_{\sigma(t)}^r \frac{1}{a(u)}du\Big(\int_u^r \frac{1}{b(v)}dv\Big)\Big] \sum_{i=1}^m q_i(r)dr = \infty, \tag{2.21}$$

where $\sigma(t) = max\{\sigma_1(t), \sigma_2(t), \dots, \sigma_m(t)\}$. Then the equation (1.1) is oscillatory.

Proof. The beginning part of the proof is similar to the proof of Theorem 2.1 until we reach at two possible cases. Suppose $x'(t)$ is eventually positive. Then, we can define

$$z(t) = \frac{b(t)(a(t)x'(t))'}{\sum_{i=1}^m f(x(\sigma_i(t)))}p(t) > 0.$$

It is obvious that $z(t) > 0$ for $t \ge t_2$ and $z'(t)$ is

$$\begin{aligned} z'(t) &= -\frac{\sum_{i=1}^m q_i(t)f(x(\sigma_i(t)))}{\sum_{i=1}^m f(x(\sigma_i(t)))}p(t) \\ &\quad + \frac{p'(t)}{p(t)}z(t) - \frac{\sum_{i=1}^m f'(x(\sigma_i(t)))x'(\sigma_i(t))\sigma_i'(t)}{\sum_{i=1}^m f(x(\sigma_i(t)))}z(t). \end{aligned}$$

Then

$$z'(t) \le -q(t)p(t) + \frac{p'(t)}{p(t)}z(t) - \frac{\sum_{i=1}^m f'(x(\sigma_i(t))x'(\sigma_i(t))\sigma_i'(t)}{\sum_{i=1}^m f(x(\sigma_i(t)))}z(t), \tag{2.22}$$

where $q(t) = \min\{q_1(t), q_2(t), \dots, q_m(t)\}$. On the other hand, since $(b(t)(a(t)x^{'}(t))^{'})^{'} \le 0$, (2.5) holds and $b'(t) \ge 0$, we can obtain

$$(a(t)x'(t))'' \le 0. \tag{2.23}$$

Using (2.23) and the equality

$$a(t)x'(t) = a(T)x'(T) + \int_T^t (a(s)x'(s))'ds \tag{2.24}$$

will lead to

$$a(t)x'(t) \ge (t-T)(a(t)x'(t))'. \tag{2.25}$$

Now using non-increasing nature of $(a(t)x'(t))'$, we obtain

$$a(\sigma_i(t))x'(\sigma_i(t)) \ge (\sigma_i(t)-T)(a(t)x'(t))' \quad for \quad i = 1, 2, \dots, m. \tag{2.26}$$

Multiplying both sides of (2.26) by

$$\frac{\sigma_i'(t)}{a(\sigma_i(t))}$$

and taking the summation from 1 to m, we have

$$\sum_{i=1}^m \sigma_i'(t)x'(\sigma_i(t)) \ge \sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))}\sigma_i'(t)(a(t)x'(t))^{'}. \tag{2.27}$$

Then, using (2.27) in (2.22), it follows that

$$z'(t) \le -q(t)p(t) + \frac{p'(t)}{p(t)}z(t) - \lambda\frac{\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))}\sigma_i'(t)}{b(t)p(t)}z^2(t),$$

and then completing the square leads to

$$z'(t) \le -q(t)p(t) + \frac{b(t)(p'(t))^2}{\lambda\sum_{i=1}^m \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))}\sigma_i'(t)4p(t)}. \tag{2.28}$$

Integrating (2.28) between T to t and letting $t \to \infty$, we see that $\lim_{t\to\infty} z(t) = -\infty$. This contradicts $z(t)$ being eventually positive.

If $x'(t)$ is eventually negative and proceeding as in the proof of Theorem 2.1 we will end up with

$$\int_{\sigma(t)}^\infty \Big[\int_{\sigma(t)}^r \frac{1}{a(u)}du\Big(\int_u^r \frac{1}{b(v)}dv\Big)\Big] \sum_{i=1}^m q_i(r)f(x(\sigma_i(r)))dr \le x(\sigma(t)),$$

where $\sigma(t) = \max\{\sigma_1(t), \sigma_1(t), \dots, \sigma_n(t)\}$. Thus we have

$$\int_{\sigma(t)}^t \Big[\int_{\sigma(t)}^r \frac{1}{a(u)}du\Big(\int_u^r \frac{1}{b(v)}dv\Big)\Big] \sum_{i=1}^m q_i(r)f(x(\sigma_i(r)))dr \le x(\sigma(t)). \tag{2.29}$$

Using the fact that $\sigma_i(t) < t$, $f(x)$ is increasing and $x(t)$ is decreasing in (2.29), we obtain

$$\int_{\sigma(t)}^t \Big[\int_{\sigma(t)}^r \frac{1}{a(u)}du\Big(\int_u^r \frac{1}{b(v)}dv\Big)\Big] \sum_{i=1}^m q_i(r)dr \le \frac{x(\sigma(t))}{f(x(\sigma(t)))}.$$

Since $x(t)$ is decreasing and positive, it is approaching a finite non-negative number as $t \to \infty$. In view of (2.21) and the last equation, it is not possible that $\lim_{t\to\infty} x(t) > 0$. Suppose $\lim_{t\to\infty} x(t) = 0$, then

$$\lim_{t\to\infty} \frac{x(\sigma(t))}{f(x(\sigma(t)))} = \lim_{t\to\infty} \frac{1}{f'(x(\sigma(t)))} = \frac{1}{f'(0)} \le \frac{1}{\lambda}.$$

This is a contradiction to (2.21) . Therefore, the proof is complete.

Theorem 2.4. *Suppose that $f'(x) \ge \lambda$ for some $\lambda > 0$ and*

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \Big[\int_{\sigma(t)}^{r} \frac{1}{a(u)} du \Big(\int_{u}^{r} \frac{1}{b(v)} dv\Big)\Big] \sum_{i=1}^{m} q_i(r) dr = \infty. \tag{2.30}$$

In addition, suppose that there exist a continuously differentiable function $p \in C([t_0,\infty),\mathbb{R})$, $p(t) > 0$ and an oscillatory function $\psi(t)$ such that

$$\int^{\infty} \Big[q(t)p(t) - \frac{b(t)(p'(t))^2}{\sum_{i=1}^{m} \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))}\sigma_i'(t) 4\lambda d p(t)}\Big] dt = \infty \tag{2.31}$$

for some $d \in (0,1)$ and for every $T \ge 0$, and

$$(b(t)(a(t)\psi'(t))')' = h(t), \quad \lim_{t\to\infty} \psi^{(i)}(t) = 0, \quad i = 0,1,2. \tag{2.32}$$

Then the equation (1.1) is almost oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of (1.1). Without loss of generality we may assume that $x(t)$ is eventually positive. Consider

$$y(t) = x(t) - \psi(t). \tag{2.33}$$

Obviously $y(t)$ is eventually positive, otherwise, $x(t) < \psi(t)$ and it is a contradiction with oscillatory behavior of $\psi(t)$. We know that,

$$(b(t)(a(t)y'(t))')' \le 0. \tag{2.34}$$

Proceeding as in the proof of Theorem 2.1, there is a $t_1 \ge 0$ such that for $t \ge t_1$

$$(a(t)y'(t))' > 0 \quad \textit{and} \quad (a(t)y'(t))'' \le 0.$$

Consider again two cases. Suppose that $y'(t)$ is eventually positive, then $y(t)$ is increasing and eventually positive. On the other hand, since $\psi(t) \to 0 \quad as \quad t \to \infty$ and $y(t) = x(t) - \psi(t)$, there exists a $t_2 \ge t_1$ such that

$$x(\sigma_i(t)) \ge d y(\sigma_i(t)) \quad for \quad t \ge t_2 \quad and \quad d \in (0,1), \quad i = 1,2,\dots,m.$$

Since f is an increasing function, we obtain

$$f(x(\sigma_i(t))) \ge f(dy(\sigma_i(t))) \quad for \quad t \ge t_2, \quad i = 1,2,\dots,m.$$

Define $z(t)$ by

$$z(t) = \frac{b(t)(a(t)y'(t))'}{\sum_{i=1}^{m} f(dy(\sigma_i(t)))} p(t),$$

then obviously $z(t) > 0$ for $t \ge t_2$ and $z'(t)$ is

$$\begin{aligned} z'(t) \quad = \quad & -\frac{\sum_{i=1}^{m} q_i(t) f(x(\sigma_i(t)))}{\sum_{i=1}^{m} f(dy(\sigma_i(t)))} p(t) + \frac{p'(t)}{p(t)} z(t) \\ & - d\frac{\sum_{i=1}^{m} f'(dy(\sigma_i(t))) y'(\sigma_i(t))\sigma_i'(t)}{\sum_{i=1}^{m} f(dy(\sigma_i(t)))} z(t). \end{aligned}$$

Then, using $f'(x) \geq \lambda > 0$, we obtain

$$z'(t) \leq -q(t)p(t) + \frac{p'(t)}{p(t)}z(t) - d\lambda \frac{\sum_{i=1}^{m} y'(\sigma_i(t))\sigma_i'(t)}{\sum_{i=1}^{m} f(dy(\sigma_i(t)))}z(t), \tag{2.35}$$

where $q(t) = min\{q_1(t), q_2(t), \dots, q_m(t)\}$. We can now show that

$$\sum_{i=1}^{m} \sigma_i'(t)y'(\sigma_i(t)) \geq \sum_{i=1}^{m} \frac{(\sigma_i(t) - T)}{a(\sigma_i(t))}\sigma_i'(t)(a(t)y'(t))' \tag{2.36}$$

as in proof of Theorem 2.1. Using (2.35) and (2.36), we have

$$z'(t) \leq -q(t)p(t) + \frac{p'(t)}{p(t)}z(t) - d\lambda \frac{\sum_{i=1}^{m} \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))}\sigma_i'(t)}{b(t)p(t)}z^2(t).$$

Completing the square in the above equation leads to

$$z'(t) \leq -q(t)p(t) + \frac{b(t)(p')^2(t)}{\sum_{i=1}^{m} \frac{(\sigma_i(t)-T)}{a(\sigma_i(t))}\sigma_i'(t)4\lambda dp(t)}. \tag{2.37}$$

Integrating (2.37) from T to t and letting $t \to \infty$, we see that $\lim_{t\to\infty} z(t) = -\infty$. This contradicts $z(t)$ being eventually positive.

Now suppose $y'(t)$ is eventually negative. Since y is eventually positive and decreasing, $\lim_{t\to\infty} y(t) = c$, where c is a nonnegative number. Therefore, $\lim_{t\to\infty} x(t) = c$. Integrating (1.1) three times as we did in the proof of Theorem 2.1, we will end up with

$$\int_{\sigma(t)}^{\infty} \Big[\int_{\sigma(t)}^{r} \frac{1}{a(u)} du \Big(\int_{u}^{r} \frac{1}{b(v)} dv \Big) \Big] \sum_{i=1}^{m} q_i(r) f(x(\sigma_i(r)))dr \leq y(t),$$

where $\sigma(t) = \max\{\sigma_1(t), \sigma_1(t), \dots, \sigma_n(t)\}$. Thus we have

$$\int_{\sigma(t)}^{t} \Big[\int_{\sigma(t)}^{r} \frac{1}{a(u)} du \Big(\int_{u}^{r} \frac{1}{b(v)} dv \Big) \Big] \sum_{i=1}^{m} q_i(r) f(x(\sigma_i(r)))dr \leq y(t). \tag{2.38}$$

Hence, we conclude that $\liminf_{t\to\infty} x(t) = 0$. But $x(t)$ is monotone, so we have $\lim_{t\to\infty} x(t) = 0$. Thus $c = 0$ and by (2.32) and (2.33) $\lim_{t\to\infty} x^{(i)}(t) = 0$, $i = 0, 1, 2$, which means that $x(t)$ is almost oscillatory. This completes the proof.

Oscillatory behavior of third order differential equations with continuous deviating arguments. Suppose that the following conditions hold unless stated otherwise

(a) $a(t) > 0$, $b(t) > 0$, $b'(t) \geq 0$, $\int^{\infty} \frac{dt}{a(t)} = \infty$, $\int^{\infty} \frac{dt}{b(t)} = \infty$,
(b) $q(t, \xi) \in C([t_0, \infty) \times [c, d], \mathbb{R})$, $q(t, \xi) > 0$,
(c) $\frac{f(x)}{x} \geq \epsilon > 0$, for $x \neq 0$, ϵ *is a constant*,
(d) $\sigma(t, \xi) \in C([t_0, \infty) \times [c, d], \mathbb{R})$, $\sigma(t, \xi) < t$, $\xi \in [c, d]$, $\sigma(t, \xi)$ is nondecreasing with respect to t and ξ and

$$\lim_{t\to\infty} \min_{\xi\in[c,d]} \sigma(t, \xi) = \infty.$$

Theorem 2.5. *If*

$$\int_{t_1}^{\infty}\int_c^d q(s,\xi)d\xi ds=\infty \tag{2.39}$$

and

$$\epsilon\int_{g(t)}^{t}\Big[\int_{g(t)}^{r}\frac{1}{a(u)}du\Big(\int_u^r\frac{1}{b(v)}dv\Big)\Big]\int_c^d q(r,\xi)d\xi dr>1, \tag{2.40}$$

where $g(t)=\sigma(t,d)$. Then the equation (1.2) is oscillatory.

Proof. Suppose that $x(t)$ is non-oscillatory solution of (1.2). Without loss of generality we may assume that $x(t)$ is eventually positive. (If $x(t)$ is eventually negative solution, it can be proved by the same arguments). From (1.2), we have

$$(b(t)(a(t)x'(t))')'=-\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))d\xi. \tag{2.41}$$

Proceeding as in the proof of Theorem 2.1, we have

$$(b(t)(a(t)x'(t))')'\le 0,$$

$$(a(t)x'(t))'>0 \quad\text{and}\quad (a(t)x'(t))''\le 0$$

for large enough t. Thus, $x(t)$, $x'(t)$ and $(a(t)x'(t))'$ are monotone and eventually one-signed. From condition (c),

$$f(x(\sigma(t,\xi)))\ge\epsilon x(\sigma(t,\xi))>0.$$

Therefore,

$$0\ge(b(t)(a(t)x'(t))')'+\epsilon\int_c^d q(t,\xi)x(\sigma(t,\xi))d\xi. \tag{2.42}$$

Now consider again two cases.

Suppose that $x'(t)$ is eventually positive, say $x'(t)>0$ for $t>t_2$. Now we can choose a constant $k>0$ such that $x(k)>0$. By (d), there exist a sufficiently large T such that $\sigma(t,\xi)>k$ for $t>T$, $\xi\in[c,d]$. Therefore,

$$x(\sigma(t,\xi))\ge x(k).$$

Thus,

$$(b(t)(a(t)x'(t))')'+\epsilon x(k)\int_c^d q(t,\xi)d\xi\le 0. \tag{2.43}$$

Integrating this last equation from t_1 to t, we get

$$b(t)(a(t)x'(t))'\le b(t_1)(a(t_1)x'(t_1))'-\epsilon x(k)\int_{t_1}^{t}\int_c^d q(s,\xi)d\xi ds. \tag{2.44}$$

Taking the limit of both sides as $t\to\infty$ and using (2.39), the last inequality above leads to a contradiction to $(a(t)x'(t))'>0$. Now suppose $x'(t)$ is eventually negative. Proceeding as in the proof of Theorem 2.1 and integrating equation (1.2) three times, we get

$$\int_t^{\infty}\Big[\int_t^r\frac{1}{a(u)}du\Big(\int_u^r\frac{1}{b(v)}dv\Big)\Big]\int_c^d q(r,\xi)f(x(\sigma(r,\xi)))d\xi dr\le x(t) \tag{2.45}$$

Using (c) in (2.45), we obtain

$$\int_t^{\infty}\Big[\int_t^r\frac{1}{a(u)}du\Big(\int_u^r\frac{1}{b(v)}dv\Big)\Big]\epsilon\int_c^d q(r,\xi)x(\sigma(r,\xi))d\xi dr\le x(t). \tag{2.46}$$

Replace t by $g(t)$ in (2.46), where $g(t) = \sigma(t, d)$, then we have

$$\epsilon \int_{g(t)}^{t} \Big[\int_{g(t)}^{r} \frac{1}{a(u)} du \Big(\int_{u}^{r} \frac{1}{b(v)} dv \Big)\Big] \int_{c}^{d} q(r,\xi) x(\sigma(r,\xi)) d\xi dr \le x(g(t)). \tag{2.47}$$

Since $x(t)$ is decreasing and positive,

$$\epsilon \int_{g(t)}^{t} \Big[\int_{g(t)}^{r} \frac{1}{a(u)} du \Big(\int_{u}^{r} \frac{1}{b(v)} dv \Big)\Big] \int_{c}^{d} q(r,\xi) d\xi dr \le 1.$$

This is a contradiction to (2.40). Therefore, the proof is complete.

Example 2.6. Consider the following functional differential equation

$$x''' + \int_{2/7\pi}^{1/2\pi} \frac{2e^{-1/\xi}}{\xi^2} x(t - \frac{1}{\xi}) d\xi = 0$$

so that $a(t) = 1$, $b(t) = 1$, $f(x) = x$, $q(t, \xi) = \frac{2e^{-1/\xi}}{\xi^2}$, $\sigma(t, \xi) = t - \frac{1}{\xi}$. We can easily see that the conditions of Theorem 2.5 are satisfied. It is easy to verify that $x(t) = e^{-t} \sin t$ is a solution of this problem.

Theorem 2.7. *Suppose (2.40) holds. In addition to that suppose there exist* $p \in C([t_0, \infty), \mathbb{R})$, $p(t) > 0$ *such that*

$$\int^{\infty} \Big[\Gamma(t)p(t) - \frac{a(\sigma(t,c))b(t)(p'(t))^2}{(\sigma(t,c) - T)\sigma'(t,c)4p(t)}\Big] dt = \infty, \tag{2.48}$$

where $\Gamma(t) = \epsilon \int_c^d q(t,\xi)d\xi$. *Then the equation (1.2) is oscillatory.*

Proof. Suppose that $x(t)$ is non-oscillatory solution of (1.2). We can assume that $x(t)$ is eventually positive. The case of $x(t)$ is eventually negative can be proved by the same arguments. Proceeding as in the proof of Theorem 2.1, we have

$$(b(t)(a(t)x'(t))')' \le 0,$$

$$(a(t)x'(t))' > 0 \quad \textit{and} \quad (a(t)x'(t))'' \le 0.$$

Thus, $x(t)$, $x'(t)$ and $(a(t)x'(t))'$ are monotone and eventually one-signed. From condition (c),

$$f(x(\sigma(t,\xi))) \ge \epsilon x(\sigma(t,\xi)) > 0.$$

$$(b(t)(a(t)x'(t))')' + \epsilon \int_{c}^{d} q(t,\xi) x(\sigma(t,\xi)) d\xi \le 0. \tag{2.49}$$

If $x'(t)$ is eventually positive, then we can define

$$z(t) = \frac{b(t)(a(t)x'(t))'}{x(\sigma(t,c))} p(t).$$

It is obvious that $z(t) > 0$ for $t \ge t_2$ and $z'(t)$ is

$$z'(t) = \frac{(b(t)(a(t)x'(t))')'}{x(\sigma(t,c))} p(t) + \frac{p'(t)}{p(t)} z(t) - \frac{x'(\sigma(t,c))\sigma'(t,c)}{x(\sigma(t,c))} z(t). \tag{2.50}$$

From proof of Theorem 2.1, we have

$$a(t)x'(t) \ge (t - T)(a(t)x'(t))'.$$

Since $(a(t)x(t))'$ is non-increasing, we have

$$a(\sigma(t,c))x'(\sigma(t,c)) \ge (\sigma(t,c) - T)(a(t)x'(t))',$$

then

$$x'(\sigma(t,c)) \geq \frac{(\sigma(t,c)-T)(a(t)x'(t))'}{a(\sigma(t,c))}. \tag{2.51}$$

Plug (2.51) in (2.50), then we obtain

$$z'(t) = \frac{(b(t)(a(t)x'(t))')'}{x(\sigma(t,c))}p(t) + \frac{p'(t)}{p(t)}z(t) - \frac{(\sigma(t,c)-T)\sigma'(t,c)}{p(t)b(t)a(\sigma(t,c))}z^2(t).$$

Completing the square leads to

$$z'(t) \leq -\Gamma(t)p(t) + \frac{b(t)a(\sigma(t,c))(p'(t))^2}{(\sigma(t,c)-T)\sigma'(t,c))4p(t)}. \tag{2.52}$$

Integrating (2.52) from T to t and letting $t \to \infty$, we see that $\lim_{t\to\infty} z(t) = -\infty$. This contradicts $z(t)$ being eventually positive.

If $x'(t)$ is eventually negative, the proof is exactly the same as in the second part of the proof of previous Theorem.

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