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ERROR ESTIMATES FOR ASYMPTOTIC SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. We establish error estimates for first-order linear systems of equations and linear second-order dynamic equations on time scales by using calculus on a time scales [1, 4, 5] and Birkhoff-Levinson's method of asymptotic solutions [3, 6, 8, 9].

1. Results

Asymptotic behavior of solutions of dynamic equations and systems on time scales was investigated in [5]. In this paper we establish error estimates of such asymptotic representations, which may be applied to the investigation of stability of dynamic equations (see f.e. [9]).

Consider the system of ordinary differential equations on time scales

$$a^{\Delta}(t) = A(t)a(t), \quad t > T, \tag{1.1}$$

where a^{Δ} is delta (Hilger) derivative, $a(t)$ is a n-vector function, and $A(t)$ is a $n \times n$ matrix function from $C_{rd}(T, \infty)$ (definition of rd-continuous functions see in [4]). A time scale is an arbitrary nonempty closed subset of the real numbers. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

The graininess function $\mu : \mathbb{T} \to [0, \infty]$ is defined by

$$\mu(t) = \sigma(t) - t.$$

We assume that $\sup \mathbb{T} = \infty$.

Suppose we can find the exact solutions of the auxiliary system

$$\psi^{\Delta}(t) = A_1(t)\psi(t), \quad t > T, \tag{1.2}$$

with the matrix function $A_1(t) \in C_{rd}(T, \infty)$ close to the matrix function A(t), which means that condition (1.5) below is satisfied. Let $\Psi(t)$ be the fundamental matrix of the system (1.2). Then the solutions of (1.1) can be represented in the form

$$a(t) = \Psi(t)(C + \varepsilon(t)), \tag{1.3}$$

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where $a(t), \varepsilon(t), C$ are the vector columns. We can consider (1.3) as a definition of the error vector function $\varepsilon(t)$. Denote

$$H(t) \equiv \left(1 + \mu(t)\Psi^{-1}(t)\Psi^{\Delta}(t)\right)^{-1} \Psi^{-1}(t) \left(A(t)\Psi(t) - \Psi^{\Delta}(t)\right). \tag{1.4}$$

Theorem 1.1. *Assume there exist an invertible and differentiable matrix function $\Psi(t) \in C_{rd}(T, \infty)$ such that $1 + \mu(t)\Psi^{-1}(t)\Psi^{\Delta}(t)$ is invertible and*

$$\int_t^{\infty} \Big(\lim_{m \searrow \mu(s)} \frac{\log(1 + m\|H(s)\|)}{m} \Delta s \Big) < \infty. \tag{1.5}$$

Then every solution of (1.1) *can be represented in form* (1.3) *and the error function $\varepsilon(t)$ can be estimated as*

$$\|\varepsilon(t)\| \le \|C\| \left(-1 + e_{\|H\|}(\infty, t)\right), \tag{1.6}$$

where $\|.\|$ is the Euclidean vector (or matrix) norm: $\|C\| = \sqrt{C_1^2 + \dots + C_n^2}$, and expression in (1.5) *usually is used to define the exponential function on time scales (see* [1, 4]*):*

$$e_{\|H\|}(\infty, t) = \exp\Big(\int_t^{\infty} \lim_{m \searrow \mu(s)} \frac{\log(1 + m\|H(s)\|)}{m} \Delta s\Big). \tag{1.7}$$

Remark 1.2. Comparing with the similar result from [5] advantage of Theorem 1.1 is that it not only proves that error vector function approaches to zero as t approaches to infinity, but inequality (1.6) also estimates the speed of that approach to zero.

¿From the estimate (1.6) it follows also that the error vector function $\varepsilon(t)$ is small when $\int_t^{\infty} \lim_{m \searrow \mu(s)} \frac{\log(1+m\|H(s)\|)}{m} \Delta s$ is small.

Proof of Theorem 1.1. Let $a(t)$ be a solution of (1.1). The substitution $a(t) = \Psi(t)u(t)$ transforms (1.1) into

$$u^{\Delta} = H(t)u(t), \quad t > T,$$

where H is defined by (1.4). By integration we get

$$u(t) = C - \int_t^b H(s)u(s)\Delta s, \quad t < s < b, \tag{1.8}$$

where the constant vector C is chosen as in (1.3). Estimating $u(t)$

$$\|u(t)\| \le \|C\| + \int_t^b \|H(s)\| \cdot \|u(s)\|\Delta s,$$

and applying Gronwall's lemma (see [4]) we have

$$\|u(t)\| \le \|C\| e_{\|H\|}(b, t). \tag{1.9}$$

From representation (1.3) and expression (1.7), we have

$$\varepsilon(t) = \Psi^{-1}(t)a(t) - C = u(t) - C = -\int_t^b H(s)u(s)\Delta s.$$

Then using (1.9) we obtain

$$\begin{aligned}\|\varepsilon(t)\| &\le \int_t^b \|H(s)u(s)\|\Delta s \\ &\le \|C\| \int_t^b \|H(s)\| \cdot e_{\|H\|}(b,s) \\ &= \|C\| \left[-1 + e_{\|H\|}(b,t)\right] \\ &\le \|C\| \left[-1 + e_{\|H\|}(\infty,t)\right].\end{aligned}$$

Note that from (1.5), it follows that

$$\lim_{t\to\infty} \left[-1 + e_{\|H\|}(\infty,t)\right] = \lim_{t\to\infty} \Big[-1 + \exp \int_t^\infty \lim_{m\searrow\mu(s)} \frac{\log(1+m\|H(s)\|)}{m} \Delta s\Big] = 0.$$

□

Consider the second-order dynamic equation on time scales

$$L[x(t)] = x^{\Delta\Delta} + p(t)x^{\Delta}(t) + q(t)x(t) = 0, \quad t > t_0 > 0,\ t \in \mathbb{T}. \tag{1.10}$$

¿From the functions $\varphi_{1,2}(t) \in C^2_{rd}(T,\infty)$ let us construct auxiliary matrix-functions

$$\begin{gathered}\Phi(t) = \begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \varphi_1^{\Delta}(t) & \varphi_2^{\Delta}(t)\end{pmatrix}, \quad H(t) = (1+\mu(t)\Phi^{-1}(t)\Phi^{\Delta}(t))^{-1}B(t), \\ B(t) = \begin{pmatrix} B_{21}(t) & B_{22}(t) \\ -B_{11}(t) & -B_{12}(t)\end{pmatrix}, \quad B_{kj}(t) \equiv \frac{\varphi_k(t)L[\varphi_j(t)]}{W(\varphi_1,\varphi_2)}, \quad j = 1,2.\end{gathered} \tag{1.11}$$

Theorem 1.3. *Let $\varphi_{1,2}(t) \in C^2_{rd}(T,\infty)$ be complex-valued functions such that*

$$\int_T^\infty \Big(\lim_{m\searrow\mu(s)} \frac{\log\Big(1+m\|\left(1+m\Phi^{-1}(t)\Phi^{\Delta}(t)\right)^{-1} B(t)\|\Big)}{m}\Big)\Delta t < \infty, \quad k,j = 1,2, \tag{1.12}$$

where $\|.\|$ is Euclidean matrix norm. Then for arbitrary constants C_1,C_2 there exist solution of (1.1) *that can be written in the form*

$$x(t) = [C_1 + \varepsilon_1(t)]\,\varphi_1(t) + [C_2 + \varepsilon_2(t)]\,\varphi_2(t), \tag{1.13}$$

$$x^{\Delta}(t) = [C_1 + \varepsilon_1(t)]\,\varphi_1^{\Delta}(t) + [C_2 + \varepsilon_2(t)]\,\varphi_2^{\Delta}(t). \tag{1.14}$$

The error vector-function $\varepsilon(t) = (\varepsilon_1(t), \varepsilon_2(t))$ is estimated as

$$\|\varepsilon(t)\| \le \|C\|\big(-1 + e_{\|H(t)\|}(\infty,t)\big), \tag{1.15}$$

where $C = (C_1, C_2)$ is an arbitrary constant vector, and the matrix function $H(t)$ is defined in (1.11).

Proof. Rewrite equation (1.10) in form (1.1):

$$a^{\Delta}(t) = A(t)a(t), \tag{1.16}$$

where

$$a(t) = \begin{pmatrix} x(t) \\ x^{\Delta}(t)\end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t)\end{pmatrix}.$$

By substitution

$$a(t) = \Phi(t)w(t), \tag{1.17}$$

in (1.16) we get

$$w^{\Delta} = H(t)w(t), \tag{1.18}$$

where $H(t)$ defined by (1.11). To apply Theorem 1.1 to system (1.16) we choose $A(t) = H(t)$ and $A_1 \equiv 0$. Then the identity matrix is fundamental solution of (1.2), so conditions (1.5) turns to (1.12). ¿From Theorem 1.1 we have

$$w(t) = C + \varepsilon(t), \quad \text{or} \quad a(t) = \Phi(t)w(t) = \Phi(t)(C + \varepsilon(t)).$$

Representations (1.13),(1.14) and estimates (1.15) follow from Theorem 1.1. □

Example 1.4. For solutions of the equation

$$x^{\Delta\Delta}(t) + \big(\gamma^2 + \frac{1}{t^2}\big)x(t) = 0, \quad t > t_0,$$

we get representations (1.13), (1.14), where

$$\varphi_1 = \cos_\gamma(t, t_0), \quad \varphi_2 = \sin_\gamma(t, t_0)$$

are trigonometric functions on time scales [4]. By direct calculations $H = O(t^{-2})$ as $t \to \infty$, and

$$|\varepsilon_j(t)| \le \|C\| \Big[-1 + \exp\Big(\int_t^\infty \lim_{m \searrow \mu(s)} \frac{\log\big(1 + C_1 m s^{-2}\big)}{m} \Delta s \Big) \Big], \quad j = 1, 2.$$

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