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A THIRD-ORDER M-POINT BOUNDARY-VALUE PROBLEM OF DIRICHLET TYPE INVOLVING A P-LAPLACIAN TYPE OPERATOR

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ABSTRACT. Let ϕ, be an odd increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$ satisfying $\phi(0) = 0$, and let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying Caratheodory's conditions. Let $\alpha_i \in \mathbb{R}$, $\xi_i \in (0,1)$, $i = 1, \dots, m-2$, $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ be given. We are interested in the existence of solutions for the m-point boundary-value problem:

$$(\phi(u''))' = f(t, u, u', u''), \quad t \in (0,1),$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u''(0) = 0,$$

in the resonance and non-resonance cases. We say that this problem is at *resonance* if the associated problem

$$(\phi(u''))' = 0, \quad t \in (0,1),$$

with the above boundary conditions has a non-trivial solution. This is the case if and only if $\sum_{i=1}^{m-2} \alpha_i \xi_i = 1$. Our results use topological degree methods. In the non-resonance case; i.e., when $\sum_{i=1}^{m-2} \alpha_i \xi_i \neq 1$ we note that the sign of degree for the relevant operator depends on the sign of $\sum_{i=1}^{m-2} \alpha_i \xi_i - 1$.

1. Introduction

In this paper we consider the boundary-value problem

$$\begin{gathered}(\phi(u''))' = f(t, u, u', u''), \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u''(0) = 0,\end{gathered} \tag{1.1}$$

where ϕ is an odd increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ with $\phi(0) = 0$ and the function $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is Caratheodory. Also $\alpha_i \in \mathbb{R}$, $\xi_i \in (0,1)$, for $i = 1, 2, \dots m-2$, are such that $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$.

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We say that (1.1) is at *resonance*, if the associated multi-point boundary-value problem

$$\begin{gathered}(\phi(u''))' = 0, \quad t \in (0,1), \\ u(0) = 0, u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u''(0) = 0,\end{gathered} \tag{1.2}$$

has a non-trivial solution.

We are interested here in the existence of solutions for the m-point boundary-value problem (1.1) in the resonance and in the non-resonance cases.

The study of multipoint second-order boundary-value problems for $\phi(u) \equiv u$ was initiated by Il'in and Moiseev in [16, 17] and has been the subject of many papers, see for example [2, 3, 8, 9, 10, 11, 12, 13, 15, 18, 19, 20, 21, 23].

More recently multipoint second-order boundary-value problems containing the p-Laplace operator or the more general operator $-(\phi(u'))'$ complemented with linear boundary conditions, have been studied in [1, 4, 6, 22, 26, 27].

Problem (1.1) is at resonance if and only if $\sum_{i=1}^{m-2} \alpha_i \xi_i = 1$, having $u(t) = \rho t$ as a non-trivial solution, where $\rho \in \mathbb{R}$ is an arbitrary constant.

Our aim in this paper is to obtain existence of solutions for problem (1.1), by using topological degree arguments. Thus, in section 2, we first derive a deformation lemma that is needed when problem (1.1) is at resonance.

In section 3 an existence theorem for problem (1.1) is derived from this lemma. Finally in section 4 we consider problem (1.1) when it is non-resonant. The crucial point here is to prove that the Leray Schauder degree of a certain operator is different from zero which is shown to be an explicit consequence of the non-resonance condition, i.e., $\sum_{i=1}^{m-2} \alpha_i \xi_i \neq 1$. In addition we obtain the interesting property that the degree of the operator changes sign when $\sum_{i=1}^{m-2} \alpha_i \xi_i$ goes from being less than one to being greater than one.

We shall denote by $C[0,1]$ (resp. $C^1[0,1]$, $C^2[0,1]$) the classical space of continuous (resp. continuously differentiable, twice continuously differentiable) real-valued functions on the interval $[0,1]$. The norm in $C[0,1]$ is denoted by $|\cdot|_\infty$. Also, we shall denote by $L^1(0,1)$ the space of real-valued (equivalence classes of) functions whose absolute value is Lebesgue integrable on $(0,1)$. The Brouwer and Leray-Schauder degree shall be respectively denoted by $\deg_B$ and $\deg_{LS}$.

2. A deformation lemma for the resonance case

We begin this section by formulating a general deformation lemma for the solvability of the boundary-value problem (1.1) in the resonance case.

Let $f^* : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1] \mapsto \mathbb{R}$ be a function satisfying Caratheodory's conditions; i.e., (i) for all $(s,r,q,\lambda) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1]$ the function $f^*(\cdot,s,r,q,\lambda)$ is measurable on $[0,1]$, (ii) for a.e. $t \in [0,1]$ the function $f^*(t,\dots,\cdot)$ is continuous on $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1]$, and (iii) for each $R > 0$ there exists a Lebesgue integrable function $\rho_R : [0,1] \mapsto \mathbb{R}$ such that $|f^*(t,s,r,q,\lambda)| \le \rho_R(t)$ for a.e. $t \in [0,1]$ and all $(s,r,q,\lambda) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times [0,1]$ with $|s| \le R$, $|r| \le R$, and $|q| \le R$. We suppose that $f(t,s,r,q) = f^*(t,s,r,q,1)$ is the given function in problem (1.1).

We, now, introduce an operator $\mathfrak{B}(u,\lambda) : C^2[0,1]\times[0,1] \mapsto \mathbb{R}$ defined for $(u,\lambda) \in C^2[0,1]\times[0,1]$ by

$$\begin{aligned}\mathfrak{B}(u,\lambda) = \lambda\Big(u(1) - \sum_{i=1}^{m-2}\alpha_i u(\xi_i)\Big)\\ + (1-\lambda)\Big(\int_0^1\int_0^s f^*(\tau,u(\tau),u'(\tau),u''(\tau),\lambda)d\tau ds\\ - \sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_0^s f^*(\tau,u(\tau),u'(\tau),u''(\tau),\lambda)d\tau ds\Big).\end{aligned} \tag{2.1}$$

For $\lambda \in [0,1]$ we consider the family of boundary-value problems:

$$\begin{gathered}(\phi(u''))' = \lambda f^*(t,u,u',u'',\lambda), \quad t\in(0,1),\\ u(0)=0, \quad u''(0)=0, \quad \mathfrak{B}(u,\lambda)=0.\end{gathered} \tag{2.2}$$

Let $\Omega \subset C^2[0,1]$ be a bounded open set. Let us set for $\rho\in\mathbb{R}$, $i_\rho(t) = \rho t$, for $t\in[0,1]$, and

$$X = \{i_\rho : \rho\in\mathbb{R}\},$$

then X is a one dimensional subspace of $C^2[0,1]$. Defining $i:\mathbb{R}\mapsto X$ by $i(\rho) = i_\rho$ it is clear that i is an isomorphism from $\mathbb{R}$ onto X.

Next let us define $F : X\mapsto\mathbb{R}$ by

$$F(i_\rho) = \int_0^1\int_0^s f^*(\tau,\rho\tau,\rho,0,0)d\tau ds - \sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_0^s f^*(\tau,\rho\tau,\rho,0,0)d\tau ds,$$

and set $\mathcal{F} = F\circ i$, then $\mathcal{F}:\mathbb{R}\mapsto\mathbb{R}$ is continuous, and is given by

$$\mathcal{F}(\rho) = \int_0^1\int_0^s f^*(\tau,\rho\tau,\rho,0,0)d\tau ds - \sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_0^s f^*(\tau,\rho\tau,\rho,0,0)d\tau ds.$$

We have the following lemma.

Lemma 2.1. *Assume that*

(i) *for $\lambda\in(0,1)$ the boundary-value problem (2.2) has no solution $u\in\partial\Omega$,*
(ii) *the equation $\mathcal{F}(\rho)=0$ has no solution for any ρ with $i_\rho(t)\in\partial\Omega\cap X$, and*
(iii) *the Brouwer degree $\deg_B(F,\Omega\cap X,0)\neq 0$.*

Then the boundary-value problem (1.1) *has at least one solution in $\overline{\Omega}$.*

Proof. If the boundary-value problem (1.1) has a solution in $\partial\Omega$, then there is nothing to prove. Accordingly, let us assume that the boundary-value problem (1.1) has no solution in $\partial\Omega$. This assumption combined with assumption (i) implies that the boundary-value problem (2.2) has no solution $u\in\partial\Omega$ for $\lambda\in(0,1]$.

Let us define an operator $\Psi^* : C^2[0,1]\times[0,1]\mapsto C^2[0,1]$ by setting for $(u,\lambda)\in C^2[0,1]\times[0,1]$

$$\begin{aligned}\Psi^*(u,\lambda)(t) = \int_0^t\Big(u'(0) + \int_0^s\phi^{-1}\Big(\lambda\int_0^r f^*(\tau,u(\tau),u'(\tau),u''(\tau),\lambda)d\tau\Big)dr\Big)ds\\ + t\mathfrak{B}(u,\lambda),\end{aligned} \tag{2.3}$$

where $\mathfrak{B}(u,\lambda)$ is as defined in equation (2.1).

We note from our assumptions that the function f^* satisfies Caratheodory's conditions so that for $(u,\lambda) \in C^2[0,1] \times [0,1]$, $f^*(t,u(t),u'(t),u''(t),\lambda) \in L^1(0,1)$. Accordingly, the function $s \in [0,1] \mapsto \int_0^s f^*(\tau,u(\tau),u'(\tau),u''(\tau),\lambda)d\tau$ is absolutely continuous on $[0,1]$. Since, now, the integrand in (2.3) is continuous on $[0,1]$ we see that the operator Ψ^* is well defined.

Next, let us suppose that $u(t)$ be a solution to the boundary-value problem (2.2) for some $\lambda \in [0,1]$. We, then, see by integrating the equation in (2.2) and using the boundary conditions in (2.2) that $u(t)$ satisfies the equation

$$u(t) = \Psi^*(u,\lambda)(t), t \in [0,1],$$

along with

$$u(0) = 0, u''(0) = 0, \mathfrak{B}(u,\lambda) = 0.$$

Conversely, let us suppose that for some $\lambda \in [0,1]$, $u(t)$, $t \in [0,1]$, satisfies the equation

$$u(t) = \Psi^*(u,\lambda)(t). \tag{2.4}$$

We first see from the equation (2.4) and the definition of $\Psi^*(u,\lambda)$ that

$$u(0) = 0.$$

Next, we obtain, by differentiating the equation (2.4) that

$$u'(t) = u'(0) + \int_0^t \phi^{-1}\Big(\lambda \int_0^r f^*(\tau,u(\tau),u'(\tau),u''(\tau),\lambda)d\tau\Big)dr + \mathfrak{B}(u,\lambda), t \in [0,1]. \tag{2.5}$$

Evaluating (2.5) at $t = 0$ we see that

$$\mathfrak{B}(u,\lambda) = 0.$$

Again, we obtain, by differentiating (2.5) that

$$u''(t) = \phi^{-1}\Big(\lambda \int_0^t f^*(\tau,u(\tau),u'(\tau),u''(\tau),\lambda)d\tau\Big). \tag{2.6}$$

Evaluating the equation (2.6) at $t = 0$ we see that

$$u''(0) = 0.$$

Also, equation (2.6) further implies that $\phi(u''(t))$ is absolutely continuous on $[0,1]$ and

$$(\phi(u''(t)))' = \lambda f^*(t,u(t),u'(t),u''(t),\lambda), t \in [0,1].$$

Thus $u(t)$, $t \in (0,1)$, is a solution to the boundary-value problem (2.2). We have, accordingly, proved that $u(t)$, $t \in (0,1)$, is a solution to the boundary-value problem (2.2) if and only if $u(t)$, $t \in [0,1]$, is a solution to the equation (2.4).

We observe that it is easy to show, using standard arguments, that $\Psi^* : C^2[0,1] \times [0,1] \mapsto C^2[0,1]$ is a completely continuous operator. If, now, $u(t) \in \partial\Omega$ is a solution to the boundary-value problem (1.1) then we are done. Accordingly, let us assume that the boundary-value problem (1.1) has no solution on $\partial\Omega$. Since, now, $f^*(t,s,r,q,1) = f(t,s,r,q)$ for all $(t,s,r,q) \in [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ we see that the assumption (i) of the lemma implies that

$$u \neq \Psi^*(u,\lambda) \quad \text{for all } u \in \partial\Omega \text{ and } \lambda \in (0,1].$$

We, next, assert that $u \neq \Psi^*(u,0)$ for all $u \in \partial\Omega$. Indeed, let $u \in \partial\Omega$ be such that $u = \Psi^*(u,0)$. It then follows from the definition of Ψ^*, as given in (2.3), that $u(t) = \rho t = i_\rho(t)$, with $\rho = u'(0) + \mathfrak{B}(u,0)$, $u'(t) = \rho + \mathfrak{B}(u,0)$, $u''(0) = 0$, $\mathfrak{B}(u,0) = 0$, $u \in \partial\Omega \cap X$, and

$$\begin{aligned}
\mathfrak{B}(u,0) &= \int_0^1 \int_0^s f^*(\tau, u(\tau), u'(\tau), u''(\tau), 0) d\tau ds \\
&\quad - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s f^*(\tau, u(\tau), u'(\tau), u''(\tau), 0) d\tau ds \\
&= \int_0^1 \int_0^s f^*(\tau, \rho\tau, \rho, 0, 0) d\tau ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} \int_0^s f^*(\tau, \rho\tau, \rho, 0, 0) d\tau ds \\
&= \mathcal{F}(\rho) = 0.
\end{aligned}$$

But this contradicts the assumption (ii) of the lemma. We thus get that

$$u \neq \Psi^*(u,\lambda) \quad \text{for all } u \in \partial\Omega \text{ and } \lambda \in [0,1].$$

Thus $\deg_{LS}(I - \Psi^*(\cdot,\lambda), \Omega, 0)$ is well defined for all $\lambda \in [0,1]$. By the homotopy invariance property of Leray-Schauder degree we obtain immediately that

$$\deg_{LS}(I - \Psi^*(\cdot,1), \Omega, 0) = \deg_{LS}(I - \Psi^*(\cdot,0), \Omega, 0) = \deg_B(I - \Psi^*(\cdot,0)|_X, \Omega_0, 0), \tag{2.7}$$

where, $\Omega_0 = \Omega \cap X$. Now since for $v \in X$

$$\big(I - \Psi^*(\cdot,0)\big)v = -i_{F(v)},$$

we have

$$\deg_{LS}(I - \Psi^*(\cdot,1), \Omega, 0) = \deg_B(-i_{F(\cdot)}, \Omega_0, 0) = -\deg_B(i_{F(\cdot)}, \Omega_0, 0).$$

Since, $i^{-1} \circ i_{F(\cdot)} \circ i = \mathcal{F}$, we obtain by using a standard formula in degree theory that

$$\deg_B(i_{F(\cdot)}, \Omega_0, 0)) = \deg_B(\mathcal{F}, i^{-1}(\Omega_0), 0)).$$

Hence, by assumption (iii) of the lemma, it follows that $\deg_{LS}(I - \Psi^*(\cdot,1), \Omega, 0) \neq 0$. Thus, the mapping $\Psi \equiv \Psi^*(\cdot,1) : C^2[0,1] \mapsto C^2[0,1]$ has at least one fixed-point in $\overline{\Omega}$ and hence the boundary value problem (1.1) has at least one solution in $\overline{\Omega}$. This completes the proof of the lemma. □

3. Existence Theorems

We shall assume that for any constants $\Lambda \geq 0$, $A > 0$ with $\Lambda < A$ it holds that

$$\tilde{\alpha}(A,\Lambda) \equiv \limsup_{z\to\infty} \frac{\phi(\frac{A+\Lambda}{A-\Lambda} z + c)}{\phi(z)} < \infty. \tag{3.1}$$

We need the following lemma in the proof of our existence theorems.

Lemma 3.1. *Let $g : [0,1] \mapsto \mathbb{R}$ be a strictly increasing (resp. strictly decreasing) function on $[0,1]$. Then the function $G : (0,1] \hookrightarrow \mathbb{R}$ defined for $t \in (0,1]$ by*

$$G(t) = \frac{1}{t} \int_0^t g(s) ds$$

is strictly increasing (resp. decreasing) function on $(0,1]$. In particular, $\int_0^1 g(s)ds - \frac{1}{t}\int_0^t g(s)ds > 0$ (resp. < 0) for every $t \in (0,1)$. Moreover, given $\alpha_i \geq 0$, $\xi_i \in (0,1)$, $i = 1, 2, \cdots, m-2$ with $\sum_{i=1}^{m-2} \alpha_i \xi_i = 1$ we have $\int_0^1 g(s)ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} g(s)ds > 0$ (resp. < 0).

Proof. Let us suppose that g is a strictly increasing function on $[0,1]$. Now we see that

$$G'(t) = \frac{g(t)}{t} - \frac{1}{t^2}\int_0^t g(s)ds = \frac{1}{t^2}(\int_0^t (g(t)-g(s))ds > 0,$$

for every $t \in (0,1]$. Accordingly, G is strictly increasing on $(0,1]$ and $\int_0^1 g(s)ds - \frac{1}{t}\int_0^t g(s)ds > 0$ for every $t \in (0,1)$. Finally, we see that

$$\begin{aligned} &\int_0^1 g(s)ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} g(s)ds \\ &= \sum_{i=1}^{m-2} \alpha_i \xi_i (\int_0^1 g(s)ds - \frac{1}{\xi_i}\int_0^{\xi_i} g(s)ds) > 0. \end{aligned}$$

Similarly G is strictly decreasing on $(0,1]$ and $\int_0^1 g(s)ds - \sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} g(s)ds < 0$ when g is a strictly decreasing function on $[0,1]$. This completes the proof of the lemma. □

Theorem 3.2. *Let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ in the boundary-value problem* (1.1) *be a continuous function and satisfies the following conditions:*

(i) *there exist non-negative functions $d_1(t)$, $d_2(t)$, $d_3(t)$, and $r(t)$ in $L^1(0,1)$ such that*

$$|f(t,u,v,w)| \le d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + d_3(t)\phi(|w|) + r(t),$$

for all $t \in [0,1]$, u, v, $w \in \mathbb{R}$,

(ii) *there exist constants $\Lambda \ge 0$, $B \ge 0$, $A > 0$ with $\Lambda < A$ and a $v_0 > 0$ such that for all v with $|v| > v_0$, all $t \in [0,1]$ and all u, $w \in \mathbb{R}$ one has*

$$|f(t,u,v,w)| \ge -\Lambda|u| + A|v| - \Lambda|w| - B,$$

(iii) *there exists an $R > 0$ such that for all ρ, with $|\rho| > R$, either*

$$\begin{gathered} \rho f(t,\rho t,\rho,0) > 0, \text{ for all } t \in [0,1], \quad \text{or} \\ \rho f(t,\rho t,\rho,0) < 0, \text{ for all } t \in [0,1]. \end{gathered}$$

Suppose, further, that

$$\tilde{\alpha}(A,\Lambda)(\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}) + \|d_3\|_{L^1(0,1)} < 1. \tag{3.2}$$

Then, given $\alpha_i \ge 0$, $\xi_i \in (0,1)$, $i = 1,2,\cdots,m-2$ with $\sum_{i=1}^{m-2}\alpha_i\xi_i = 1$ the boundary value problem (1.1) *has at least one solution in $u(t) \in C^2[0,1]$.*

Proof. We first choose an $\varepsilon > 0$ be such that

$$(\tilde{\alpha}(A,\Lambda) + \varepsilon)(\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}) + \|d_3\|_{L^1(0,1)} < 1,$$

which is possible to do, in view of (3.2). We consider the family of boundary-value problems:

$$\begin{gathered} (\phi(u''(t)))' = \lambda f(t,u(t),u'(t),u''(t)), \quad t \in (0,1), \lambda \in [0,1], \\ u(0) = 0, \quad \mathfrak{B}(u,\lambda) = 0, \quad u''(0) = 0, \end{gathered} \tag{3.3}$$

where $\mathfrak{B}(u,\lambda)$ is as defined in (2.1). Let $u(t)$ be a solution to the boundary-value problem (3.3) for some $\lambda \in (0,1)$. Then either there exists a $t_0 \in [0,1]$ such that

$$|u'(t_0)| \le v_0 \tag{3.4}$$

or $|u'(t)| > v_0$ for all $t \in [0,1]$. In case, $|u'(t)| > v_0$ for all $t \in [0,1]$, we claim that there exists a $\tau_0 \in [0,1]$ such that $f(\tau_0, u(\tau_0), u'(\tau_0), u''(\tau_0)) = 0$. Indeed, let us suppose that $f(t, u(t), u'(t), u''(t)) \neq 0$ for all $t \in [0,1]$. It then follows from the continuity of $f(t, u(t), u'(t), u''(t))$ on the interval $[0,1]$ either $f(t, u(t), u'(t), u''(t)) > 0$ for all $t \in [0,1]$ or $f(t, u(t), u'(t), u''(t)) < 0$ for all $t \in [0,1]$. Let us first suppose that $f(t, u(t), u'(t), u''(t)) > 0$ for all $t \in [0,1]$. It then follows from the boundary condition in (2.4) that

$$\begin{aligned}
&\lambda\Big[\int_0^1 \Big(u'(0) + \int_0^s \phi^{-1}\Big(\lambda\int_0^r f(\tau, u(\tau), u'(\tau), u''(\tau))d\tau\Big)dr\Big)ds \\
&- \sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} \Big(u'(0) + \int_0^s \phi^{-1}\Big(\lambda\int_0^r f(\tau, u(\tau), u'(\tau), u''(\tau))d\tau\Big)dr\Big)ds\Big] \\
&+ (1-\lambda)\Big[\int_0^1\int_0^r f(\tau, u(\tau), u'(\tau), u''(\tau))d\tau ds \\
&- \sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i}\int_0^r f(\tau, u(\tau), u'(\tau), u''(\tau))d\tau dr\Big] \\
&= 0.
\end{aligned} \tag{3.5}$$

We, next, see that the functions

$$\begin{gathered}
\int_0^t \Big(u'(0) + \int_0^s \phi^{-1}\Big(\lambda\int_0^r f(\tau, u(\tau), u'(\tau), u''(\tau))d\tau\Big)dr\Big)ds, \\
\int_0^s\int_0^r f(\tau, u(\tau), u'(\tau), u''(\tau))d\tau dr
\end{gathered}$$

are strictly increasing functions on $(0,1]$, in view of our assumption

$$f(t, u(t), u'(t), u''(t)) > 0$$

for all $t \in [0,1]$. We then get from Lemma 3.1 and (3.5) that $0 > 0$, a contradiction. Similarly, the supposition $f(t, u(t), u'(t), u''(t)) < 0$ for all $t \in [0,1]$ leads to the contradiction $0 < 0$. Hence, there must exist a $\tau_0 \in [0,1]$ such that

$$f(\tau_0, u(\tau_0), u'(\tau_0), u''(\tau_0)) = 0, \tag{3.6}$$

proving the claim. We next see from (3.6) and assumption (ii) that

$$|u'(\tau_0)| \le \frac{B}{A} + \frac{\Lambda}{A}\|u\|_\infty + \frac{\Lambda}{A}\|u''\|_\infty. \tag{3.7}$$

Thus we see from (3.4) and (3.7) that there exists a $\tau_1 \in [0,1]$ (either t_0 or τ_0) such that

$$|u'(\tau_1)| \le v_0 + \frac{B}{A} + \frac{\Lambda}{A}\|u\|_\infty + \frac{\Lambda}{A}\|u''\|_\infty. \tag{3.8}$$

It then follows from the equation $u'(t) = u'(\tau_1) + \int_{\tau_1}^t u''(s)ds$ and (3.8) that

$$\|u'\|_\infty \le \frac{A+\Lambda}{A-\Lambda}\|u''\|_\infty + \frac{Av_0 + B}{A-\Lambda}. \tag{3.9}$$

Next, we see by integrating the equation in (3.3) from 0 to $t \in [0,1]$ and noting $u''(0) = 0$, that

$$\phi(u''(t)) = \lambda\int_0^t f(\tau, u(\tau), u'(\tau), u''(\tau))d\tau. \tag{3.10}$$

It now follows from equations (3.10), (3.8) using assumption (i), the fact that $u(0) = 0$ implies $\|u\|_\infty \le \|u'\|_\infty$ that

$$\begin{aligned}
&\phi(|u''(t)|) \\
&\le \phi(\|u\|_\infty)\|d_1\|_{L^1(0,1)} + \phi(\|u'\|_\infty)\|d_2\|_{L^1(0,1)} + \phi(\|u''\|_\infty)\|d_3\|_{L^1(0,1)} + \|r\|_{L^1(0,1)} \\
&\le (\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)})\phi(\frac{A+\Lambda}{A-\Lambda}\|u''\|_\infty + \frac{Av_0+B}{A-\Lambda}) \\
&\quad + \|d_3\|_{L^1(0,1)})\phi(\|u''\|_\infty) + \|r\|_{L^1(0,1)} \\
&\le ((\tilde\alpha(A,\Lambda)+\varepsilon)(\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}) + \|d_3\|_{L^1(0,1)})\phi(\|u''\|_\infty) \\
&\quad + C_\varepsilon(\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}) + \|r\|_{L^1(0,1)},
\end{aligned}$$

and hence

$$\begin{aligned}
\phi(\|u''\|_\infty) \le ((\tilde\alpha(A,\Lambda)+\varepsilon)(\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}) + \|d_3\|_{L^1(0,1)})\phi(\|u''\|_\infty) \\
+ C_\varepsilon(\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}) + \|r\|_{L^1(0,1)}.
\end{aligned} \tag{3.11}$$

It now follows from (3.2), the estimates (3.11), (3.9) and $\|u\|_\infty \le \|u'\|_\infty$ that there exists an $R_0 > R$, where R is as in assumption (iii), such that the family of boundary value problems (3.3) have no solution on the boundary of a bounded open set $\Omega = B(0,\widetilde R) \subset C^2[0,1]$, for every $\widetilde R \ge R_0$. Accordingly, we see that the family of boundary value problems (3.3) satisfy condition (i) of Lemma 2.1. Next, we see from assumption (iii) and Lemma 3.1 for all ρ, $|\rho| > R$, that

$$\int_0^1\int_0^s f(\tau,\rho\tau,\rho,0)d\tau ds - \sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_0^s f(\tau,\rho\tau,\rho,0)d\tau ds$$

is strictly positive or strictly negative. Accordingly, we see that $f^*(t,u,v,w,\lambda) = f(t,u,v,w)$ satisfies the condition (ii) of Lemma 2.1.

Finally, we again see from assumption (iii), the continuity in $\rho \in \mathbb{R}$ of the function

$$\psi(\rho) = \int_0^1\int_0^s f(\tau,\rho\tau,\rho,0)d\tau ds - \sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}\int_0^s f(\tau,\rho\tau,\rho,0)d\tau ds$$

and the assumption that $\widetilde R > R$, that $F(i_{\widetilde R}(t))$ and $F(i_{-\widetilde R}(t))$ have opposite signs. It follows immediately that $F(i_\rho(t)) = 0$ for an odd number of $\rho \in (-\widetilde R, \widetilde R)$ which implies that the Brouwer degree $\deg_B(F, \Omega\cap X, 0) \ne 0$. Thus the condition (iii) of Lemma 2.1 is also satisfied. Thus it follows from Lemma 2.1 that the boundary value problem (1.1) has at least one solution in $\overline\Omega$. This completes the proof of the theorem. □

4. A result for the non-resonance case

In this section we will consider problem (1.1) in the non-resonance case. Problem (1.1) is in the non-resonance case if problem (1.2) has only the trivial solution. This holds if and only if the α_i, ξ_i satisfy $\sum_{i=1}^{m-2}\alpha_i\xi_i \ne 1$. We assume henceforth that α_i, ξ_i satisfy this condition. Notice that we do not assume a sign condition on the $\alpha_i's$. In addition, we shall assume that for any σ, $0<\sigma<1$, it holds that

$$\tilde\alpha(\sigma) = \limsup_{z\to\infty}\frac{\phi(\frac{1}{1-\sigma}z)}{\phi(z)} < \infty. \tag{4.1}$$

Let us set $\xi_{m-1} = 1$, $\alpha_{m-1} = -1$, $\sigma_{ij} = \alpha_i(\xi_i - \xi_j)$ for $i \neq j$ and $\sigma_{jj} = \sum_{i=1}^{m-1} \alpha_i \xi_j$ for $i, j = 1, 2, \cdot, \cdot, \cdot, m-1$. We note that the assumption $\sum_{i=1}^{m-2} \alpha_i \xi_i \neq 1$ is equivalent to $\sum_{i=1}^{m-1} \alpha_i \xi_i \neq 0$. Also, for each $j = 1, 2, \cdot, \cdot, \cdot, m-1$ we have

$$\sum_{i=1}^{m-1} \sigma_{ij} = \sum_{i=1, i\neq j}^{m-1} \sigma_{ij} + \sigma_{jj} = \sum_{i=1, i\neq j}^{m-1} \alpha_i(\xi_i - \xi_j) + \sum_{i=1}^{m-1} \alpha_i \xi_j = \sum_{i=1}^{m-1} \alpha_i \xi_i \neq 0.$$

It follows that

$$\sum_{i=1}^{m-1} (\sigma_{ij})^+ \neq \sum_{i=1}^{m-1} (\sigma_{ij})^-,$$

for $j = 1, 2, \cdot, \cdot, \cdot, m-1$, where for $\alpha \in \mathbb{R}$, $\alpha^+ = \max(\alpha, 0)$ and $\alpha^- = \max(-\alpha, 0)$. Let us set

$$\sigma^* = \begin{cases} \min\{\frac{\sum_{i=1}^{m-1}(\sigma_{ij})^+}{\sum_{i=1}^{m-1}(\sigma_{ij})^-}, \frac{\sum_{i=1}^{m-1}(\sigma_{ij})^-}{\sum_{i=1}^{m-1}(\sigma_{ij})^+}\} & \text{if } \sum_{i=1}^{m-1}(\sigma_{ij})^+ \neq 0 \text{ and} \\ & \quad \sum_{i=1}^{m-1}(\sigma_{ij})^- \neq 0 \text{ for all } j, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Note that $0 \leq \sigma^* < 1$. The main result of this section is the following theorem.

Theorem 4.1. *Let $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ be a function satisfying Caratheodory's conditions such that the following condition holds:*
there exist non-negative functions $d_1(t)$, $d_2(t)$, $d_3(t)$, and $r(t)$ in $L^1(0,1)$ such that

$$|f(t, u, v, w)| \leq d_1(t)\phi(|u|) + d_2(t)\phi(|v|) + d_3(t)\phi(|w|) + r(t),$$

for a. e. $t \in [0,1]$ and all $u, v, w \in \mathbb{R}$. Suppose, further,

$$\tilde{\alpha}(\sigma^*)(\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}) + \|d_3\|_{L^1(0,1)} < 1, \tag{4.3}$$

where σ^* is as defined in (4.2) and $\tilde{\alpha}$ is as defined in (4.1).
Then, the boundary-value problem (1.1) *has at least one solution $u \in C^2[0,1]$.*

We need the following variant of an a priori estimate from [14] in the proof of Theorem 4.1 and present this in the following lemma.

Lemma 4.2. *Let $u \in C^1[0,1]$, be such that $u'' \in L^\infty(0,1)$ and satisfies*

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),$$

with $\sum \alpha_i \xi_i \neq 1$. If $\sum_{i=1}^{m-1}(\sigma_{ij})^+ \neq 0$, and $\sum_{i=1}^{m-1}(\sigma_{ij})^- \neq 0$ for all j, then

$$\|u'\|_\infty \leq \frac{1}{1-\sigma^*}\|u''\|_\infty. \tag{4.4}$$

If one of $\sum_{i=1}^{m-1}(\sigma_{ij})^+$, $\sum_{i=1}^{m-1}(\sigma_{ij})^-$ is zero for some $j = 1, 2, \dots, m-1$, then $u'(\eta_0) = 0$ for some $\eta_0 \in [0,1]$, and

$$\|u'\|_\infty \leq \|u''\|_\infty. \tag{4.5}$$

Proof. We first, note, that the assumption

$$u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i)$$

is equivalent to

$$\sum_{i=1}^{m-1} \alpha_i u(\xi_i) = 0,$$

with $\xi_{m-1} = 1$, $\alpha_{m-1} = -1$ and the non-resonant condition $\sum_{i=1}^{m-2} \alpha_i \xi_i \neq 1$ is equivalent to $\sum_{i=1}^{m-1} \alpha_i \xi_i \neq 0$.

Next, for each $j = 1, 2, \cdot, \cdot, \cdot, m-1$ we have $u(\xi_j) = \xi_j u'(\eta_{jj})$ for some $\eta_{jj} \in [0,1]$. Also for $i, j = 1, 2, \cdot, \cdot, \cdot, m-1$ with $i \neq j$ we have $u(\xi_i) - u(\xi_j) = u'(\eta_{ij})(\xi_i - \xi_j)$ for some $\eta_{ij} \in [0,1]$. Accordingly,

$$\begin{aligned}
\sum_{i=1, i\neq j}^{m-1} \alpha_i u'(\eta_{ij})(\xi_i - \xi_j) &= \sum_{i=1, i\neq j}^{m-1} \alpha_i (u(\xi_i) - u(\xi_j)) \\
&= -\sum_{i=1}^{m-1} \alpha_i u(\xi_j) = -\sum_{i=1}^{m-1} \alpha_i \xi_j u'(\eta_{jj}),
\end{aligned}$$

using the mean-value theorem and the assumptions $u(0) = 0$, $\sum_{i=1}^{m-1} \alpha_i u(\xi_i) = 0$ (equivalently, $u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i)$). We thus get $\sum_{i=1}^{m-1} \sigma_{ij} u'(\eta_{ij}) = 0$, and hence $\sum_{i=1}^{m-1} (\sigma_{ij})^+ u'(\eta_{ij}) = \sum_{i=1}^{m-1} (\sigma_{ij})^- u'(\eta_{ij})$. So there must exist χ_j^1 and χ_j^2 in $[0,1]$ such that

$$\Big(\sum_{i=1}^{m-1} (\sigma_{ij})^+\Big) u'(\chi_j^1) = \Big(\sum_{i=1}^{m-1} (\sigma_{ij})^-\Big) u'(\chi_j^1). \tag{4.6}$$

If one of $\sum_{i=1}^{m-1} (\sigma_{ij})^+$, $\sum_{i=1}^{m-1} (\sigma_{ij})^-$ is zero for some $j = 1, 2, \dots, m-1$ then it follows from (4.6) that there is an $\eta_0 \in [0,1]$ (indeed one of χ_j^1 or χ_j^2) such that $u'(\eta_0) = 0$ and the estimate (4.5) is immediate.

Next, suppose that $\sum_{i=1}^{m-1} (\sigma_{ij})^+ \neq 0$ and $\sum_{i=1}^{m-1} (\sigma_{ij})^- \neq 0$ for every $j = 1, 2, \dots, m-1$. Then either $u'(\chi_j^1) = u'(\chi_j^1) = 0$ for some $j = 1, 2, \dots, m-1$, in which case the estimate (4.5) is immediate, or $u'(\chi_j^1) \neq u'(\chi_j^1)$ for every $j = 1, 2, \dots, m-1$. It follows that there exist $\eta_1, \eta_2 \in [0,1]$ with $u'(\eta_1) \neq u'(\eta_2)$ such that

$$u'(\eta_1) = \sigma^* u'(\eta_2). \tag{4.7}$$

The estimate (4.4) is now immediate from (4.1), (4.7) and the equation

$$u'(t) = u'(\eta_1) + \int_{\eta_1}^{t} u'' ds.$$

This completes the proof of the lemma. □

Proof of Theorem 4.1. We consider the family of boundary-value problems:

$$\begin{gathered}
(\phi(u''(t)))' = \lambda f(t, u(t), u'(t), u''(t)), \quad t \in (0,1), \lambda \in [0,1], \\
u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad u''(0) = 0.
\end{gathered} \tag{4.8}$$

Also, we define an operator $\Psi^*: C^2[0,1]\times[0,1]\mapsto C^2[0,1]$ by setting for $(u,\lambda)\in C^2[0,1]\times[0,1]$

$$\begin{aligned}\Psi^*(u,\lambda)=&\int_0^t\Big(u'(0)+\int_0^s\phi^{-1}\Big(\lambda\int_0^r f^*(\tau,u(\tau),u'(\tau),u''(\tau),\lambda)d\tau\Big)dr\Big)ds\\&+t\Big(u(1)-\sum_{i=1}^{m-2}\alpha_i u(\xi_i)\Big).\end{aligned}$$

Following standard arguments, it can be proved that Ψ^* is a completely continuous operator. Furthermore reasoning in an entirely similar way as we did in the proof of Lemma 2.1 it can be proved that u is a solution to the family of boundary-value problems (4.8) if and only if u is a fixed point for the operator $\Psi^*(\cdot,\lambda)$; i.e., u satisfies

$$u=\Psi^*(u,\lambda).$$

We will show next that there is a constant $R>0$ independent of $\lambda\in[0,1]$ such that if u satisfies (4.8) for some $\lambda\in[0,1]$ then $\|u\|_{C^2[0,1]}<R$.

We note first that if u satisfies

$$u=\Psi^*(u,0),$$

then we must have $u=0$. Indeed from the definition of Ψ^* or from problem (4.8), it follows that $u(t)=\rho t$ with $\rho=u'(0)=u'(t)$, for all $t\in[0,1]$. Then from the second boundary condition in (4.8), and the assumption $\sum_{i=1}^{m-2}\alpha_i\xi_i\neq 1$, we find that $\rho=0$, implying that $u(t)=0$ for all $t\in[0,1]$.

In the rest of the argument we will assume that $\lambda\in(0,1]$. Also we will suppose that $\sigma^*>0$ since the proof for the case $\sigma^*=0$ is simpler.

Let us choose $\varepsilon>0$ such that

$$(\tilde\alpha(\sigma^*)+\varepsilon)(\|d_1\|_{L^1(0,1)}+\|d_2\|_{L^1(0,1)})+\|d_3\|_{L^1(0,1)}<1, \tag{4.9}$$

which can be done in view of the assumption (4.3). Next, we have from the definition of $\tilde\alpha$, as given in (4.1), that there exists a constant C^1_ε such that

$$\phi(\frac{1}{1-\sigma^*}z)\le(\tilde\alpha(\sigma^*)+\varepsilon)\phi(z)+C^1_\varepsilon,\text{ for all } z. \tag{4.10}$$

Let, now, u be a solution of the family of boundary-value problems (4.8). Then $u\in C^2[0,1]$ with $\phi(u''(t))$ absolutely continuous on $[0,1]$ and satisfies

$$u(0)=0,u(1)=\sum_{i=1}^{m-2}\alpha_i u(\xi_i),u''(0)=0.$$

We, now, use the estimates

$$\|u\|_\infty\le\|u'\|_\infty,\|u'\|_\infty\le\frac{1}{1-\sigma^*}\|u''\|_\infty,\phi(\|u''\|_\infty)\le\|(\phi(u''))'\|_{L^1(0,1)} \tag{4.11}$$

and the inequality (4.10) to get

$$\begin{aligned}
&\|(\phi(u''))'\|_{L^1(0,1)} \\
&\le \phi(\|u\|_\infty)\|d_1\|_{L^1(0,1)} + \phi(\|u'\|_\infty)\|d_2\|_{L^1(0,1)} \\
&\quad + \phi(\|u''\|_\infty)\|d_3\|_{L^1(0,1)} + \|r\|_{L^1(0,1)} \\
&\le (\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)})\phi(\frac{1}{1-\sigma^*}\|u''\|_\infty) \\
&\quad + \phi(\|u''\|_\infty)\|d_3\|_{L^1(0,1)} + \|r\|_{L^1(0,1)} \\
&\le \Big(\tilde{\alpha}(\sigma^*) + \varepsilon\Big)(\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)})\phi(\|u''\|_\infty) + \|d_3\|_{L^1(0,1)}\phi(\|u''\|_\infty) + C_\varepsilon \\
&\le [(\tilde{\alpha}(\sigma^*) + \varepsilon\Big)(\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}) + \|d_3\|_{L^1(0,1)}]\|(\phi(u''))'\|_{L^1(0,1)} + C_\varepsilon,
\end{aligned}$$

where

$$C_\varepsilon = \|r\|_{L^1(0,1)} + C_\varepsilon^1(\|d_1\|_{L^1(0,1)} + \|d_2\|_{L^1(0,1)}).$$

It, now, follows from (4.9) that there exists a constant $R_0 > 0$, independent of $\lambda \in (0,1]$ such that if u is a solution of the family of boundary-value problems (4.8) then

$$\|(\phi(u''))'\|_{L^1(0,1)}) \le R_0.$$

This, combined with (4.11) gives that there exist a constant $R > 0$ such that

$$\|u\|_{C^2[0,1]} < R.$$

This in turn implies that $\deg_{LS}(I - \Psi^*(\cdot,\lambda), B(0,R), 0)$ is well defined for all $\lambda \in [0,1]$, where $B(0,R)$ is the ball with center 0 and radius R in $C^2[0,1]$.

In what follows we will use the notation of section 2, thus X will denote the one dimensional subspace of $C^2[0,1]$ given by $X = \{i_\rho : \rho \in \mathbb{R}\}$, $i_\rho(t) = \rho t$ and $i : \mathbb{R} \mapsto X$ is the isomorphism from $\mathbb{R}$ onto X given by $i(\rho) = i_\rho$. Let us define the function $G : \mathbb{R} \mapsto \mathbb{R}$ by

$$G(\rho) = \Big(\sum_{i=1}^{m-2} \alpha_i \xi_i - 1\Big)\rho, \tag{4.12}$$

for $w \in X$, $w(t) = \rho t$ for some $\rho \in \mathbb{R}$. Now, since

$$(I - \Psi^*(\cdot,0))(w) = i_{G(\rho)},$$

it is easy to see that

$$G = i^{-1} \circ (I - \Psi^*(\cdot,0))|_X \circ i,$$

and hence, by the homotopy invariance property of Leray-Schauder degree, it follows that

$$\begin{gathered}
\deg_{LS}(I - \Psi^*(\cdot,1), B(0,R), 0) = \deg_{LS}(I - \Psi^*(\cdot,0), B(0,R), 0) \\
\deg_B(I - \Psi^*(\cdot,0)|_X, X \cap B(0,R), 0) = \deg_B(G, (-R,R), 0).
\end{gathered}$$

Thus taking into account (4.12), we obtain the interesting formulas for the degree

$$\deg_{LS}(I - \Psi^*(\cdot,1), B(0,R), 0) = \begin{cases} 1 & \text{if } \sum_{i=1}^{m-2} \alpha_i \xi_i > 1 \\ -1 & \text{if } \sum_{i=1}^{m-2} \alpha_i \xi_i < 1. \end{cases}$$

Hence if $\sum_{i=1}^{m-2} \alpha_i \xi_i \ne 1$ we have that $\deg_{LS}(I - \Psi^*(\cdot,1), B(0,R), 0) \ne 0$ and there is a $u \in B(0,R)$ that satisfies

$$u = \Psi^*(\cdot,1),$$

equivalently u is a solution to the boundary-value problem (4.1). This completes the proof of the theorem. □

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