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ON THE SOLUTION OF DIFFERENTIAL EQUATIONS WITH DELAYED AND ADVANCED ARGUMENTS

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Abstract. In this work, we construct solutions to differential difference equations with delayed and advanced arguments. We use a step derivative so that a special condition on the initial function assures the existence and uniqueness of the solution.

1. Introduction

The differential difference delayed equations and the differential difference advanced equations have been studied widely; see for example [4, 3, 7, 1]. Applications of this equations can be found in physics, biology, economy, and so on, [5, 7, 8, 6]. However as far as we have researched, there are only a few studies on the differential equations with delayed and advanced arguments [11, 10].

From a strictly mathematical point of view, we are interested on the study of the system of equations

$$x'(t) = Ax(t-\omega) + Bx(t+\omega) + Cx(t), \tag{1.1}$$

where $x(t)$ is a vector-value function in $\mathbb{R}^n$, A, B and C are arbitrary $n \times n$ matrices, and ω an real number. However, in this article, we analyze the simpler scalar equation

$$x'(t) = x(t-1) + x(t+1). \tag{1.2}$$

and leave the study of (1.1) for a future research.

To obtain a solution of (1.2), we define the function $x(t)$ initially on some interval of $\mathbb{R}$. Then we construct the solution using the step derivative method, which is an analog to the step integration method [2, 3]. Then we prove the existence, uniqueness, and smoothness of the solution.

2. Construction of the solution

In this section we construct the solution of the differential difference equation (1.2), using the step derivatives method, that provides an iterative formula. Consider the differential difference equation (1.2), with $t \geq 0$, $x : [-1, +\infty) \to \mathbb{R}$, and

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$x(t)$ differentiable in $[0,+\infty)$. Equation (1.2) is rewritten as

$$x(t+1) = x'(t) - x(t-1)$$

or equivalently as

$$x(t) = x'(t-1) - x(t-2). \tag{2.1}$$

From this equation, it follows that in order to find the solution $x(t)$ on the interval $[m, m+1]$ it is necessary to know its value on the interval $[m-2, m]$, with m a positive integer. In particular, to determine the solution on the interval $[1,2]$, it is necessary to know it on the interval $[-1,1]$.

Accordingly we define $x(t)$ for $t \in [-1,1]$ as

$$x(t) = \varphi(t) = \begin{cases} \varphi_1(t), & t \in [-1,0] \\ \varphi_2(t), & t \in [0,1], \end{cases} \tag{2.2}$$

where the function φ is taken initially in the space $C_{[-1,1]}$ (because $x(t)$ is differentiable and therefore continuous in $[0,+\infty)$).

After a formal procedure, for $t \in (1,2)$,

$$x(t) = \varphi_2'(t-1) - \varphi_1(t-2) = \varphi'(t-1) - \varphi(t-2)\,. \tag{2.3}$$

For $t \in (2,3)$,

$$\begin{aligned} x(t) &= x'(t-1) - x(t-2) \\ &= \frac{d}{dt}(\varphi_2'(t-2) - \varphi_1(t-3)) - \varphi(t-2) \\ &= \varphi_2''(t-2) - \varphi_1'(t-3) - \varphi_2(t-2) \\ &= \varphi''(t-2) - \varphi'(t-3) - \varphi(t-2)\,. \end{aligned} \tag{2.4}$$

For $t \in (3,4)$,

$$\begin{aligned} x(t) &= x'(t-1) - x(t-2) \\ &= \frac{d}{dt}(\varphi_2''(t-3) - \varphi_1'(t-4) - \varphi_2(t-3)) - \varphi_2'(t-3) + \varphi_1(t-4) \\ &= \varphi_2'''(t-3) - \varphi_1''(t-4) - 2\varphi_2'(t-3) + \varphi_1(t-4) \\ &= \varphi'''(t-3) - \varphi''(t-4) - 2\varphi'(t-3) + \varphi(t-4), \end{aligned}$$

and so forth. Due to the fact, that in each interval the solution $x(t)$ is expressed by means of increasing order derivatives of the function φ, it is necessary to take φ in $C^\infty_{[-1,1]}$. From the above expressions for the solution $x(t)$ it follows that this solution can be written via the following iterative formulas, where l is a natural number: On the interval $(2l-1, 2l)$,

$$x(t) = \sum_{k=0}^{l-1} (c_{2k}\varphi^{(2k)}(t-2l) + c_{2k+1}\varphi^{(2k+1)}(t-(2l-1))), \tag{2.5}$$

and on the interval $(2l, 2l+1)$,

$$x(t) = \sum_{k=0}^{l} c_{2k}\varphi^{(2k)}(t-2l) + \sum_{k=0}^{l-1} c_{2k+1}\varphi^{(2k+1)}(t-(2l+1))\,, \tag{2.6}$$

where c_{2k}, c_{2k+1}, $k = 0, 1, 2, \dots l$ are constants.

The proofs of (2.5) and (2.6) are done by induction (on l): The case $l = 1$, i.e., $t \in (1,2)$ and $t \in (2,3)$ are the already proven: (2.3) and (2.4).

Now we assume formulas (2.5) y (2.6) hold for $t \in (2l-1, 2l)$ and $t \in (2l, 2l+1)$ respectively.
First we deal with formula (2.5) in the interval $(2l+1, 2l+2)$:

$$x(t) = \sum_{k=0}^{l} (c_{2k}\varphi^{(2k)}(t-(2l+2)) + c_{2k+1}\varphi^{(2k+1)}(t-(2l+1))) \,. \tag{2.7}$$

Using (2.1), (2.5) and (2.6) it follows that

$$\begin{aligned}
x(t) &= x'(t-1) - x(t-2) \\
&= \sum_{k=0}^{l} c_{2k}\varphi^{(2k+1)}(t-1-2l) + \sum_{k=0}^{l-1} c_{2k+1}\varphi^{(2k+2)}(t-1-(2l+1)) \\
&+ \sum_{k=0}^{l-1} (c_{2k}\varphi^{(2k)}(t-2-2l) + c_{2k+1}\varphi^{(2k+1)}(t-2-(2l-1))) \\
&= \sum_{k=0}^{l} (c_{2k} + c_{2k+1})\varphi^{(2k+1)}(t-(2l+1)) + \sum_{k=0}^{l} c_{2k}\varphi^{(2k)}(t-(2l+2)) \\
&\quad + \sum_{r=1}^{l} c_{2r-1}\varphi^{(2r)}(t-(2l+2)) \\
&= \sum_{k=0}^{l} ((c_{2k} + c_{2k-1})\varphi^{(2k)}(t-(2l+2)) + (c_{2k} + c_{2k+1})\varphi^{(2k+1)}(t-(2l+1)))
\end{aligned}$$

where $k+1 = r$, $c_{2l+1} = c_{2l} = c_{-1} = 0$. This shows formula (2.7).

Now we prove formula (2.6) in the interval $(2l+2, 2l+3)$:

$$x(t) = \sum_{k=0}^{l+1} c_{2k}\varphi^{(2k)}(t-(2l+2)) + \sum_{k=0}^{l} c_{2k+1}\varphi^{(2k+1)}(t-(2l+3)) \,. \tag{2.8}$$

Using (2.1), (2.7) and (2.6) it follows that

$$\begin{aligned}
x(t) &= x'(t-1) - x(t-2) \\
&= \sum_{k=0}^{l} c_{2k}\varphi^{(2k+1)}(t-1-(2l+2)) + \sum_{k=0}^{l} c_{2k+1}\varphi^{(2k+2)}(t-1-(2l+1)) \\
&\quad + \sum_{k=0}^{l} c_{2k}\varphi^{(2k)}(t-2-2l) + \sum_{k=0}^{l-1} c_{2k+1}\varphi^{(2k+1)}(t-2-(2l+1)) \\
&= \sum_{k=0}^{l} c_{2k+1}\varphi^{(2k+2)}(t-(2l+2)) + \sum_{k=0}^{l} c_{2k}\varphi^{(2k)}(t-(2l+2)) \\
&\quad + \sum_{k=0}^{l} c_{2k}\varphi^{(2k+1)}(t-(2l+3)) + \sum_{k=0}^{l-1} c_{2k+1}\varphi^{(2k+1)}(t-(2l+3)) \\
&= \sum_{r=1}^{l+1} c_{2r-1}\varphi^{(2r)}(t-(2l+2)) + \sum_{k=0}^{l} c_{2k}\varphi^{(2k)}(t-(2l+2))
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=0}^{l} c_{2k}\varphi^{(2k+1)}(t-(2l+3))+\sum_{k=0}^{l-1} c_{2k+1}\varphi^{(2k+1)}(t-(2l+3))\\
&=\sum_{k=0}^{l+1}(c_{2k-1}+c_{2k})\varphi^{(2k)}(t-(2l+2))+\sum_{k=0}^{l}(c_{2k}+c_{2k+1})\varphi^{(2k+1)}(t-(2l+3))
\end{aligned}
$$

where $k+1=r$, $c_{2l+2}=c_{2l+1}=c_{-1}=0$. This shows formula (2.8).

That formulas (2.5) and (2.6) hold and coincide in the boundary points of each interval $(m, m+1)$ will be proven in the next section.

Remark 2.1. The solution $x(t)$ may be extended to the left by rewriting (1.2) as

$$x(t-1)=x'(t)-x(t+1)\,.$$

In this case we obtain expressions for $x(t)$ analogous to (2.5) and (2.6).

3. Existence and uniqueness of the solution

In this section we give necessary and sufficient conditions to assure the existence and uniqueness of the solution to problem (1.2)-(2.2).

Theorem 3.1. *The solution $x(t)$ of (1.2) satisfying the initial condition (2.2) with φ in $C^\infty_{[-1,1]}$, exists and is differentiable, if and only if*

$$\varphi^{(n+1)}(0)=\varphi^{(n)}(-1)+\varphi^{(n)}(1),$$

for $n=0,1,2,\dots$.

Proof. Since φ belongs to $C^\infty_{[-1,1]}$, for each interval $(m, m+1)$ the function $x(t)$ exists and is infinitely many times differentiable. In order to prove the continuity of $x(t)$ and the existence of its derivative in the end points m and $m+1$ (and therefore the existence of $x(t)$ in such points) it is necessary to prove the equalities

$$x^{(i)}(m^+)=x^{(i)}(m^-),\quad i=0,1;\ m=1,2,\dots, \tag{3.1}$$

where

$$
\begin{aligned}
x^{(i)}(m^+) &:= \lim_{\varepsilon\to 0,\ \varepsilon>0} x^{(i)}(m+\varepsilon)\,,\\
x^{(i)}(m^-) &:= \lim_{\varepsilon\to 0,\ \varepsilon>0} x^{(i)}(m-\varepsilon).
\end{aligned} \tag{3.2}
$$

By induction on the natural number k, we will prove the claim:

$$x(m^+)=x(m^-),\ \ m=1,2,\dots k\,, \tag{3.3}$$

$$x'(m^+)=x'(m^-),\ \ m=1,2,\dots k-1\,, \tag{3.4}$$

if and only if

$$\varphi^{(n+1)}(0)=\varphi^{(n)}(-1)+\varphi^{(n)}(1),\quad n=1,2,\dots k-1\,. \tag{3.5}$$

But if (3.3) and (3.4) hold, then it follows the equivalence

$$x'(k^+)=x'(k^-) \tag{3.6}$$

if and only if

$$\varphi^{(k+1)}(0)=\varphi^{(k)}(-1)+\varphi^{(k)}(1). \tag{3.7}$$

For the case $k=1$, by formula (2.3) at point $m=1$ it is valid

$$x(1^+)=\varphi'(0^+)-\varphi(-1^+)\,.$$

From $x(1^-) = \varphi(1^-) = \varphi(1)$ it follows that $x(1^+) = x(1^-)$ is equivalent to

$$\varphi'(0^+) = \varphi(-1^+) + \varphi(1^-).$$

From the derivatives of (2.3) it follows

$$x'(1^+) = \varphi''(0^+) - \varphi'(-1^+)\,,$$

and from (2.2) follows $x'(1^-) = \varphi'(1^-)$. Therefore $x'(1^+) = x'(1^-)$ if and only if

$$\varphi''(0^+) = \varphi'(-1^+) + \varphi'(1^-).$$

Assume now the claim holds up to $k = s$. According to the inductive assumption, formula (3.5) is satisfied for $k = s$. Formula (3.4) for $k = s + 1$ implies $x'(s^+) = x'(s^-)$, which is (3.6) for $k = s$, but according to the inductive assumption this is equivalent to (3.7) for $k = s$, which means $\varphi^{(s+1)}(0) = \varphi^{(s)}(-1) + \varphi^{(s)}(1)$. It follows, that formulas (3.3) and (3.4) for $k = s + 1$ imply

$$\varphi^{(n+1)}(0) = \varphi^{(n)}(-1) + \varphi^{(n)}(1), \quad n = 1, 2, \dots s.$$

Next will be verified, that (3.6) and (3.7) are equivalent for $k = s + 1$, provided that (3.3) and (3.4) hold for $k = s + 1$. By computing the derivatives of formula (2.1) one reaches $x'(t) = x''(t - 1) - x'(t - 2)$. Therefore,

$$x'((s+1)^+) = x''(s^+) - x'((s-1)^+), \quad x'((s+1)^-) = x''(s^-) - x'((s-1)^-).$$

Formula (3.4) for $m = s-1$ implies $x'((s-1)^+) = x'((s-1)^-)$ and the late implies

$$x'((s+1)^+) = x'((s+1)^-) \Longleftrightarrow x''(s^+) = x''(s^-) \tag{3.8}$$

Then after the inductive assumption, $x'(s^+) = x'(s^-) \Longleftrightarrow \varphi^{(s+1)}(0) = \varphi^{(s)}(-1) + \varphi^{(s)}(1)$.

Formulas for $x'(t)$ and $x''(t)$ are obtained for each interval $(m, m + 1)$ by computing the simple and double derivatives of (2.5) or (2.6) respectively; therefore the equality $x''(s^+) = x''(s^-)$ in terms of φ has the same shape than $x'(s^+) = x'(s^-)$, with the only light difference, that the derivatives of φ appear increased in one degree. Since $x'(s^+) = x'(s^-)$ if and only if $\varphi^{(s+1)}(0) = \varphi^{(s)}(-1) + \varphi^{(s)}(1)$, it follows $x''(s^+) = x''(s^-)$ if and only if $\varphi^{(s+2)}(0) = \varphi^{(s+1)}(-1) + \varphi^{(s+1)}(1)$. Claim (3.8) implies $x'((s+1)^+) = x'((s+1)^-)$ if and only if $\varphi^{(s+2)}(0) = \varphi^{(s+1)}(-1)+\varphi^{(s+1)}(1)$, with which the equivalence has been proven.

On the other hand, suppose (3.5) is true for $k = s + 1$ which means

$$\varphi^{(n+1)}(0) = \varphi^{(n)}(-1) + \varphi^{(n)}(1), \quad n = 1, 2, \dots s.$$

The formulas (3.7) and (3.5) hold for $k = s$, therefore after the inductive assumption formulas (3.3) and (3.4) are true for $k = s$, hence formulas (3.6) and (3.7) are equivalent for $k = s$.

From formula (2.1) we have

$$x((s+1)^+) = x'(s^+) - x((s-1)^+), \quad x((s+1)^-) = x'(s^-) - x((s-1)^-).$$

Since $x'(s^+) = x'(s^-)$ and $x((s-1)^+) = x((s-1)^-)$, then $x((s+1)^+) = x((s+1)^-)$. Equalities $x'(s^+) = x'(s^-)$ and $x((s+1)^+) = x((s+1)^-)$ together with (3.3) and (3.4) for $k = s$ imply, that (3.3) and (3.4) are both true for $k = s + 1$. $\square$

Remark 3.1. In [9], we have obtained non-trivial functions in the set

$$\{\varphi \in C^{\infty}_{[-1,1]} : \varphi^{(n+1)}(0) = \varphi^{(n)}(-1) + \varphi^{(n)}(1),\, n = 0, 1, 2, \dots\}.$$

Remark 3.2. If no condition such as those in Theorem 3.1 are imposed on the function φ, then the function $x(t)$, defined by (2.5) and (2.6), may be discontinuous at the integer points.

Corollary 3.1. *If for an initial function $\varphi \in C^{\infty}_{[-1,1]}$ there exists a differentiable solution $x(t)$ for $t \geq 0$, of* (1.2) *with the initial condition* (2.2)*, then this solution belongs to the space $C^{\infty}_{[-1,+\infty)}$.*

Proof. Since there exists a differentiable solution to problem (1.2) and (2.2), we have
$$\varphi^{(n+1)}(0) = \varphi^{(n)}(-1) + \varphi^{(n)}(1), \quad n = 0, 1, \dots .$$
Then formula (3.5) holds, as well as its equivalent formulas (3.3) and (3.4), therefore the equivalence between (3.6) and (3.7) follows. Since in each interval $(m, m+1)$ the formulas for $x'(t)$ and $x^{(i)}(t)$ are reached by differentiating formula (2.5) or (2.6) according to the case, one or i-times respectively, then the equality $x^{(i)}(k^+) = x^{(i)}(k^-)$ in terms of φ has the same shape than the equality $x'(k^+) = x'(k^-)$ in terms of φ, but now with the derivatives of φ increased in an i-order degree. Hence
$$\varphi^{(k+i)}(0) = \varphi^{(k+i-1)}(-1) + \varphi^{(k+i-1)}(1) \text{ for any natural number } (k+i)\,,$$
due to the equivalence between (3.6) and (3.7). But this is equivalent to $x^{(i)}(k^+) = x^{(i)}(k^-)$, for all $i = 0, 1, 2\dots$ and each natural number k, which means that $x(t) \in C^{\infty}_{[-1,+\infty)}$. □

Remark 3.3. For certain initial functions, we can define the semigroup associated to the solutions $x(t)$ of (1.2). This is another way to build solutions on the whole real line, which has been developed in [9].

Theorem 3.2. *Let $\varphi \in C^{\infty}_{[-1,1]}$. If a solution $x(t)$ of equation* (1.2)-(2.2) *exists and is differentiable, then the solution is unique.*

Proof. On the open intervals the solution coincides with (2.5) or (2.6), while in the integer points the solution is obtained uniquely due to the continuity of $x(t)$. □

References

[1] Aftabizedeh A. R., Wiener J.; *Oscillatory and periodic solutions of advanced differential equations with piecewise constant argument*, In Nonlinear Analysis and Applications. V. Lakshmikanthan, ed. Marcel Dekker Inc., N. Y. and Basel, 31-38, (1987).

[2] Bellman R., Cooke K.; *Differential-difference equations*, Editorial Mir, Moscu, (1967).

[3] Driver R. D.; *Ordinary and Delay Differential Equations*, Applied Mathematical Sciences 20, Springer Verlag, Berlin (1977).

[4] Elgoltz L. E., Norkin C. B.; *Introduction to the Theory of Differential Equations with Deviating Arguments*, Nauka, Moscow (in Russian) (1971).

[5] Gopalsamy K.; *Stability and Oscillations in Delay Differential Equations of Populations Dynamics*, Kluwer, Dordrecht (1992).

[6] Gyory I., Ladas G.; *Oscillation theory of delay differential equations with applications*, Oxford University Press, Oxford, (1991).

[7] Hale J. K.; *Theory of functional differential equations*, Springer-Verlag, New York (1977).

[8] Hale J. K.,Verduyn Lunel S. M.; *Introduction to Functional Differential Equations*, Springer-Verlag, New York (1993).

[9] Iakovleva V., Vanegas C.; *Spectral analysis of strongly continuous semigroups associated to solutions of differential delay-advance equations*, preprint (2004).

[10] Kenneth L., Wiener J.; *A Survey of Differential Equations with Piecewise Continuous Arguments.* Delay Differential Equations and Dynamical Systems, eds. S. Busenberg and M. Mantelli. Lectures Notes in Math. 1475. Springer Verlag, Berlin, 1-15, (1991).

[11] Ladas G.; *Oscillations of equations with piecewise constant mixed arguments*. In Differential Equations and Applications. II Aftabizedeh, ed. Ohio University Press, Athens, 64-69, (1989).

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