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STRUCTURE OF THE SOLUTION SET FOR TWO-POINT BOUNDARY-VALUE PROBLEMS

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ABSTRACT. We present some results on the structure of the set of solutions of a two-point problem for a class of quasilinear differential equations. These equations involve nonlinearities expressed by a combination of powers which are allowed to be singular at 0. Also we point out some open questions.

1. INTRODUCTION

Let $p \in]1,+\infty[$, and let $f :]0,+\infty[\to \mathbb{R}$ be a continuous function. We consider the quasilinear two-point problem

$$\begin{gathered} -(|u'|^{p-2}u')' = f(u) \quad \text{in }]0,1[, \\ u > 0 \quad \text{in }]0,1[, \\ u(0) = u(1) = 0. \end{gathered} \tag{1.1}$$

In the following, a solution to problem (1.1) will be understood in the weak sense. By definition, a function $u \in W_0^{1,p}(]0,1[)$ is a weak solution to (1.1) if

$$\int_0^1 (|u'|^{p-2}u'v' - f(u)v)dt = 0$$

for all $v \in W_0^{1,p}(]0,1[)$. By regularity results, a solution to (1.1) is at least of class C^1 in $[0,1]$.

It is well known that problem (1.1) has at most one solution when the condition

$$\text{the function } t \in]0,+\infty[\to f(t)t^{1-p} \text{ is strictly decreasing in }]0,+\infty[\tag{1.2}$$

holds (see for instance [12, 15]). One of the simplest function satisfying this condition is

$$f(t) = \lambda t^{s-1}, \quad t > 0,$$

where $s \in]0,p[$ and $\lambda > 0$. In this case, we know that a solution u exists [15], and it can be explicitly computed by quadratures. In particular, one has

$$u(t) = \begin{cases} G^{-1}(t), & \text{for } 0 \le t \le 1/2, \\ G^{-1}(1-t), & \text{for } 1/2 \le t \le 1, \end{cases}$$

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where

$$G(x) = c_1 \int_0^x (c_2^s - \tau^s)^{-1/p} d\tau, \quad x \in [0, c_2]$$

and c_1, c_2 are positive constants depending on s, p, λ.

Then, it is quite natural to ask ourselves what happens when the function $f(t) = \lambda t^{s-1}$ is perturbed by a term which makes condition (1.2) no longer satisfied. Among the cases studied in the literature, we point out the following two ones:

(1) $f(t) = \lambda t^{s-1} + t^{q-1}$, with $0 < s < p < q$;
(2) $f(t) = \lambda t^{s-1} - t^{r-1}$, with $0 < r < s < p$.

As we shall see in both cases (1) and (2), the existence and/or uniqueness may not hold for all $\lambda > 0$. Case (1) is within the framework of concave-convex positive nonlinearities, while case (2) is a typical convex-concave nonlinearities which is negative exactly in a bounded right neighborhood of zero. The behavior on varying of the parameter λ of the solution set of problem (1.1) associated to the nonlinearity defined in (1) is quite different, in fact the opposite, with respect to that associated to the nonlinearity defined in (2).

In Sections 2 and 3, we present some results concerning the solution set of problem (1.1) for f given by (1) and (2), respectively. More precisely, we describe how the solution set behaves when varying of λ. Some extensions to other classes of nonlinearities as well as to problems in higher dimension are presented in Section 4.

2. Behavior of the solution set for $f(t) = \lambda t^{s-1} + t^{q-1}$

Let $p \in]1, +\infty[$, $s \in]0, p[$, $q \in]p, +\infty[$ and $\lambda \in]0, +\infty[$. In this section we consider the quasilinear problem

$$\begin{gathered} -(|u'|^{p-2}u')' = \lambda u^{s-1} + u^{q-1} \quad \text{in }]0,1[, \\ u > 0 \quad \text{in }]0,1[, \\ u(0) = u(1) = 0. \end{gathered} \tag{2.1}$$

This problem is the one-dimensional quasilinear version of the Dirichlet problem

$$\begin{gathered} -\Delta u = \lambda u^{s-1} + u^{q-1} \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \end{gathered} \tag{2.2}$$

where Ω is a bounded open domain in $\mathbb{R}^n$, $s \in]1, 2[$ and $q \in]2, +\infty[$. Problem (2.2) was studied in [1], where the following result was established:

Theorem 2.1 ([1, Theorem 2.3])**.** *There exists $\Lambda > 0$ such that problem* (2.2) *admits:*

- *at least one solution, for $\lambda \in]0, \Lambda]$,*
- *at least two solutions, for $\lambda \in]0, \Lambda[$ and with $q \le \frac{2n}{n-2}$, if $n \ge 3$,*
- *no solution for $\lambda > \Lambda$.*

In the same paper, the authors proposed to study, on varying of λ, the exact structure of the solution set of problem (2.2) in the one-dimensional case. This question was addressed in [16] for the quasilinear case, and in [14] for the semilinear case. In both these papers, a complete description of the solution set was given. In the semilinear case, the result obtained in [14] gives also some additional qualitative properties of the solutions. Since we are only interested in studying the structure of the solution set, here it is sufficient to report the main result of [16]

Theorem 2.2 ([16, Theorem 1]). *Assume* $p \in]1,+\infty[$, $s \in [1,p[$ *and* $q \in]p,+\infty[$. *There exists* $\Lambda > 0$ *such that problem* (2.1) *admits:*

- *exactly two solutions for* $\lambda \in]0,\Lambda[$,
- *exactly one solution, for* $\lambda = \Lambda$,
- *no solution for* $\lambda \in]\Lambda,+\infty[$.

This theorem is proved by using the so called "shooting method" which allows to convert a two-point problem into an algebraic equation. In particular, if we consider problem (2.1), we can see that there exists a one to one correspondence between the set of solutions of (2.1) and the set of solutions of the equation (in the unknown $c \in \mathbb{R}_+$)

$$T(c) = \frac{1}{2}\Big(\frac{p-1}{p}\Big)^{1/p}\lambda^{\frac{1}{p}\cdot\frac{q-p}{q-s}} \tag{2.3}$$

where

$$T(c) := \int_0^c \Big(\frac{c^s}{s} + \frac{c^q}{q} - \frac{t^s}{s} - \frac{t^q}{q}\Big)^{-1/p} dt, \quad c > 0,$$

is the so called *time map* associated to the problem. In addiction, for each solution $c_0 > 0$ of equation (2.3), the corresponding solution $u : [0,1] \to [0,c_0]$ to (2.1) is implicitly defined by

$$\int_0^{u(x)} \Big(\frac{c_0^s}{s} + \frac{c_0^q}{q} - \frac{t^s}{s} - \frac{t^q}{q}\Big)^{-1/p} dt = \Big(\frac{p}{p-1}\Big)^{1/p}\lambda^{\frac{1}{p}\frac{q-p}{q-s}}x, \quad x \in [0,\frac{1}{2}],$$
$$u(x) = u(1-x), \quad x \in]1/2,1].$$

Thus, solving problem (2.1) is equivalent to solving equation (2.3) in $\mathbb{R}_+$. The number of solutions of (2.3) can be computed by studying the profile of T. In [16], the authors find that T has the following profile shown in Figure 1 (from which Theorem 2.2 easily follows). By using the same method, in [17] it is proved that conclusion of Theorem 2.2 holds also in the singular case $s \in]0,1[$.

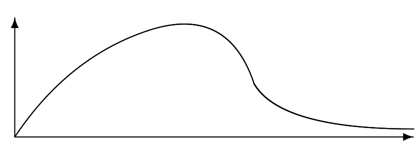

FIGURE 1. Profile of the time map T

We notice that, being

$$f(t) = \lambda t^{s-1} + t^{q-1} > 0, \quad \text{for } t > 0,$$

every positive solution u to (2.1) satisfies the Hopf boundary condition

$$u'(0) > 0, \quad u'(1) < 0,$$

that is u belongs to the interior $\mathcal{P}$ of the positive cone of $C^1([0,1])$, defined by

$$\mathcal{P} := \{u \in C^1([0,1]) : u > 0 \text{ in }]0,1[,\ u'(0) > 0,\ u'(1) < 0\}. \tag{2.4}$$

This property does not hold if we consider the nonlinearity $f(t) = \lambda t^{s-1} - t^{r-1}$ defined in (2). We deal with this case in the next section. We will see that, with this nonlinearity, problem (1.1) admits solutions belonging to the set

$$\mathcal{P}_0 := \{u \in C^1([0,1]) : u > 0 \text{ in }]0,1[,\ u'(0) = u'(1) = 0\} \tag{2.5}$$

for some value of the parameter λ.

3. Behavior of the solution set for $f(t) = \lambda t^{s-1} - t^{r-1}$

Let $p \in]1,+\infty[$, $s \in]0,p[$, $r \in]0,s[$ and $\lambda \in]0,+\infty[$. Let us consider the problem

$$\begin{gathered} -(|u'|^{p-2}u')' = \lambda u^{s-1} - u^{r-1}, \quad \text{in }]0,1[, \\ u > 0, \quad \text{in }]0,1[, \\ u(0) = u(1) = 0. \end{gathered} \tag{3.1}$$

We will see that an exact multiplicity result analogous to Theorem 2.2 holds for problem (3.1). A substantial difference, in this case, is that, contrarily to the conclusion of Theorem 2.2, a solution exists for λ large, and does not exist for λ small. More precisely, there exists $\Lambda > 0$ such that solutions exist for $\lambda \geq \Lambda$ and do not exist for $\lambda \in]0,+\Lambda[$. As quoted above, another difference to be point out is that, due to the particular structure of the nonlinearity f, which is negative and not Lipschitz continuous near 0, the Hopf boundary condition $u'(0) > 0$, $u'(1) < 0$ might not be true for a solution u to (3.1). Indeed, we will see that for a certain value of the parameter λ, a solution u satisfying $u'(0) = u'(1) = 0$ exists.

The time map T associated to problem (3.1) has the expression

$$T(c) = \int_0^c \Big(\frac{c^s}{s} - \frac{c^r}{r} - \frac{t^s}{s} + \frac{t^r}{r}\Big)^{-1/p} dt, \quad c \geq t(r) := \big(\frac{s}{r}\big)^{\frac{1}{s-r}}.$$

Here, $t(r)$ is the unique positive solution of the equation $\frac{t^s}{s} - \frac{t^r}{r} = 0$. As for problem (2.1), to each solution $c_0 \in [t(r),+\infty[$ of the equation

$$T(c) = \frac{1}{2}\Big(\frac{p-1}{p}\Big)^{1/p} \lambda^{\frac{1}{p}\cdot\frac{p-s}{s-r}}, \tag{3.2}$$

corresponds a unique solution $u : [0,1] \to [0,c_0]$, implicitly defined by

$$\begin{gathered} \int_0^{u(x)} \Big(\frac{c_0^s}{s} + \frac{c_0^q}{q} - \frac{t^s}{s} - \frac{t^q}{q}\Big)^{-1/p} dt = \big(\frac{p}{p-1}\big)^{1/p} \lambda^{\frac{1}{p}\frac{p-s}{s-r}} x, \quad x \in [0,\frac{1}{2}], \\ u(x) = u(1-x), \quad x \in]1/2,1]. \end{gathered}$$

For the nonsingular case $r > 1$, a complete description of the solution set of problem (3.1) was given in [10], where the following result was established.

Theorem 3.1 ([10, Theorem 1]). *Assume $p \in]1,\infty[$ $s \in]1,p[$ and $r \in]1,s[$. There exist two positive constants Λ_1, Λ_2, with $\Lambda_1 < \Lambda_2$, such that problem* (3.1) *admits:*

- *no solution if $\lambda \in]0,\Lambda_1[$;*
- *a unique solution u_λ if $\lambda \in \{\Lambda_1\} \cup]\Lambda_2,+\infty[$, such that $u_\lambda \in \mathcal{P}$;*
- *exactly two solutions u_λ, v_λ if $\lambda \in]\Lambda_1,\Lambda_2]$, such that $u_\lambda, v_\lambda \in \mathcal{P}$, if $\lambda < \Lambda_2$, and $u_\lambda \in \mathcal{P}, v_\lambda \in \mathcal{P}_0$, if $\lambda = \Lambda_2$.*

Here, $\mathcal{P}$ and $\mathcal{P}_0$ are the sets defined in 2.4 and 2.5, respectively. It is interesting noticing that the solution $v_{\Lambda_2} \in \mathcal{P}_0$ yields, for $\lambda > \Lambda_2$, a continuum of nonnegative solutions to problem (3.1) compactly supported in $]0,1[$. We get these solutions by putting

$$u(x) = (b-a)^{\frac{p}{p-r}} v_{\Lambda_2}\big(\frac{x-a}{b-a}\big), \quad \text{if } x \in [a,b]$$
$$u(x) = 0, \quad \text{if } x \in [0,1] \setminus [a,b]$$

for each couple of numbers $a, b \in]0,1[$, such that $b - a = \left(\frac{\Lambda_2}{\lambda}\right)^{\frac{1}{p}\frac{p-r}{s-r}}$.

Concerning the singular case $r \in]0,1[$, we can mention the results proved in [13] for $p = 2$ and $s = 1$ and in [17] for $p = 2$ and $1 < s < 2$, summarized by the following Theorem.

Theorem 3.2. *Assume $p = 2$, $s \in [1,2[$ and $r \in]0,1[$. There exists $\Lambda_1 > 0$ and, if $r \in]1-\frac{s}{2},1[$, there exists $\Lambda_2 \in]\lambda_1,+\infty[$ such that problem (3.1) admits:*

- *no solution if $\lambda \in]0,\Lambda_1[$,*
- *a unique solution if $r \in]0,1-\frac{s}{2}]$ and $\lambda \in [\Lambda_1,+\infty[$,*
- *a unique solution if $r \in]1-\frac{s}{2},1[$ and $\lambda \in \{\Lambda_1\} \cup]\Lambda_2,+\infty[$,*
- *exactly two solutions if $r \in]1-\frac{s}{2},1[$ and $\lambda \in]\Lambda_1,\Lambda_2]$.*

Note that Theorem 3.2 highlights a dependence on the exponent r of the number of the solutions. We will see how this dependence derives from the behavior of the time map near the endpoint $t_0(r)$ of its domain.

With some restriction, the quasilinear singular case was addressed in [11], where problem (3.1) was studied for $p \in]1,+\infty[$, $s \in]\frac{p}{p+1},p[$ and $r \in]0,s[$. In this setting, a complete description of the set of solutions is given by the following result

Theorem 3.3 ([11, Theorem 2])**.** *Assume $p \in]1,+\infty[$, $s \in]\frac{p}{p+1},2[$ and $r \in [\frac{p}{p+1},s[$. There exists $\Lambda_1 > 0$ and $\Lambda_2 \in]\Lambda_1,+\infty[$ such that problem (3.1) admits:*

- *no solution if $\lambda \in]0,\Lambda_1[$,*
- *a unique solution u_λ, if $\lambda \in \{\Lambda_1\} \cup]\Lambda_2,+\infty[$, such that $u_\lambda \in \mathcal{P}$,*
- *exactly two solutions u_λ, v_λ, if $\lambda \in]\Lambda_1,\Lambda_2]$, such that $u_\lambda, v_\lambda \in \mathcal{P}$, if $\lambda < \Lambda_2$, and $u_\lambda \in \mathcal{P}, v_\lambda \in \mathcal{P}_0$, if $\lambda = \Lambda_2$.*

Thus, for $r \geq \frac{p}{p+1}$, Theorem 3.3 extends Theorem 3.1 to the case of singular exponents. Actually, [11, Theorem 2] gives also the following partial information concerning the case $r \in]0,\frac{p}{p+1}[$.

Theorem 3.4 ([11, Theorem 2])**.** *Assume $p \in]1,+\infty[$, $s \in]\frac{p}{p+1},p[$ and $r \in]0,\frac{p}{p+1}[$. Then, there exist $\delta_1, \delta_2 > 0$, with $\delta_1 + \delta_2 \leq \frac{p}{p+1}$, such that*

- *if $r \in]\frac{p}{p+1} - \delta_1, \frac{p}{p+1}[$, the same conclusion as Theorem 3.3 holds;*
- *if $r \in]0,\delta_2[$, there exists $\Lambda_1 > 0$ such that problem (3.1) admits no solution for $\lambda \in]0,\Lambda_1[$ and a unique solution u_λ, if $\lambda \in [\Lambda_1,+\infty[$, such that $u_\lambda \in \mathcal{P}$.*

Theorems 3.1 and 3.4 are all consequences of the way the profile of the time map varies in dependence of the exponent r. Figure 2 illustrates the various profiles of the time map obtained in [10, 11]

Note that, for $p = 2$ and $s \in [1,p[$ and $r \in]0,s[$, the results of [16, 17], says that $\delta_1 + \delta_2 = \frac{p}{p+1} = \frac{2}{3}$, with $\delta_1 = 1 - \frac{s}{2}$ and $\delta_2 = \frac{s}{2} - \frac{1}{3}$. We also point out that Theorems 3.1–3.4 give no information in the case $s \in]0,\frac{p}{p+1}[$

From the results presented in this section, the following questions naturally arise:

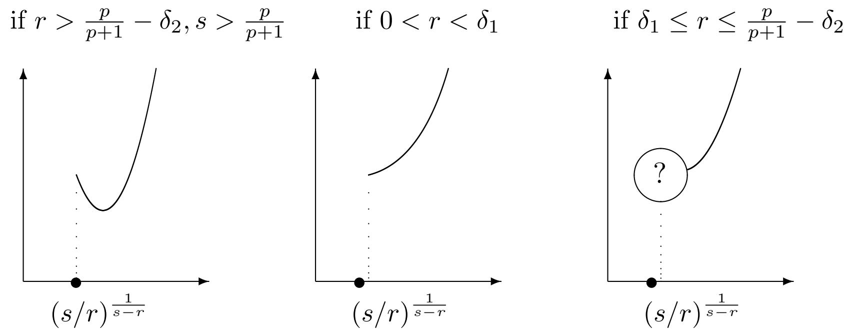

FIGURE 2. Profile of the time map in dependance of r

(1) When $p \in]1, +\infty[$ and $s \in]\frac{p}{p+1}, p[$, is it true, in light of Theorem 3.2, that the numbers δ_1, δ_2 in Theorem 3.4 are related by $\delta_1 + \delta_2 = \frac{p}{p+1}$?
(2) What happens when $s \in]0, \frac{p}{p+1}[$?

An answer to these questions would allow to complete the study of the set of solutions of problem (3.1), for all $p \in]1, +\infty[$, $s \in]0, p[$, $r \in]0, s[$, and $\lambda > 0$.

Of course, the question is knowing the profile of the time map near 0 on varying of the exponent r. Indeed, we can see that the number of the solutions of equation (3.2) (which amounts to the number of solutions of (3.1)) depends on the way the profile of the time map T "starts" from endpoint $t(r) := (s/r)^{\frac{1}{s-r}}$ of its domain. In particular, since in [11] it is proved that T has at most a critical point in $]t(r), +\infty[$, what we needs is knowing when T is increasing or decreasing near $t(r)$ according to the values of r. Routine arguments show that

- T is of class C^1 in $]t(r), +\infty[$;
- there exists (finite or infinite) the limit $\lim_{c \to t(r)} T'(c)$.

So, in view of the above considerations, we are led to study the sign of the extended real function

$$\xi(r) = \lim_{c \to t(r)} T'(c), \quad r \in]0, s[.$$

From [16, 17], it is known that for $p = 2$ and $s \in [1, 2[$, one has

- $\xi(r) > 0$, in $]0, 1 - \frac{s}{2}[$,
- $\xi(r) = 0$, at $r = 1 - \frac{s}{2}$,
- $\xi(r) < 0$, in $]1 - \frac{s}{2}, s[$.

For the quasilinear case $p \in]1, +\infty[$, by [10, 11] we know that

(i) $\xi(r) = -\infty$, if $r \in [\frac{p}{p+1}, s[$,
(ii) $\xi(r) \in]0, +\infty[$, if r is near 0,
(iii) $\xi(r) \in] - \infty, 0[$, if r is less than and near $\frac{p}{p+1}$.

The sign of $\xi(r)$ described above is deduced by properties of hypergeometric functions. This approach seems not working in the uncovered cases. By using a different approach, in [7] the sign of $\xi(r)$ has been determined for each $p \in]1, +\infty[$, $s \in]0, p[$, and $r \in]0, s[$. Let us outline the idea introduced in [7]. Set $\tau_{s,p} = \min\{s, \frac{p}{p+1}\}$. After noticing that

$$r \in]0, \tau_{s,p}[\to \xi(r), \tag{3.3}$$

is a C^1 real function, in [7] it is proved that there exists a function

$$\gamma :]0, \min\{s, \frac{p}{p+1}\}[\to \mathbb{R}$$

satisfying

$$\gamma(r)\xi(r) - \xi'(r) > 0, \quad \text{for all } r \in]0, \tau_{s,p}[$$

Then, setting

$$\phi(r) = \gamma(r)\xi(r) - \xi'(r), \quad r \in]0, \tau_{s,p}[,$$

and solving the previous equation for ξ, one has

$$\xi(r) = e^{\int_{r_0}^{r} \gamma(\sigma) d\sigma} \Big(k - \int_{r_0}^{r} \phi(\sigma) e^{-\int_{r_0}^{\sigma} \gamma(\tau) d\tau} d\sigma \Big)$$

for some $k \in \mathbb{R}$ and $r_0 \in]0, \tau_{s,p}[$. This clearly implies that ξ may change sign at most only once in $]0, \tau_{s,p}[$. Therefore, if $s \geq \frac{p}{p+1}$, recalling (i)–(iii), one infers that there exists $r^* = r^*(s) \in]0, \frac{p}{p+1}[$ such that

$$\xi^{-1}(]0, +\infty[) =]0, r^*[, \quad \xi^{-1}(0) = r^*, \quad \xi^{-1}(] - \infty, 0[) =]r^*, \frac{p}{p+1}[,$$
$$\xi(r) = -\infty, \quad \text{if } r \in [\frac{p}{p+1}, s[.$$

When $s \leq \frac{p}{p+1}$, to know whether or not ξ changes sign in $]0, s[$, one needs to study the behavior of ξ near s. To this end, in [7] the authors prove that there exists a positive constant k such that

$$\lim_{r \to s^-} (s - r)^{1/p} \xi(r) = k.$$

Hence, ξ is positive near s, and thus in the whole interval $]0, s[$.

As a consequence of these facts, we have the following result which completes the study of the set of solutions of problem (3.1).

Theorem 3.5 ([7, Theorem 1])**.** *Let $p > 1$, $s \in]0, p[$ and $r \in]0, s[$. Then, there exists $\Lambda_1 > 0$ and, for each $s \in [\frac{p}{p+1}, p[$, there exists $r^*(s) \in]0, \frac{p}{p+1}[$ with the following properties:*

- *if $s \in [\frac{p}{p+1}, p[$ and $r \in]r^*(s), \frac{p}{p+1}[$, there exists $\Lambda_2 \in]\Lambda_1, +\infty[$ such that problem* (3.1) *admits:*
  (a) *a unique solution if either $\lambda \in \{\Lambda_1\} \cup]\Lambda_2, +\infty[$;*
  (b) *exactly two solutions if $\lambda \in]\Lambda_1, \Lambda_2]$;*
- *if $s \in [\frac{p}{p+1}, p[$, $r \in]0, r^*(s)]$ and $\lambda \in [\Lambda_1, +\infty[$, problem* (3.1) *admits a unique solution;*
- *if $s \in]0, \frac{p}{p+1}[$ and $\lambda \in [\Lambda_1, +\infty[$ problem* (3.1) *admits a unique solution;*
- *if $\lambda \in]0, \Lambda_1[$, problem* (3.1) *admits no solution .*

Remark 3.6. Similarly to Theorem 3.1, the solutions corresponding to each $\lambda \in [\Lambda_1, +\infty[\setminus \{\Lambda_2\}$ and one of the solutions corresponding to $\lambda = \Lambda_2$ belong to $\mathcal{P}$. While, the other solution corresponding to $\lambda = \Lambda_2$ belongs to $\mathcal{P}_0$. In the nonsingular case $r > 1$, this yields, in same way as for Theorem 3.1, the existence of a continuum of nonnegative solutions for each $\lambda \in]\Lambda_2, +\infty[$.

4. Perturbations from problem (3.1)

In this section we present some results on the effects that certain perturbation terms yield on number of solutions of problem (3.1).

Let $p \in]1, +\infty[$ and let λ_p be the first eigenvalue of the one dimensional p-Laplacian in $]0, 1[$, with Dirichlet boundary conditions. The explicit expression of λ_p is given by

$$\lambda_p :== (p-1)(2\pi)^p \Big(p \sin\frac{\pi}{p}\Big)^{-p}.$$

We first investigate the effect of adding the resonance term $\lambda_p t$ in the nonlinearity $f(t) = \lambda t^{s-1} - t^{r-1}$, where $s \in]0, p[$, $r \in]0, s[$ and $\lambda > 0$.

The following result, proved in [8] and reported here in an equivalent statement (which we can easily get by rescaling u), gives a complete answer for the nonsingular case $r > 1$

Theorem 4.1. *Let $p > 1$, $s \in]1, p[$ and $r \in]1, s[$. Then, there exists $\Lambda_1 > 0$ such that the problem*

$$\begin{gathered} -(|u'|^{p-2}u')' = \lambda_p u^{p-1} + \lambda u^{s-1} - u^{r-1} \quad \text{in }]0,1[, \\ u > 0 \quad \text{in }]0,1[, \\ u(0) = u(1) = 0 \end{gathered} \tag{4.1}$$

admits

- *a unique solution u_λ, for $\lambda \in]0, \Lambda_1]$, such that $u_\lambda \in \mathcal{P}$;*
- *a unique solution u_λ, for $\lambda = \Lambda_1$, such that $u_\lambda \in \mathcal{P}_0$;*
- *no solution for $\lambda > \lambda_1$.*

So, by perturbing problem (3.1) with the resonance term $\lambda_p u^{p-1}$, we get an opposite behavior of the solution set on varying of λ. In addiction, when a solution to (4.1) exists, it is unique.

The proof of Theorem 4.1 is again based on the shooting method. However, differently to the proofs of the results presented so far, in this case the parameter λ is involved in the expression of the time map T_λ, which is given by:

$$T_\lambda(c) = \int_0^c \Big(\frac{\lambda_p}{p}c^p + \frac{\lambda}{s}c^s - \frac{c^r}{r} - \frac{\lambda_p}{p}t^p - \frac{\lambda}{s}t^s + \frac{t^r}{r}\Big)^{-1/p} dt, \quad c > t(\lambda).$$

where $t(\lambda) > 0$ is the unique solution of the equation $\frac{\lambda_p}{p}t^p + \frac{\lambda}{s}t^s - \frac{1}{r}t^r = 0$. The number of solutions of problem (4.1) amounts exactly to the number of solutions of the equation

$$T_\lambda(c) = \xi_p := \frac{1}{2}\big(\frac{p}{p-1}\big)^{1/p}.$$

The conclusion of Theorem 4.1 derives from the profile time map T_λ, depicted in Figure 3 for $\lambda < \Lambda_1$, $\lambda = \Lambda_1$ and $\lambda > \Lambda_1$:

Remark 4.2. If Λ_1 is as in Theorem 4.1, then for $\lambda \in]\Lambda_1, +\infty[$, we can show that there exists a continuum of nonnegative solutions compactly supported in $]0, 1[$. Nevertheless, in this case, these solutions cannot be obtained by rescaling the solution that belongs to $\mathcal{P}_0$, as in Theorems 3.1 and 3.5. Instead, they are obtained (see [8]) by showing that for each $\lambda \in]\Lambda_1, +\infty[$, there exists $\delta \in]0, 1[$ such

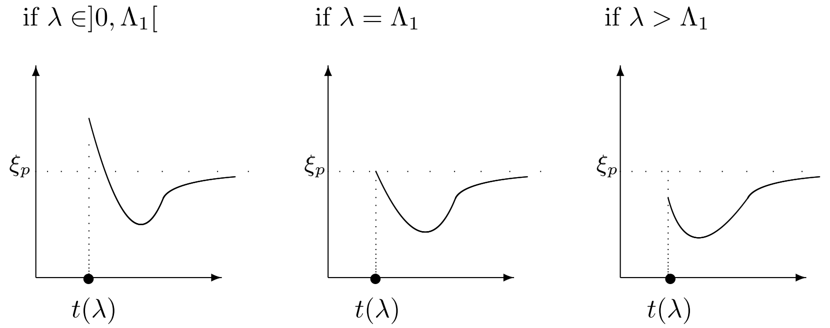

FIGURE 3. Profile of the time map associated with (4.1)

that for each compact interval $[a, b] \subset]0, 1[$, with $b - a = \delta$, there is a (unique) solution v to the problem

$$\begin{gathered} -(|u'|^{p-2}u')' = \lambda_p u^{p-1} + \lambda u^{s-1} - u^{r-1} \quad \text{in }]a, b[, \\ u > 0 \quad \text{in }]a, b[, \\ u(a) = u(b) = u'(a) = u'(b) = 0. \end{gathered}$$

Then, we get a continuum of nonnegative solutions compactly supported in $]0, 1[$ to problem (P_λ) on varying of $[a, b] \subset]0, 1[$, with $b - a = \delta$, by considering the zero extension of v to the whole $]0, 1[$.

Of course, a question worth of investigation is to study the solution set of problem (4.1) in the singular cases $r \in]0, 1[$ or $s \in]0, 1[$. The approach could be similar as that of Theorem 3.5, but the fact that there is no way to drop out the dependence of the time map from the parameter λ makes the argument more complicated. However, some evidence leads to conjecture that the same conclusion would hold.

We now pass to consider what effect a $(p-1)$-superlinear perturbation yields on problem (3.1). Let $p, s, r, q, \sigma, \lambda$ be positive numbers, with $1 < r < s < p < q$. We are going to consider the problem

$$\begin{gathered} -(|u'|^{p-2}u')' = \sigma u^{q-1} + \lambda u^{s-1} - u^{r-1} \quad \text{in }]0, 1[, \\ u > 0 \quad \text{in }]0, 1[, \\ u(0) = u(1) = 0 \end{gathered}$$

Setting $v = \sigma^{\frac{1}{q-p}} u$, $\rho = \lambda \sigma^{\frac{p-s}{q-p}}$, and $\mu = \sigma^{\frac{p-r}{q-p}}$, this problem can be reformulated as

$$\begin{gathered} -(|v'|^{p-2}v')' = v^{q-1} + \rho v^{s-1} - \mu v^{r-1} \quad \text{in }]0, 1[, \\ v > 0 \quad \text{in }]0, 1[, \\ v(0) = v(1) = 0, \end{gathered} \tag{4.2}$$

Problem (4.2) has been considered in [9] (see also [3] for the N-dimensional case).

The time map associated to (4.2) has a somewhat complicate structure and an exact multiplicity result seems quite hard to obtain in this case. Some information are provided by the following result, proved in [9].

Theorem 4.3 ([9, Theorems 2.7 and 2.9])**.** *The set $S \subset \mathbb{R}^2$ defined by*

$$S := \{(\rho, \mu) \in \mathbb{R}^2_+ : (4.2) \textit{ admits at least three solutions belonging to } \mathcal{P}\}$$

has nonempty interior. Moreover, there exists $\rho^* \in]0, +\infty]$ *and, for each* $\rho \in]0, \rho^*[$, *there exist at least two numbers* $\mu_1(\rho)$ *and* $\mu_2(\rho)$ *such that* $(P_{\rho,\mu_i(\rho)})$ *admits at least a solution belonging to* $\mathcal{P}_0$, *for* $i = 1, 2$.

Besides investigating the exact structure of the solution set of problem (4.2) with λ instead of μ, on varying of ρ, λ, it would be interesting to give an answer to the following questions suggested by the conclusion of Theorem 4.3:

(1) Is ρ^* finite, or is not finite?
(2) What is the structure of the set of solutions belonging to $\mathcal{P}_0$?
(3) What about the singular cases $r \in]0, 1[$ or $s \in]0, 1[$?

Concerning the second question, we conjecture that there are exactly two curves $\mu_1, \mu_2 :]0, \rho^*[\to]0, +\infty[$ with no common points such that

(a) for each $\rho \in]0, \rho^*[$ and $i = 1, 2$, problem (4.2), with $\mu_i(\rho)$ instead of μ, admits a unique solution in $\mathcal{P}_0$,
(b) $\{(\rho, \mu) \in \mathbb{R}^2_+ \colon \text{(4.2)has solutions in } \mathcal{P}_0\} = \text{graph}(\mu_1) \cup \text{graph}(\mu_2)$.

Finally, we give some extensions of the results presented so far to the N-dimensional case. Let Ω be an open smooth bounded domain in $\mathbb{R}^N$. Let us consider the N-dimensional version of problem (3.1) in the semilinear case $p = 2$

$$\begin{gathered} -\Delta u = \lambda u^{s-1} - u^{r-1}, \quad \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

where $s \in]0, 2[$, $r \in]0, s[$ and $\lambda \in]0, +\infty[$. This problem was considered in [2] for the nonsingular case $r > 1$, in [5] for the singular case $r \in]0, 1[$ and $s \in [1, 2[$, and in [6] for the "double" singular case $s \in]0, 1[$, $r \in]0, s[$. The results obtained in these papers, proved via variational and approximation techniques, say that there exists $\Lambda > 0$ such that the problem admits at least a solution for $\lambda \in]\Lambda, +\infty[$ and no solution for $\lambda \in]0, \Lambda[$. For $\lambda > \Lambda$ the existence of a nonzero and nonnegative solution is also ensured, but the multiplicity of positive solutions is still an open problem, at least for general bounded domains. For $\lambda = \Lambda$ and $r > 1$, it is proved in [2] that there exists a nonzero and nonnegative solution. However nothing is said about the positivity of this solution as well as its possible uniqueness (as in the one dimensional case). In the singular case, for $\lambda = \Lambda$, it is an open question even the existence of nonzero and nonnegative solutions.

The last result we present concerns the problem

$$\begin{gathered} -\Delta u = \lambda_1 u + \lambda u^{s-1} - u^{r-1}, \quad \text{in } \Omega, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered}$$

which is the perturbation of the previous problem with the resonant term $\lambda_1 u$, where λ_1 is the first eigenvalue of the Laplacian on Ω. The following recent result, proved in [4] and reported here in an equivalent statement which one obtains by rescaling u, highlights, as in the one-dimensional case, an opposite behavior with respect to the unperturbed problem.

Theorem 4.4. *Let* $s \in]1, 2[$ *and* $r \in]1, s[$. *For each* $\lambda > 0$, *there exists a nonzero and nonnegative solution to the problem*

$$-\Delta u = \lambda_1 u + \lambda u^{s-1} - u^{r-1}, \quad \text{in } \Omega,$$

$$u = 0, \quad \text{on } \partial\Omega.$$

Moreover, there exists $\Lambda > 0$ *such that, for each* $\lambda \in]0, \Lambda[$*, every nonnegative and nonzero solutions belongs to* $\mathcal{P}$*.*

It is worth pointing out that the Strong Maximum Principle stated by this result holds for a nonlinearity f which is neither positive nor Lipschitz continuous near 0, that is f does not satisfy the sufficient condition typically used to get the validity of the Strong Maximum Principle for nonnegative solutions of nonlinear elliptic Dirichlet problem.

Open questions connected to this last result are its possible extensions to singular cases as well as to more general nonlinearities of the form $\lambda_1 t + \lambda f(t)$.

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