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# NONLINEAR MULTIDIMENSIONAL PARABOLIC-HYPERBOLIC EQUATIONS

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*Dedicated to Jacqueline Fleckinger on the occasion of an international conference in her honor*

Abstract. This paper deals with the coupling of a quasilinear parabolic problem with a first order hyperbolic one in a multidimensional bounded domain $\Omega$. In a region $\Omega_p$ a diffusion-advection-reaction type equation is set while in the complementary $\Omega_h \equiv \Omega\backslash\Omega_p$, only advection-reaction terms are taken into account. Suitable transmission conditions at the interface $\partial\Omega_p \cap \partial\Omega_h$ are required. We find a weak solution characterized by an entropy inequality on the whole domain.

## 1. Introduction

We are interested in a coupling of a quasilinear parabolic equation with an hyperbolic first-order one in a bounded domain $\Omega$ of $\mathbb{R}^n$, $n \geq 1$. The main motivation for considering this problem is the study of infiltration processes in an heterogeneous porous media. For instance, in a stratified subsoil made up of layers with different geological characteristics, the effects of diffusivity may be negligible in some layers. Such a coupled problem occurs also in fluid-dynamical theory for viscous-compressible flows around a rigid profile so that near this profile the viscosity effects have to be taken into account while at a distance they can be neglected. Another example arises in heat transfer studies as mentioned in [6].

We consider the case of two layers, that is sufficient. Then, the geometrical configuration is such that:

$\overline{\Omega} = \overline{\Omega_h} \cup \overline{\Omega_p}$; $\Omega_h$ and $\Omega_p$ are two disjoint bounded domains with Lipschitz boundaries denoted by $\Gamma_l = \partial\Omega_l$, $l \in \{h,p\}$ and $\Gamma_{hp} = \Gamma_h \cap \Gamma_p$. In addition we set $Q = ]0,T[\times\Omega$ and for $l$ in $\{h,p\}$, $Q_l = ]0,T[\times\Omega_l$, $\Sigma_l = ]0,T[\times\Gamma_l$. Now, for $q$ in $[0,n+1]$, $\mathcal{H}^q$ is the $q$-dimensional Hausdorff measure over $\mathbb{R}^{n+1}$ and for $l$ in $\{h,p\}$, $\nu_l$ is the outward normal unit vector defined $\mathcal{H}^n$-a.e. on $\Sigma_l$. So the interface, denoted by $\Sigma_{hp} = ]0,T[\times\Gamma_{hp}$, is such that $\mathcal{H}^n(\overline{\Sigma_{hp}} \cap (\overline{\Sigma_l\backslash\Sigma_{hp}})) = 0$.

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Now, due to a combination of conservation laws and Darcy's law, the physical model is described as follows:

For any positive and finite real $T$, find a measurable and bounded function $u$ on $Q$ such that,

$$\partial_t u - \sum_{i=1}^{n} \partial_{x_i}(f(u)\partial_{x_i} P) + g(t,x,u) = 0 \quad \text{in } Q_h, \tag{1.1}$$

$$\partial_t u - \sum_{i=1}^{n} \partial_{x_i}(f(u)\partial_{x_i} P) + g(t,x,u) = \Delta\phi(u) \quad \text{in } Q_p, \tag{1.2}$$

$$u = 0 \quad \text{on } ]0,T[\times\partial\Omega, \tag{1.3}$$

$$u(0,.) = u_0 \quad \text{on } \Omega. \tag{1.4}$$

Then, suitable conditions on $u$ across the interface $\Sigma_{hp}$ must be added. As for the linear problem studied by F. Gastaldi and *al.* in [6] or for the one dimensional nonlinear problem studied by G. Aguilar and *al.* in [2], these transmission conditions include the continuity property of the flux through the interface formally written here as

$$-f(u)\nabla P.\nu_h = (\nabla\phi(u) + f(u)\nabla P).\nu_p \quad \text{on } \Sigma_{hp}. \tag{1.5}$$

Let us mention that this problem has already been studied by the authors in [1] for a nondecreasing flux function $f$ when $\nabla P.\nu_h \leq 0$ a.e. on $\Gamma_{hp}$. Here, we still consider a nondecreasing flux function $f$, but we give an existence and uniqueness result holding even when $\nabla P.\nu_h \geq 0$ a.e. on $\Gamma_{hp}$.

1.1. **Assumptions and notation.** The pressure $P$ is a known stationary function belonging to $W^{2,+\infty}(\Omega)$ and such that $\Delta P = 0$ which is not restrictive as soon as (1.1) and (1.2) include some reaction terms. In addition,

$$\nabla P.\nu_h \text{ has a constant sign all along } \Gamma_{hp}. \tag{1.6}$$

The reaction function $g$ belongs to $W^{1,+\infty}(]0,T[\times\Omega\times\mathbb{R})$ and we set

$$M'_g = \operatorname*{ess\,sup}_{(t,x,u)\in]0,T[\times\Omega\times\mathbb{R}} |\partial_u g(t,x,u)| \quad \text{and} \quad M_0 = \operatorname*{ess\,sup}_{]0,T[\times\Omega} |g_h(t,x,0)|.$$

The initial data $u_0$ belongs to $L^\infty(\Omega)$. Thus we can define the nondecreasing time-depending function

$$M : t \in [0,T] \to M(t) = \|u_0\|_{L^\infty(\Omega)} e^{M'_g t} + M_0 \frac{e^{M'_g t} - 1}{M'_g}. \tag{1.7}$$

To simplify we write $M = M(T)$.

Now, we assume local hypotheses on $f$ and $\phi$.

(i) The flux function $f$ is a nondecreasing Lipschitzian function on $[-M,M]$ with constant $M'_f$ and such that $f(0) = 0$. To express the boundary conditions on the frontier of the hyperbolic area, we introduce the nonnegative function $\mathcal{F}$ defined on $[-M,M]^3$ by

$$\mathcal{F}(a,b,c) = \frac{1}{2}\{|f(a) - f(b)| - |f(c) - f(b)| + |f(a) - f(c)|\}. \tag{1.8}$$

(ii) $\phi$ is an increasing Lipschitzian function on $[-M,M]$ such that $\phi^{-1}$ is Hölder continuous and $\phi(0) = 0$.

(iii) $f \circ \phi^{-1}$ is Hölder continuous with exponent $\theta$ in $[1/2, +\infty[$ that is there exists a positive constant $\mathcal{C}$ such that

$$\forall (x,y) \in [-M, M]^2, \quad |(f \circ \phi^{-1})(x) - (f \circ \phi^{-1})(y)| \leq \mathcal{C}|x-y|^{\theta}. \tag{1.9}$$

**Remark 1.1.** The monotonicity of $f$ and the condition (1.6) involve that

if a.e. on $\Gamma_{hp}, \nabla P.\nu_h \leq 0$, then $\Sigma_{hp}$ is included in the set of outward characteristics for the first-order operator in the hyperbolic domain and along the interface the information is leaving the hyperbolic domain. This property has been used in [1] to split the problem by first considering the behavior of a solution in the hyperbolic area and then in the parabolic one;

if a.e. on $\Gamma_{hp}, \nabla P.\nu_h \geq 0$, then $\Sigma_{hp}$ is included in the set of inward characteristics for the first-order operator in the hyperbolic domain and along the interface the information is now entering the hyperbolic domain. This property will also be used to first consider the behavior of a solution in the parabolic area and then in the hyperbolic one.

At last, for any positive real $\mu$, $\operatorname{sgn}_\mu$ is the Lipschitzian approximation of the function *sgn* defined by:

$$\forall x \in [0, +\infty[, \quad \operatorname{sgn}_\mu(x) = \min(\frac{x}{\mu}, 1), \quad \operatorname{sgn}_\mu(-x) = -\operatorname{sgn}_\mu(x). \tag{1.10}$$

For the rest of this work, $\sigma$ (resp. $\bar{\sigma}$) denotes the variable on $\Sigma_l$ (resp. $\Gamma_l$), $l \in \{h, hp, p\}$. This way, for any $t$ of $[0,T]$, $\sigma = (t, \bar{\sigma})$.

1.2. **Functional spaces.** In the sequel, $W(0,T)$ is the Hilbert space

$$W(0,T) \equiv \{v \in L^2(0,T; H_0^1(\Omega)); \partial_t v \in L^2(0,T; H^{-1}(\Omega))\}$$

equipped with the norm $\|w\|_{W(0,T)} = \left(\|\partial_t w\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|\nabla w\|^2_{L^2(Q)^n}\right)^{1/2}$ and $V$ is the Hilbert space

$$V = \{v \in H^1(\Omega_p), v = 0 \text{ a.e. on } \Gamma_p \backslash \Gamma_{hp}\}$$

equipped with the norm $\|v\|_V = \|\nabla v\|_{L^2(\Omega_p)^n}$.

We denote $\langle ., . \rangle$ the pairing between $V$ and $V'$.

At last $BV(\mathcal{O})$ with $\mathcal{O} = \Omega_h$ or $\mathcal{O} = Q_h$ is the space of summable functions $v$ with bounded total variation on $\mathcal{O}$ where the total variation is given by

$$TV_{\mathcal{O}}(v) = \sup\Big\{\int_{\mathcal{O}} v(x) div\Phi(x)dx, \ \Phi \in (\mathcal{D}(\mathcal{O}))^p, \|\Phi\|_{(L^\infty(\mathcal{O}))^p} \leq 1\Big\}$$

where $p$ is the dimension of the open set $\mathcal{O}$. Moreover, we denote by $\gamma v$ the trace on $\Gamma_{hp}$ or $\Sigma_{hp}$ of a function $v$ belonging to $BV(\mathcal{O})$.

The concept of a weak entropy solution to (1.1)-(1.5) is defined in Section 2 through an entropy inequality in the whole domain, the boundary conditions on the outer frontier of the hyperbolic area being expressed by referring to [8]. Then, we show some properties of such a solution in the hyperbolic area and in the parabolic one. The proof of the existence result is given in Section 3 and the uniqueness property is established in Section 4.

## 2. The Entropy Formulation

2.1. **Weak entropy solution.** The definition of a weak entropy solution to (1.1)-(1.5) has to include an entropy criterion in $Q_h$ where the quasilinear first-order hyperbolic operator is set. Problem (1.1)-(1.5) can be viewed as an evolutional problem for a quasilinear parabolic equation that *strongly degenerates* in a fixed subdomain $Q_h$ of $Q$. As in [2] or [1], we propose a weak formulation through a global entropy inequality in the whole $Q$, the latter giving rise to a variational equality in the parabolic domain and to an entropy inequality in the hyperbolic one so as to ensure the uniqueness.

**Definition 2.1.** A function $u$ is a weak entropy solution to the coupling problem (1.1)-(1.5) if $u \in L^\infty(Q)$, $\phi(u) \in L^2(0,T;V)$ and for all $\varphi \in \mathcal{D}(Q)$, $\varphi \geq 0$, for all $k \in \mathbb{R}$,

$$\begin{aligned} &\int_Q |u-k|\partial_t\varphi \,dx\,dt - \int_{Q_p} \nabla|\phi(u)-\phi(k)|.\nabla\varphi\,dx\,dt \\ &- \int_Q |f(u)-f(k)|\nabla P.\nabla\varphi\,dx\,dt - \int_Q \operatorname{sgn}(u-k)g(t,x,u)\varphi\,dx\,dt \geq 0, \end{aligned} \tag{2.1}$$

for all $\zeta \in L^1(\Sigma_h\backslash\Sigma_{hp})$, $\zeta \geq 0$, for all $k \in \mathbb{R}$,

$$\operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_h\backslash\Sigma_{hp}} \mathcal{F}(u(\sigma+\tau\nu_h),0,k)\nabla P(\bar\sigma).\nu_h\zeta d\mathcal{H}^n \leq 0, \tag{2.2}$$

$$\operatorname*{ess\,lim}_{t\to 0^+} \int_\Omega |u(t,x)-u_0(x)|dx = 0. \tag{2.3}$$

2.2. **An entropy inequality in the hyperbolic zone.** We derive from (2.1) and (2.2) an entropy inequality in the hyperbolic domain.

**Proposition 2.2.** *Let $u$ be a weak entropy solution to the coupling problem* (1.1)-(1.5). *Then for any real $k$ and any $\varphi$ of $\mathcal{D}(]0,T[\times\mathbb{R}^n)$, $\varphi \geq 0$,*

$$\begin{aligned} &-\int_{Q_h} (|u-k|\partial_t\varphi - |f(u)-f(k)|\nabla P.\nabla\varphi - \operatorname{sgn}(u-k)g(t,x,u)\varphi)\,dx\,dt \\ &\leq \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_{hp}} |f(u(\sigma+\tau\nu_h))-f(k)|\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n \\ &\quad + \int_{\Sigma_h\backslash\Sigma_{hp}} |f(k)|\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n \\ &\quad - \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_h\backslash\Sigma_{hp}} |f(u(\sigma+\tau\nu_h))|\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n. \end{aligned} \tag{2.4}$$

*Proof.* From (2.1) it comes that for $\varphi$ in $\mathcal{D}(Q_h)$, $\varphi \geq 0$,

$$\int_{Q_h} (|u-k|\partial_t\varphi - |f(u)-f(k)|\nabla P.\nabla\varphi - \operatorname{sgn}(u-k)g(t,x,u)\varphi)\,dx\,dt \geq 0. \tag{2.5}$$

First, by referring to F.Otto's works in [8], we deduce from (2.5) that, for any real $k$ and any $\beta$ in $L^1(\Sigma_h)$, the following limit exists:

$$\operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_h} |f(u(\sigma+\tau\nu_h))-f(k)|\nabla P(\bar\sigma).\nu_h\beta(\sigma)d\mathcal{H}^n. \tag{2.6}$$

Then, it results from (2.5) (see [8]) that, for any real $k$ and any $\varphi$ in $\mathcal{D}(]0,T[\times\mathbb{R}^n)$, $\varphi \geq 0$,

$$
\begin{aligned}
&-\int_{Q_h}(|u-k|\partial_t\varphi - |f(u)-f(k)|\nabla P.\nabla\varphi - \operatorname{sgn}(u-k)g(t,x,u)\varphi)\,dx\,dt\\
&\leq \operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h}|f(u(\sigma+\tau\nu_h))-f(k)|\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n.
\end{aligned}
$$

To conclude we share the frontier of $\Omega_h$ into $\Gamma_{hp}$ and $\Gamma_h\backslash\Gamma_{hp}$ and we use boundary condition (2.2) on $\Sigma_h\backslash\Sigma_{hp}$. □

2.3. **A variational equality in the parabolic zone.** We give now some information on the regularity for $\partial_t u$ in $Q_p$ and we derive from (2.1) a variational equality satisfied by any weak entropy solution $u$ to the coupling problem (1.1)-(1.5).

**Proposition 2.3.** *Let $u$ be a weak entropy solution to the coupling problem (1.1)-(1.5). Then $\partial_t u$ belongs to $L^2(0,T;V')$. Furthermore, for any $\varphi$ in $L^2(0,T;V)$,*

$$
\begin{aligned}
&\int_0^T\langle\partial_t u,\varphi\rangle dt + \int_{Q_p}\nabla\phi(u).\nabla\varphi\,dx\,dt + \int_{Q_p}f(u)\nabla P.\nabla\varphi\,dx\,dt\\
&+\int_{Q_p}g(t,x,u)\varphi\,dx\,dt + \operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}}f(u(\sigma+\tau\nu_h))\nabla P(\bar\sigma).\nu_h\varphi d\mathcal{H}^n = 0.
\end{aligned}
\tag{2.7}
$$

**Remark 2.4.** This proposition is proved in [1, Proposition 3.4] independently of any condition on the hyperbolic characteristics on $\Sigma_{hp}$.

## 3. The Existence Result

In this section, we will prove the existence of a weak entropy solution.

**Theorem 3.1.** *The coupling problem* (1.1)–(1.5) *has at least a weak entropy solution.*

To construct a weak entropy solution to Problem (1.1)-(1.5), we work successively in the hyperbolic domain and in the parabolic one or vice-versa. Indeed, thanks to Remark 1.1, when a.e. on $\Gamma_{hp}$ $\nabla P.\nu_h \leq 0$, we can begin by working in the hyperbolic zone while, when a.e. on $\Gamma_{hp}$ $\nabla P.\nu_h \geq 0$, we can begin by working in the parabolic area.

3.1. **Waves going from $Q_h$ to $Q_p$.** In this section we suppose that, a.e. on $\Gamma_{hp}$, $\nabla P.\nu_h \leq 0$. The existence of a weak entropy solution to Problem (1.1)-(1.5) is already proved in [1] by the viscosity method. Here, we give a different proof of this result.

First, thanks to [8], there exists one and only one function $w_h$ in $L^\infty(Q_h)$ such that for all $\varphi \in \mathcal{D}(Q_h)$, $\varphi \geq 0$, for all $k \in \mathbb{R}$,

$$
\int_{Q_h}(|w_h-k|\partial_t\varphi - |f(w_h)-f(k)|\nabla P.\nabla\varphi - \operatorname{sgn}(w_h-k)g(t,x,w_h)\varphi)\,dx\,dt \geq 0, \tag{3.1}
$$

for all $\zeta \in L^1(\Sigma_h)$, $\zeta \geq 0$, for all $k \in \mathbb{R}$,

$$
\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h}\mathcal{F}(w_h(\sigma+\tau\nu_h),0,k)\nabla P(\bar\sigma).\nu_h\zeta d\mathcal{H}^n \leq 0, \tag{3.2}
$$

$$\operatorname*{ess\,lim}_{t\to 0^+}\int_{\Omega_h}|w_h(t,x)-u_0(x)|dx=0. \tag{3.3}$$

Then, thanks to [5], there exists one and only one function $w_p$ in $L^\infty(Q_p)$ such that $\phi(w_p)\in L^2(0,T;V)$, $\partial_t w_p\in L^2(0,T;V')$ and for all $\varphi\in L^2(0,T;V)$,

$$\begin{aligned}&\int_0^T\langle\partial_t w_p,\varphi\rangle dt+\int_{Q_p}\nabla\phi(w_p).\nabla\varphi\,dx\,dt+\int_{Q_p}f(w_p)\nabla P.\nabla\varphi\,dx\,dt\\&+\int_{Q_p}g(t,x,w_p)\varphi\,dx\,dt+\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}}f(w_h(\sigma+\tau\nu_h))\nabla P(\bar\sigma).\nu_h\varphi d\mathcal{H}^n=0,\end{aligned} \tag{3.4}$$

$$\operatorname*{ess\,lim}_{t\to 0^+}\int_{\Omega_p}|w_p(t,x)-u_0(x)|dx=0. \tag{3.5}$$

Indeed the mapping

$$\varphi\longmapsto -\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}}f(w_h(\sigma+\tau\nu_h))\nabla P(\bar\sigma).\nu_h\varphi d\mathcal{H}^n$$

belongs to $L^\infty(0,T;V')$. Therefore to prove Theorem 3.1, we are going to establish the following lemma.

**Lemma 3.2.** *Let $u$ be defined by $u=w_h$ in $Q_h$ and $u=w_p$ in $Q_p$. Then $u$ is a weak entropy solution to the coupling problem* (1.1)-(1.5).

*Moreover if $u_{0|\Omega_h}$ belongs to $BV(\Omega_h)$, then $u_{|Q_h}$ belongs to $BV(Q_h)$ and*

$$\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}}|f(u(\sigma+\tau\nu_h))-f(\gamma u(\sigma))|d\mathcal{H}^n=0$$

*where $\gamma u(\sigma)$ is the trace on $\Sigma_{hp}$ in the BV-sense of the BV-function $u_{|Q_h}$.*

*Proof.* First note that $u\in L^\infty(Q)$ and $\phi(u)\in L^2(0,T;V)$. Let $\varphi$ be in $\mathcal{D}(Q)$, $\varphi\geq 0$ and let $k$ be in $\mathbb{R}$. As in the proof of Proposition 2.2, we derive from (3.1) the following inequality

$$\begin{aligned}&\int_{Q_h}(|w_h-k|\partial_t\varphi-|f(w_h)-f(k)|\nabla P.\nabla\varphi-\operatorname{sgn}(w_h-k)g(t,x,w_h)\varphi)\,dx\,dt\\&\geq-\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}}|f(w_h(\sigma+\tau\nu_h))-f(k)|\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n.\end{aligned} \tag{3.6}$$

Then, we choose in (3.4) the test-function $\varphi\operatorname{sgn}_\mu(\phi(w_p)-\phi(k))$. It follows:

$$\begin{aligned}&-\int_0^T\langle\partial_t w_p,\operatorname{sgn}_\mu(\phi(w_p)-\phi(k))\varphi\rangle dt\\&-\int_{Q_p}\operatorname{sgn}_\mu(\phi(w_p)-\phi(k))\ \nabla(\phi(w_p)-\phi(k)).\nabla\varphi\,dx\,dt\\&-\int_{Q_p}\operatorname{sgn}_\mu(\phi(w_p)-\phi(k))(f(w_p)-f(k))\nabla P.\nabla\varphi\,dx\,dt\\&-\int_{Q_p}g(t,x,w_p)\operatorname{sgn}_\mu(\phi(w_p)-\phi(k))\varphi\,dx\,dt\\&=\int_{Q_p}\operatorname{sgn}'_\mu(\phi(w_p)-\phi(k))\ \nabla(\phi(w_p)-\phi(k)).\nabla(\phi(w_p)-\phi(k))\varphi\,dx\,dt\end{aligned}$$

$$
\begin{aligned}
&+\int_{Q_p} {\operatorname{sgn}'}_\mu(\phi(w_p)-\phi(k))(f(w_p)-f(k))\nabla P.\nabla(\phi(w_p)-\phi(k))\varphi\,dx\,dt\\
&+\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}} f(w_h(\sigma+\tau\nu_h))\nabla P.\nu_h \operatorname{sgn}_\mu(\phi(w_p)-\phi(k))\varphi d\mathcal{H}^n\\
&+\int_{\Sigma_{hp}} \operatorname{sgn}_\mu(\phi(w_p)-\phi(k))f(k)\nabla P.\nu_p\varphi d\mathcal{H}^n. \qquad (3.7)
\end{aligned}
$$

Thanks to (1.9) and to the Cauchy-Scharwz inequality there exists a positive constant $\mathcal{C}$ such that :

$$
\begin{aligned}
&\int_{Q_p} {\operatorname{sgn}'}_\mu(\phi(w_p)-\phi(k))\ \nabla(\phi(w_p)-\phi(k)).\nabla(\phi(w_p)-\phi(k))\varphi\,dx\,dt\\
&+\int_{Q_p} {\operatorname{sgn}'}_\mu(\phi(w_p)-\phi(k))(f(w_p)-f(k))\nabla P.\nabla(\phi(w_p)-\phi(k))\varphi\,dx\,dt\\
&\ge -\mathcal{C}\int_{Q_p}|\phi(w_p)-\phi(k)|^{2\theta}{\operatorname{sgn}'}_\mu(\phi(w_p)-\phi(k))\varphi\,dx\,dt,
\end{aligned}
$$

and the term in the right-hand side goes to 0 with $\mu$ as $\theta\ge 1/2$ thanks to the Lebesgue's bounded convergence theorem.

In the first term of (3.7), we use an integration by parts formula based on a convexity inequality (see e.g. [5], the Mignot-Bamberger Lemma) to obtain

$$
-\int_0^T\langle\partial_t w_p,\operatorname{sgn}_\mu(\phi(w_p)-\phi(k))\varphi\rangle dt=\int_{Q_p}\Big(\int_k^{w_p}\operatorname{sgn}_\mu(\phi(r)-\phi(k))dr\Big)\partial_t\varphi\,dx\,dt.
$$

Therefore, we are able to pass to the limit in (3.7) when $\mu$ approaches $0^+$ in all the integrals over $Q_p$. For the one on $\Sigma_{hp}$, we argue from (3.1) and [8] that (2.6) is valid for $w_h$. Therefore there exists $\theta$ in $L^\infty(\Sigma_{hp})$ such that for any $\beta$ in $L^1(\Sigma_{hp})$,

$$
\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}} f(w_h(\sigma+\tau\nu_h))\nabla P(\bar\sigma).\nu_h\beta(\sigma)d\mathcal{H}^n=\int_{\Sigma_{hp}}\theta(\sigma)\beta(\sigma)d\mathcal{H}^n. \qquad (3.8)
$$

Therefore, we can use that

$$
\lim_{\mu\to 0^+}\int_{\Sigma_{hp}}\theta(\sigma)\operatorname{sgn}_\mu(\phi(w_p)-\phi(k))\varphi d\mathcal{H}^n=\int_{\Sigma_{hp}}\theta(\sigma)\operatorname{sgn}(\phi(w_p)-\phi(k))\varphi d\mathcal{H}^n.
$$

After all, we obtain

$$
\begin{aligned}
&\int_{Q_p}(|w_p-k|\partial_t\varphi-|f(w_p)-f(k)|\nabla P.\nabla\varphi-\operatorname{sgn}(w_p-k)g(t,x,w_p)\varphi)\,dx\,dt\\
&-\int_{Q_p}\nabla|\phi(w_p)-\phi(k)|.\nabla\varphi\,dx\,dt \qquad (3.9)\\
&\ge \operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}}(f(w_h(\sigma+\tau\nu_h))-f(k))\operatorname{sgn}(w_p(\sigma)-k)\nabla P.\nu_h\varphi d\mathcal{H}^n
\end{aligned}
$$

where in (3.9), $w_p(\sigma)$ is defined as $\phi^{-1}(\phi(w_p(\sigma)))$ and belongs to $L^\infty(\Sigma_{hp})$. By adding the inequalities (3.6) and (3.9), we obtain

$$
\begin{aligned}
&\int_Q|u-k|\partial_t\varphi\,dx\,dt-\int_{Q_p}\nabla|\phi(u)-\phi(k)|.\nabla\varphi\,dx\,dt\\
&-\int_Q|f(u)-f(k)|\nabla P.\nabla\varphi\,dx\,dt-\int_Q\operatorname{sgn}(u-k)g(t,x,u)\varphi\,dx\,dt
\end{aligned}
$$

$$\geq \operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}}(f(w_h(\sigma+\tau\nu_h))-f(k))\operatorname{sgn}(w_p(\sigma)-k)\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n$$
$$-\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}}|f(w_h(\sigma+\tau\nu_h))-f(k)|\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n.$$

Now by using the condition $\nabla P.\nu_h \leq 0$ a.e. on $\Gamma_{hp}$, we derive that $u$ satisfies Inequality (2.1).

At last, thanks to (3.2), (3.3) and (3.5), we can conclude that $u$ is a weak entropy solution to the coupling problem (1.1)-(1.5).

Now, if $u_{0|\Omega_h}$ belongs to $BV(\Omega_h)$, it results from [3] and [8] that $w_{h|Q_h}$ belongs to $BV(Q_h)$. Therefore $u_{|Q_h}$ belongs to $BV(Q_h)$ and thanks to the properties of the trace operator from $BV(Q_h)$ into $L^1(\Sigma_h)$

$$\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h}|f(u(\sigma+\tau\nu_h))-f(\gamma u(\sigma))|d\mathcal{H}^n=0$$

where $\gamma u(\sigma)$ is the trace on $\Sigma_h$ in the BV-sense of the BV-function $u_{|Q_h}$. □

3.2. **Waves going from $Q_p$ to $Q_h$.** In this section we suppose that a.e. on $\Gamma_{hp}$, $\nabla P.\nu_h \geq 0$.

First, thanks to [5], there exists one and only one function $w_p$ in $L^\infty(Q_p)$ such that $\phi(w_p)\in L^2(0,T;V)$, $\partial_t w_p \in L^2(0,T;V')$ and for all $\varphi\in L^2(0,T;V)$,

$$\begin{aligned}&\int_0^T\langle\partial_t w_p,\varphi\rangle dt+\int_{Q_p}\nabla\phi(w_p).\nabla\varphi\,dx\,dt+\int_{Q_p}f(w_p)\nabla P.\nabla\varphi\,dx\,dt\\&+\int_{Q_p}g(t,x,w_p)\varphi\,dx\,dt+\int_{\Sigma_{hp}}f(w_p(\sigma))\nabla P(\bar\sigma).\nu_h\varphi d\mathcal{H}^n=0,\end{aligned}\tag{3.10}$$

$$\operatorname*{ess\,lim}_{t\to 0^+}\int_{\Omega_p}|w_p(t,x)-u_0(x)|dx=0.\tag{3.11}$$

In (3.10), $w_p(\sigma)$ is defined as $\phi^{-1}(\phi(w_p(\sigma)))$ and belongs to $L^\infty(\Sigma_{hp})$.

Then, thanks to [8], there exists one and only one function $w_h$ in $L^\infty(Q_h)$ such that for all $\varphi\in\mathcal{D}(Q_h)$, $\varphi\geq 0$, for all $k\in\mathbb{R}$,

$$\int_{Q_h}(|w_h-k|\partial_t\varphi-|f(w_h)-f(k)|\nabla P.\nabla\varphi-\operatorname{sgn}(w_h-k)g(t,x,w_h)\varphi)\,dx\,dt\geq 0;\tag{3.12}$$

for all $\zeta\in L^1(\Sigma_h)$, $\zeta\geq 0$, for all $k\in\mathbb{R}$,

$$\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_h\setminus\Sigma_{hp}}\mathcal{F}(w_h(\sigma+\tau\nu_h),0,k)\nabla P(\bar\sigma).\nu_h\zeta d\mathcal{H}^n\leq 0,\tag{3.13}$$

$$\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}}\mathcal{F}(w_h(\sigma+\tau\nu_h),w_p(\sigma),k)\nabla P(\bar\sigma).\nu_h\zeta d\mathcal{H}^n\leq 0,\tag{3.14}$$

$$\operatorname*{ess\,lim}_{t\to 0^+}\int_{\Omega_h}|w_h(t,x)-u_0(x)|dx=0.\tag{3.15}$$

Therefore to prove Theorem 3.1, we establish the following lemma.

**Lemma 3.3.** *Let $u$ be defined by $u=w_h$ in $Q_h$ and $u=w_p$ in $Q_p$. Then $u$ is a weak entropy solution to the coupling problem* (1.1)-(1.5)*.*

*Moreover*

$$\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}}|f(u(\sigma+\tau\nu_h))-f(u(\sigma))|d\mathcal{H}^n=0$$

*where $u(\sigma)$ is defined as $\phi^{-1}(\phi(u(\sigma)))$.*

*Proof.* First $u \in L^\infty(Q)$ and $\phi(u) \in L^2(0,T;V)$. Now let $\varphi$ be in $\mathcal{D}(Q)$, $\varphi \geq 0$ and let $k$ be in $\mathbb{R}$.

Following the proof of (3.9) in Lemma 3.2, we deduce from (3.10) that

$$\begin{aligned}
&\int_{Q_p} (|w_p - k|\partial_t\varphi - |f(w_p) - f(k)|\nabla P.\nabla\varphi - \operatorname{sgn}(w_p - k)g(t,x,w_p)\varphi)\,dx\,dt \\
&- \int_{Q_p} \nabla|\phi(w_p) - \phi(k)|.\nabla\varphi\,dx\,dt \\
&\geq \int_{\Sigma_{hp}} |f(w_p(\sigma)) - f(k)|\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n.
\end{aligned} \tag{3.16}$$

Moreover, Inequality (3.6) is still satisfied by $w_h$. By adding the inequalities (3.6) and (3.16) we obtain

$$\begin{aligned}
&\int_Q |u-k|\partial_t\varphi\,dx\,dt - \int_{Q_p} \nabla|\phi(u) - \phi(k)|.\nabla\varphi\,dx\,dt \\
&- \int_Q |f(u) - f(k)|\nabla P.\nabla\varphi\,dx\,dt - \int_Q \operatorname{sgn}(u-k)g(t,x,u)\varphi\,dx\,dt \\
&\geq \int_{\Sigma_{hp}} |f(w_p(\sigma)) - f(k)|\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n \\
&\quad - \operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_{hp}} |f(w_h(\sigma + \tau\nu_h)) - f(k)|\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n.
\end{aligned}$$

Then, thanks to (3.14) and to the condition $\nabla P.\nu_h \geq 0$ a.e. on $\Gamma_{hp}$, we obtain that $u$ satisfies Inequality (2.1).

Now, thanks to (3.13), (3.15) and (3.11), we conclude that $u$ is a weak entropy solution to the coupling problem (1.1)-(1.5).

At last, it results from [7] that as $\Sigma_{hp}$ is included in the set of inward characteristics for the first order operator, the solution $w_h$ of Problem (3.12)-(3.15) satisfies

$$\operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_{hp}} |f(w_h(\sigma + \tau\nu_h)) - f(w_p(\sigma))|d\mathcal{H}^n = 0.$$

□

## 4. The Uniqueness Property

We have seen in Lemma 3.2 or Lemma 3.3 that Problem (1.1)-(1.5) has at least a weak entropy solution $u$ for which there exists $\theta \in L^1(\Sigma_{hp})$, $|\theta| \leq M$ and

$$\operatorname*{ess\,lim}_{\tau\to 0^-} \int_{\Sigma_{hp}} |f(u(\sigma + \tau\nu_h)) - f(\theta(\sigma))|\nabla P.\nu_h d\mathcal{H}^n = 0. \tag{4.1}$$

Indeed, when $\nabla P.\nu_h \leq 0$ a.e. on $\Gamma_{hp}$, as soon as $u_{0|\Omega_h}$ belongs to $BV(\Omega_h)$ then (4.1) is satisfied with $\theta = \gamma u$ where $\gamma u$ is the trace on $\Sigma_{hp}$ in the BV-sense of the BV-function $u_{|Q_h}$. When $\nabla P.\nu_h \geq 0$ a.e. on $\Gamma_{hp}$, (4.1) is satisfied with $\theta = u$ where $u$ is defined as $\phi^{-1}(\phi(u))$ and $\phi(u)$ is the trace on $\Sigma_{hp}$ of $\phi(u)_{|Q_p}$.

In this section, we prove the uniqueness property in the class of weak entropy solutions satisfying (4.1). Indeed, we have justified that Problem (1.1)-(1.5) admits such a solution (under the additional hypothesis $u_{0|\Omega_h}$ belongs to $BV(\Omega_h)$ when a.e. on $\Gamma_{hp}\nabla P.\nu_h \leq 0$).

4.1. **Preliminaries.** To use the method of doubling variables, we introduce a sequence of mollifiers $(W_\delta)_{\delta>0}$ on $\mathbb{R}^{n+1}$ defined by

$$\forall\delta>0,\, \forall r=(t,x)\in\mathbb{R}^{n+1},\, W_\delta(r)=\varpi_\delta(t)\prod_{i=1}^{n}\varpi_\delta(x_i),$$

where $(\varpi_\delta)_{\delta>0}$ is a standard sequence of mollifiers on $\mathbb{R}$. We will use classical results on the Lebesgue set of a summable function on $Q$ and a similar property on $\Sigma$ proved in [9]:

**Lemma 4.1.** *Let $v$ and $w$ be in $L^\infty(Q_h)$ such that (2.5) and (4.1) hold. Then for any continuous function $\varphi$ on $\overline{Q_h}$,*

$$\begin{aligned}&\lim_{\delta\to0^+}\int_{Q_h}\int_{\Sigma_h\setminus\Sigma_{hp}}|f(v(r))|\nabla P(\bar\sigma).\nu_h\varphi(\frac{\tilde\sigma+r}{2})\mathcal{W}_\delta(\tilde\sigma-r)d\mathcal{H}^n_{\tilde\sigma}dr\\&=\frac12\operatorname*{ess\,lim}_{\tau\to0^-}\int_{\Sigma_h\setminus\Sigma_{hp}}|f(v(\sigma+\tau\nu_h))|\nabla P(\bar\sigma)\nu_h\varphi(\sigma)d\mathcal{H}^n,\end{aligned}$$

$$\begin{aligned}&\lim_{\delta\to0^+}\int_{Q_h}\operatorname*{ess\,lim}_{\tau\to0^-}\int_{\Sigma_h\setminus\Sigma_{hp}}|f(v(\sigma+\tau\nu_h))|\nabla P(\bar\sigma).\nu_h\varphi(\frac{\sigma+\tilde r}{2})\mathcal{W}_\delta(\sigma-\tilde r)d\mathcal{H}^n_\sigma d\tilde r\\&=\frac12\operatorname*{ess\,lim}_{\tau\to0^-}\int_{\Sigma_h\setminus\Sigma_{hp}}|f(v(\sigma+\tau\nu_h))|\nabla P(\bar\sigma).\nu_h\varphi(\sigma)d\mathcal{H}^n,\end{aligned}$$

*and*

$$\begin{aligned}&\lim_{\delta\to0^+}\int_{Q_h}\int_{\Sigma_{hp}}|f(\theta_v(\sigma))-f(w(\tilde r))|\nabla P(\bar\sigma).\nu_h\varphi(\frac{\sigma+\tilde r}{2})W_\delta(\sigma-\tilde r)d\mathcal{H}^n_\sigma d\tilde r\\&=\frac12\int_{\Sigma_{hp}}|f(\theta_v(\sigma))-f(\theta_w(\sigma))|\nabla P(\bar\sigma)\nu_h\varphi(\sigma)d\mathcal{H}^n\end{aligned}$$

*where $\theta_v$ (resp. $\theta_w$) is defined by (4.1) for $v$ (resp. $w$).*

4.2. **The uniqueness theorem.**

**Lemma 4.2.** *Let $u_1$, $u_2$ be two weak solutions to* (1.1)-(1.5) *for initial data respectively $u_{0,1}$, $u_{0,2}$ and such that* (4.1) *holds with $f(\theta_i)\nabla P.\nu_h=f(u_i)\nabla P.\nu_h$, for $i=1,2$, when $\nabla P.\nu_h\ge0$ a.e. on $\Gamma_{hp}$. Then, for a.e. $t$ of $[0,T]$,*

$$\int_\Omega|u_1(t,.)-u_2(t,.)|dx\le e^{M'_g\,t}\int_\Omega|u_{0,1}-u_{0,2}|dx.$$

**Theorem 4.3.** *Let $u_0$ be in $L^\infty(\Omega)$. The coupling problem* (1.1)-(1.5) *admits at most one weak entropy solution $u$ such that* (4.1) *holds with $f(\theta)\nabla P.\nu_h=f(u)\nabla P.\nu_h$ when $\nabla P.\nu_h\ge0$ a.e. on $\Gamma_{hp}$.*

*Moreover, for initial data $u_{0,1}$ and $u_{0,2}$ in $L^\infty(\Omega)$ the corresponding weak entropy solutions $u_1$ and $u_2$ to* (1.1)-(1.5) *are such that for a.e. $t$ of $[0,T]$,*

$$\int_\Omega|u_1(t,.)-u_2(t,.)|dx\le e^{M'_g\,t}\int_\Omega|u_{0,1}-u_{0,2}|dx.$$

*Proof of Lemma 4.2.* (i) We first compare the two solutions $u_1$ and $u_2$ in the parabolic zone. The lack of regularity of the time partial derivative of any weak entropy solution to (1.1)-(1.5) requires a doubling of the time variable.

Therefore, let $\chi$ be a nonnegative element of $\mathcal{D}(0,T)$. We consider $\delta$ a positive real small enough for $\alpha_\delta:(\tilde t,t)\longmapsto\alpha_\delta(\tilde t,t)=\chi((t+\tilde t)/2)\varpi_\delta((t-\tilde t)/2)$ to belong to $\mathcal{D}(]0,T[\times]0,T[)$. Then, for $\mu > 0$, in (2.7) for $u_1$ written in variables $(t,x)$ we consider $\varphi(t,x) = \operatorname{sgn}_\mu(\phi(u_1)(t,x) - \phi(u_2)(\tilde{t},x))\alpha_\delta(\tilde{t},t)$ and in (2.7) written in variables $(\tilde{t},x)$ for $u_2$, we consider $\varphi(\tilde{t},x) = -\operatorname{sgn}_\mu(\phi(u_1)(t,x)-\phi(u_2)(\tilde{t},x))\alpha_\delta(\tilde{t},t)$. To simplify the writing, we add a "tilde" superscript to any function in the $\tilde{t}$ variable. Moreover, thanks to (4.1) we observe that in (2.7), for $i = 1,2$,

$$\operatorname*{ess\,lim}_{\tau\to 0^-}\int_{\Sigma_{hp}} f(u_i(\sigma+\tau\nu_h))\nabla P(\bar{\sigma}).\nu_h\varphi d\mathcal{H}^n = \int_{\Sigma_{hp}} f(\theta_i(\sigma))\nabla P.\nu_h\varphi d\mathcal{H}^n.$$

Then, by adding up, it comes:

$$\begin{aligned}
&\int_0^T\int_0^T\langle\partial_t u_1 - \partial_{\tilde{t}}\tilde{u}_2, \operatorname{sgn}_\mu(\phi(u_1)-\phi(\tilde{u}_2))\rangle\alpha_\delta dt d\tilde{t}\\
&+\int_{]0,T[\times Q_p}\nabla(\phi(u_1)-\phi(\tilde{u}_2)).\nabla\operatorname{sgn}_\mu(\phi(u_1)-\phi(\tilde{u}_2))\ \alpha_\delta\ \ dx\,dtd\tilde{t}\\
&+\int_{]0,T[\times Q_p}(f(u_1)-f(\tilde{u}_2))\nabla P.\nabla\operatorname{sgn}_\mu(\phi(u_1)-\phi(\tilde{u}_2))\ \alpha_\delta\ \ dx\,dtd\tilde{t}\\
&+\int_{]0,T[\times Q_p}(g(t,x,u_1)-g(\tilde{t},x,\tilde{u}_2))\operatorname{sgn}_\mu(\phi(u_1)-\phi(\tilde{u}_2))\ \alpha_\delta\ \ dx\,dtd\tilde{t}\\
&= -\int_0^T\int_{\Sigma_{hp}} f(\theta_1(\sigma))\nabla P.\nu_h\operatorname{sgn}_\mu(\phi(u_1)-\phi(u_2(\tilde{\sigma})))\alpha_\delta d\mathcal{H}^n_\sigma d\tilde{t}\\
&\quad+\int_0^T\int_{\Sigma_{hp}} f(\theta_2(\tilde{\sigma}))\nabla P.\nu_h\operatorname{sgn}_\mu(\phi(u_1)-\phi(u_2(\tilde{\sigma})))\alpha_\delta d\mathcal{H}^n_{\tilde{\sigma}} dt.
\end{aligned}\tag{4.2}$$

In the left-hand side, we use the calculus of the proof of Lemma 3.2. So, we are able to pass to the limit in (4.2) when $\mu$ approaches $0^+$. Therefore,

$$\begin{aligned}
&-\int_{]0,T[\times Q_p}|u_1-\tilde{u}_2|(\partial_t\alpha_\delta+\partial_{\tilde{t}}\alpha_\delta)\,dx\,dtd\tilde{t}\\
&\le\int_{]0,T[\times Q_p}|g(t,x,\tilde{u}_2)-g(\tilde{t},x,\tilde{u}_2)|\alpha_\delta\,dx\,dtd\tilde{t}\\
&\quad-\int_0^T\int_{\Sigma_{hp}}(f(\theta_1(\sigma))-f(\theta_2(\tilde{\sigma})))\nabla P.\nu_h\operatorname{sgn}_\mu(\phi(u_1)-\phi(u_2(\tilde{\sigma})))\alpha_\delta d\mathcal{H}^n_\sigma d\tilde{t}.
\end{aligned}$$

Now, we come back to the definition of $\alpha_\delta$ to express the sum $\partial_t\alpha_\delta+\partial_{\tilde{t}}\alpha_\delta$. Then we are able to take the limit with respect to $\delta$ through the notion of the Lebesgue's set of a summable function on $]0,T[$. Therefore, as $g$ is Lipschitzian, for any $\chi$ in $\mathcal{D}(0,T)$, $\chi\ge 0$,

$$\begin{aligned}
&-\int_{Q_p}|u_1-u_2|\chi'(t)\,dx\,dt\\
&\le M'_g\int_{Q_p}|u_1-u_2|\chi(t)\,dx\,dt\\
&\quad-\int_{\Sigma_{hp}}(f(\theta_1(\sigma))-f(\theta_2(\sigma)))\nabla P.\nu_h\operatorname{sgn}(\phi(u_1)-\phi(u_2))\chi(t)d\mathcal{H}^n.
\end{aligned}\tag{4.3}$$

(ii) Now, we work in the hyperbolic domain. We use a doubling method for all the variables . Let $\psi$ be such that $\psi\equiv\chi\zeta$ where $\chi$ is a function in $\mathcal{D}(0,T)$, $\chi\ge 0$, as in Part (i) and $\zeta$ is in $\mathcal{D}(\mathbb{R}^n)$ such that: $\zeta\ge 0$, $\zeta\equiv 1$ on $Q_h$. We consider $\delta$ a positive real small enough in order that the mapping $(\tilde{t},t) \longmapsto \chi((t+\tilde{t})/2)w_\delta((t-\tilde{t})/2)$ belongs to $\mathcal{D}(]0,T[\times]0,T[)$. Then, for any positive $\delta$, we define the function $\Psi_\delta$ in $]0,T[\times\mathbb{R}^n\times]0,T[\times\mathbb{R}^n$ by $\Psi_\delta(r,\tilde{r}) = \chi((t+\tilde{t})/2)\zeta((x+\tilde{x})/2)W_\delta(r-\tilde{r})$.

Due to Proposition 2.2, Inequality (2.4) holds for $u_1$ and $u_2$. We choose in (2.4) written for $u_1$ in variables $(t,x)$,

$$k = \tilde{u}_2 \equiv u_2(\tilde{t},\tilde{x}) \quad \text{and} \quad \varphi(t,x) = \Psi_\delta(t,x,\tilde{t},\tilde{x})$$

and in (2.4) written for $u_2$ in variables $(\tilde{t},\tilde{x})$,

$$k = u_1(t,x) \quad \text{and} \quad \varphi(\tilde{t},\tilde{x}) = \Psi_\delta(t,x,\tilde{t},\tilde{x}).$$

By integrating over $Q_h$ and adding up, it comes by using (4.1):

$$\begin{aligned}
&-\int_{Q_h\times Q_h} (|u_1-\tilde{u}_2|(\partial_t\Psi_\delta+\partial_{\tilde{t}}\Psi_\delta) - |f(u_1)-f(\tilde{u}_2)|(\nabla P.\nabla_x\Psi_\delta + \nabla\tilde{P}.\nabla_{\tilde{x}}\Psi_\delta)\,dr\,d\tilde{r} \\
&+\int_{Q_h\times Q_h} \operatorname{sgn}(u_1-\tilde{u}_2)(g(t,x,u_1)-g(\tilde{t},x,\tilde{u}_2))\Psi_\delta\,dr\,d\tilde{r} \\
&\le \int_{Q_h}\int_{\Sigma_h\setminus\Sigma_{hp}} |f(\tilde{u}_2)|\nabla_x P.\nu_h\Psi_\delta(\sigma,\tilde{r})d\mathcal{H}^n_\sigma d\tilde{r} \\
&\quad + \int_{Q_h}\int_{\Sigma_h\setminus\Sigma_{hp}} |f(u_1)|\nabla_{\tilde{x}}\tilde{P}.\nu_h\Psi_\delta(r,\tilde{\sigma})d\mathcal{H}^n_{\tilde{\sigma}} dr \\
&\quad - \int_{Q_h}\operatorname*{ess\,lim}_{\tau\to0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} |f(u_1(\sigma+\tau\nu_h))|\nabla_x P.\nu_h\Psi_\delta(\sigma,\tilde{r})d\mathcal{H}^n_\sigma d\tilde{r} \\
&\quad - \int_{Q_h}\operatorname*{ess\,lim}_{\tau\to0^-}\int_{\Sigma_h\setminus\Sigma_{hp}} |f(u_2(\tilde{\sigma}+\tau\nu_h))|\nabla_{\tilde{x}}\tilde{P}.\nu_h\Psi_\delta(r,\tilde{\sigma})d\mathcal{H}^n_{\tilde{\sigma}} dr \\
&\quad + \int_{Q_h}\int_{\Sigma_{hp}} |f(\theta_1(\sigma))-f(\tilde{u}_2)|\nabla_x P.\nu_h\Psi_\delta(\sigma,\tilde{r})d\mathcal{H}^n_\sigma d\tilde{r} \\
&\quad + \int_{Q_h}\int_{\Sigma_{hp}} |f(\theta_2(\tilde{\sigma}))-f(u_1)|\nabla_{\tilde{x}}\tilde{P}.\nu_h\Psi_\delta(r,\tilde{\sigma})d\mathcal{H}^n_{\tilde{\sigma}} dr.
\end{aligned} \tag{4.4}$$

Then through a classical reasoning we pass to the limit with $\delta$ on the left-hand side of (4.4). On the right-hand side, we refer to Lemma 4.1. It comes:

$$\begin{aligned}
-\int_{Q_h} |u_1-u_2|\chi'(t)\,dx\,dt &\le -\int_{Q_h} \operatorname{sgn}(u_1-u_2)(g(t,x,u_1)-g(t,x,u_2))\chi(t)\,dx\,dt \\
&\quad + \int_{\Sigma_{hp}} |f(\theta_1(\sigma))-f(\theta_2(\sigma))|\nabla_x P.\nu_h\chi(t)d\mathcal{H}^n.
\end{aligned}$$

The Lipschitz condition for $g$ provides: for any $\chi$ of $\mathcal{D}(0,T)$, $\chi\ge0$,

$$\begin{aligned}
-\int_{Q_h} |u_1-u_2|\chi'(t)\,dx\,dt &\le \int_{\Sigma_{hp}} |f(\theta_1(\sigma))-f(\theta_2(\sigma))|\nabla_x P.\nu_h\chi(t)d\mathcal{H}^n \\
&\quad + M'_g\int_{Q_h} |u_1-u_2|\chi(t)\,dx\,dt.
\end{aligned} \tag{4.5}$$

By adding inequalities (4.3) and (4.5), we obtain

$$-\int_Q |u_1-u_2|\chi'(t)\,dx\,dt$$

$$\leq M'_g \int_Q |u_1 - u_2|\chi(t)\,dx\,dt + \int_{\Sigma_{hp}} |f(\theta_1(\sigma)) - f(\theta_2(\sigma))|\nabla_x P.\nu_h \chi(t) d\mathcal{H}^n$$
$$- \int_{\Sigma_{hp}} (f(\theta_1(\sigma)) - f(\theta_2(\sigma)))\nabla P.\nu_h \operatorname{sgn}(\phi(u_1) - \phi(u_2))\chi(t) d\mathcal{H}^n.$$

Therefore, when a.e. on $\Gamma_{hp}$, $\nabla P.\nu_h \leq 0$, we have

$$|f(\theta_1(\sigma)) - f(\theta_2(\sigma))|\nabla P.\nu_h \leq (f(\theta_1(\sigma)) - f(\theta_2(\sigma))) \operatorname{sgn}(\phi(u_1) - \phi(u_2))\nabla P.\nu_h.$$

Now, when a.e. on $\Gamma_{hp}$, $\nabla P.\nu_h \geq 0$, a.e. on $\Sigma_{hp}$,

$$f(\theta_i(\sigma))\nabla P.\nu_h = f(u_i(\sigma))\nabla P.\nu_h,\ i = 1, 2.$$

As a consequence, a.e. on $\Sigma_{hp}$,

$$(f(\theta_1(\sigma)) - f(\theta_2(\sigma)))\nabla P.\nu_h \operatorname{sgn}(\phi(u_1) - \phi(u_2)) = |f(\theta_1(\sigma)) - f(\theta_2(\sigma))|\nabla P.\nu_h.$$

At last in both cases, we have for any $\chi$ of $\mathcal{D}(0,T)$, $\chi \geq 0$,

$$- \int_Q |u_1 - u_2|\chi'(t)\,dx\,dt \leq M'_g \int_Q |u_1 - u_2|\chi(t)\,dx\,dt.$$

When $\chi$ is the element of a sequence approximating $\mathbb{I}_{[0,t]}$, $t$ being given outside a set of measure zero, the desired inequality of Lemma 4.2 is obtained thanks to the initial condition (2.3) for $u_1$ and $u_2$ and to the Gronwall Lemma. □

**Comments.** In this paper we have looked for solutions to the coupling problem (1.1)-(1.5). We have proved an existence and uniqueness result when along the interface all the charasteristics have the same behaviour. Either there are all leaving the hyperbolic domain, either there are all entering this domain. In the first case, we refer the reader to [1] for a study without condition (4.1) by means of the vanishing viscosity method and the notion of process solutions [4].

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