Electronic Journal of Qualitative Theory of Differential Equations
2023, No. 7, 1–10; https://doi.org/10.14232/ejqtde.2023.1.7 www.math.u-szeged.hu/ejqtde/

An exact bifurcation diagram for a p–q Laplacian boundary value problem

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Received 28 June 2022, appeared 2 March 2023
Communicated by John R. Graef

Abstract. We study positive solutions to the p–q Laplacian two-point boundary value problem:

$$\begin{cases} -\mu[(u')^{p-1}]' - [(u')^{q-1}]' = \lambda u(1-u) \quad \text{on } (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

when $p = 4$ and $q = 2$. Here $\lambda > 0$ is a parameter and $\mu \geq 0$ is a weight parameter influencing the higher-order diffusion term. When $\mu = 0$ (the Laplacian case) the exact bifurcation diagram for a positive solution is well-known, namely, when $\lambda \leq \pi^2$ there are no positive solutions, and for $\lambda > \pi^2$ there exists a unique positive solution $u_{\lambda,\mu}$ such that $\|u_{\lambda,\mu}\|_\infty \to 0$ as $\lambda \to \pi^2$ and $\|u_{\lambda,\mu}\|_\infty \to 1$ as $\lambda \to \infty$. Here, we will prove that for all $\mu > 0$ similar bifurcation diagrams preserve, and they all bifurcate from $(\lambda, u) = (\pi^2, 0)$. Our results are established via the method of sub-super solutions and a quadrature method. We also present computational evaluations of these bifurcation diagrams for various values of μ and illustrate how they evolve when μ varies.

Keywords: positive solutions, p–q Laplacian, Dirichlet boundary conditions, exact bifurcation diagram.

2020 Mathematics Subject Classification: 34B08, 34B18.

1 Introduction

We analyze positive solutions to the boundary value problem:

$$\begin{cases} -\mu[(u')^{p-1}]' - [(u')^{q-1}]' = \lambda f(u) \quad \text{on } (0,1), \\ u(0) = 0 = u(1) \end{cases} \tag{1.1}$$

when $p = 4$ and $q = 2$. Here we will choose f to be a smooth function such that $f(0) = 0$, and $\lambda > 0$, $\mu \geq 0$ are parameters, with μ influencing the higher-order diffusion term. Study of p–q

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Laplacian problems have been of interest in the literature (see [1–4,6]) as they arise as steady states of reaction-diffusion processes when the diffusion involved is of a certain nonlinear class. See [6] in particular, where they note that equations of this type arise in biophysics, plasma physics, and chemical reactor design. However, our motivation of this study is purely mathematical. We will begin with the case $\mu = 0$ when the exact bifurcation diagram for positive solutions is known, and then prove that for all $\mu > 0$ similar bifurcation diagrams preserve, and that they all bifurcate from the branch of trivial solutions at the same point where the bifurcation occurs in the case $\mu = 0$. In particular, in this study we choose

$$f(s) = s(1-s); \qquad s \in [0,1],$$

for which when $\mu = 0$ it is well-known that the bifurcation diagram of positive solutions is exact (see [5,7]) of the form:

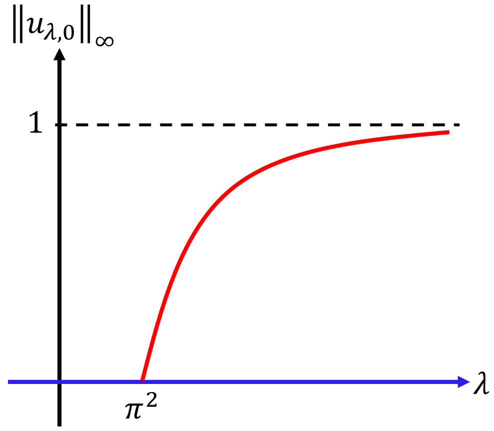

Figure 1.1: A prototypical bifurcation diagram of positive solutions for (1.1) when $f(s) = s(1-s)$ and $\mu = 0$.

Namely, for $\lambda \leq \pi^2$ there are no positive solutions, and for $\lambda > \pi^2$, there is a unique positive solution $u_{\lambda,0}$ such that $\|u_{\lambda,0}\|_\infty \to 0$ as $\lambda \to \pi^2$ and $\|u_{\lambda,0}\|_\infty \to 1$ as $\lambda \to \infty$. Here, we extend the study for the case $\mu > 0$. In particular, we prove:

Theorem 1.1. *Let $\mu > 0$ be fixed. Then for $\lambda \leq \pi^2$, (1.1) has no positive solution, and for $\lambda > \pi^2$, (1.1) has a unique positive solution $u_{\lambda,\mu}$ such that $\|u_{\lambda,\mu}\|_\infty \to 0$ as $\lambda \to \pi^2$ and $\|u_{\lambda,\mu}\|_\infty \to 1$ as $\lambda \to \infty$. Further, for $\lambda > \pi^2$, if $\mu_2 > \mu_1$ then $u_{\lambda,\mu_1}(x) \geq u_{\lambda,\mu_2}(x)$ for all $x \in [0,1]$.*

Remark 1.2. Theorem 1.1 establishes that for each $\mu > 0$, a similar exact bifurcation diagram for positive solutions to the case when $\mu = 0$ preserves and each bifurcates from $(\lambda, u) = (\pi^2, 0)$ (see Figure 1.2).

Remark 1.3. Our analysis uses the relationship (2.3), which determines the bifurcation diagram. The derivation of (2.3) uses $p = 4$ and $q = 2$ (see the proof of Lemma 2.2). Establishing such a result for any $p > q > 1$ is an open problem. Further, our analysis is restricted to the specific f we chose.

We prove our results by the method of sub-super solutions (see [4]) and via using the quadrature method discussed in [2] (an extension of the quadrature method first introduced for the case $\mu = 0$ in [5]). In Section 2 we present preliminaries, in Section 3 we prove Theorem 1.1, and in Section 4 we compute the bifurcation diagrams numerically for several values of μ and demonstrate their evolution as μ varies.

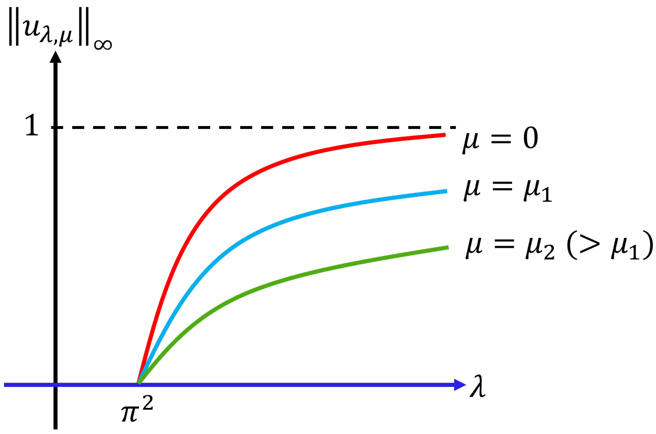

Figure 1.2: Prototypical bifurcation diagrams of positive solutions for (1.1) when $\mu \geq 0$.

2 Preliminaries

In this section, we introduce definitions of a subsolution and a supersolution of (1.1) and state a sub-supersolution theorem that will be used to prove our existence result for positive solutions. We also state a result via a quadrature method which we will use in our analysis (combined with an existence result obtained via sub-supersolutions) to establish exact details on the bifurcation diagram for positive solutions.

By a subsolution of (1.1) we mean $\psi \in C^2((0,1)) \cap C([0,1])$ that satisfies

$$\begin{cases} -\mu[(\psi')^3]' - \psi'' \leq \lambda f(\psi) & \text{on } (0,1), \\ \psi(0) \leq 0, \psi(1) \leq 0. \end{cases} \tag{2.1}$$

By a supersolution of (1.1) we mean $Z \in C^2((0,1)) \cap C([0,1])$ that satisfies

$$\begin{cases} -\mu[(Z')^3]' - Z'' \geq \lambda f(Z) & \text{on } (0,1), \\ Z(0) \geq 0, Z(1) \geq 0. \end{cases} \tag{2.2}$$

Then the following result holds:

Lemma 2.1. *Let ψ and Z be a subsolution and a supersolution of (1.1) respectively such that $\psi \leq Z$. Then (1.1) has a solution $u \in C^2((0,1)) \cap C([0,1])$ such that $u \in [\psi, Z]$.*

Proof. See [4]. □

Lemma 2.2. *Let $\lambda, \mu > 0$ be fixed and $\rho \in (0,1)$. Then (1.1) has a positive solution with $\|u_{\lambda,\mu}\|_\infty = \rho$ if and only if λ and ρ satisfy*

$$G(\lambda, \rho) = \int_0^\rho \frac{ds}{\sqrt{\sqrt{12\mu\lambda[F(\rho) - F(s)] + 1} - 1}} = \frac{1}{2\sqrt{3\mu}}, \tag{2.3}$$

where $F(s) = \int_0^s f(z)dz$.

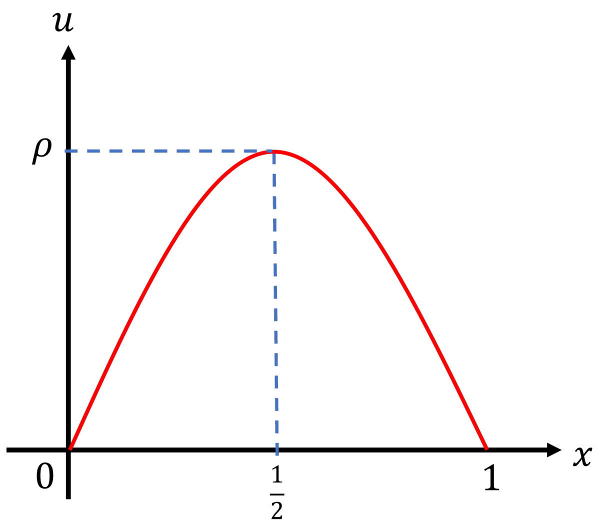

Figure 2.1: A prototypical shape of a positive solution to (1.1).

Proof. (See also [3].) Suppose $u_{\lambda,\mu}$ is a positive solution of (1.1) with $\|u_{\lambda,\mu}\|_\infty = \rho$. Since (1.1) is autonomous, $u_{\lambda,\mu}$ must be symmetric about $x = \frac{1}{2}$, increasing on $(0, \frac{1}{2})$, and decreasing on $(\frac{1}{2}, 1)$. See Figure 2.1.

Multiplying the differential equation in (1.1) by $u'_{\lambda,\mu}(x)$ for $x \in [0, \frac{1}{2}]$, we get

$$-\mu u'_{\lambda,\mu}(x)[(u'_{\lambda,\mu}(x))^3]' - u'_{\lambda,\mu}(x)[u'_{\lambda,\mu}(x)]' = u'_{\lambda,\mu}(x)\lambda f(u_{\lambda,\mu}(x)), \tag{2.4}$$

which can be written as

$$\frac{-3\mu}{4}[(u'_{\lambda,\mu}(x))^4]' - \frac{1}{2}[(u'_{\lambda,\mu}(x))^2]' = \lambda[F(u_{\lambda,\mu}(x))]'; \qquad x \in \left[0, \frac{1}{2}\right]. \tag{2.5}$$

Integrating (2.5) with respect to x over $\left[0, \frac{1}{2}\right]$, we obtain

$$3\mu[u'_{\lambda,\mu}(x)]^4 + 2[u'_{\lambda,\mu}(x)]^2 = 4\lambda[F(\rho) - F(u_{\lambda,\mu}(x))]; \qquad x \in \left[0, \frac{1}{2}\right]. \tag{2.6}$$

Solving (2.6) for $[u'_{\lambda,\mu}(x)]^2$, we obtain

$$[u'_{\lambda,\mu}(x)]^2 = \frac{\sqrt{12\mu\lambda[F(\rho) - F(u_{\lambda,\mu}(x))] + 1} - 1}{3\mu}; \qquad x \in \left[0, \frac{1}{2}\right].$$

Since $u'_{\lambda,\mu}(x) > 0$ for $x \in \left[0, \frac{1}{2}\right]$, it follows that

$$u'_{\lambda,\mu}(x) = \frac{\sqrt{\sqrt{12\mu\lambda[F(\rho) - F(u_{\lambda,\mu}(x))] + 1} - 1}}{\sqrt{3\mu}}; \qquad x \in \left[0, \frac{1}{2}\right]. \tag{2.7}$$

Integrating (2.7) with respect to x over $\left[0, \frac{1}{2}\right)$, we obtain

$$\frac{x}{\sqrt{3\mu}} = \int_0^{u_{\lambda,\mu}(x)} \frac{ds}{\sqrt{\sqrt{12\mu\lambda[F(\rho) - F(s)] + 1} - 1}}; \qquad x \in \left[0, \frac{1}{2}\right), \tag{2.8}$$

and letting $x \to \left(\frac{1}{2}\right)^-$, we obtain (2.3):

$$G(\lambda,\rho) = \int_0^\rho \frac{ds}{\sqrt{\sqrt{12\mu\lambda[F(\rho)-F(s)]+1}-1}} = \frac{1}{2\sqrt{3\mu}}.$$

Conversely, suppose λ and $\rho \in (0,1)$ are such that (2.3) is satisfied. Then for each $x \in \left[0,\frac{1}{2}\right)$, we can find a unique $u_{\lambda,\mu}(x)$ satisfying (2.8). We can now extend this $u_{\lambda,\mu}$ on $[0,1]$ such that $u_{\lambda,\mu}\left(\frac{1}{2}\right) = \rho$ and $u_{\lambda,\mu}(x) = u_{\lambda,\mu}(1-x)$ for $x \in \left(\frac{1}{2},1\right]$. With the aid of the Implicit Function Theorem, we can show that $u_{\lambda,\mu} \in C^2((0,1)) \cap C([0,1])$ and then it is easy to show it satisfies (1.1). Hence, (2.3) determines the bifurcation diagram of positive solutions $u_{\lambda,\mu}$ for (1.1) with $\|u_{\lambda,\mu}\|_\infty = \rho \in (0,1)$. $\square$

Remark 2.3. If $\mu = 0$, (1.1) becomes the boundary value problem:

$$\begin{cases} -u'' = \lambda f(u) & \text{on } (0,1), \\ u(0) = 0 = u(1) \end{cases} \tag{2.9}$$

and by the quadrature method described in [5], the bifurcation diagram for positive solutions of (2.9) is determined by

$$\lambda = 2\left\{\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}\right\}^2; \qquad \rho \in (0,1). \tag{2.10}$$

3 Proof of Theorem 1.1

Claim: Nonexistence of positive solutions for $\lambda \le \pi^2$.

Suppose $u_{\lambda,\mu} > 0; (0,1)$ is a solution to (1.1) for $\lambda \le \pi^2$. Multiplying each term of the differential equation by $\sin(\pi x)$ and integrating on $(0,1)$, we have

$$-\mu\int_0^1 [(u'_{\lambda,\mu}(x))^3]'\sin(\pi x)dx - \int_0^1 u''_{\lambda,\mu}(x)\sin(\pi x)dx = \lambda\int_0^1 u_{\lambda,\mu}(x)[1-u_{\lambda,\mu}(x)]\sin(\pi x)dx. \tag{3.1}$$

Equivalently, we have

$$-\mu\int_0^1 [(u'_{\lambda,\mu}(x))^3]'\sin(\pi x)dx + \lambda\int_0^1 [u_{\lambda,\mu}(x)]^2\sin(\pi x)dx = (\lambda-\pi^2)\int_0^1 u_{\lambda,\mu}(x)\sin(\pi x)dx. \tag{3.2}$$

Since $\lambda \le \pi^2$, we have

$$(\lambda-\pi^2)\int_0^1 u_{\lambda,\mu}(x)\sin(\pi x)dx \le 0. \tag{3.3}$$

However,

$$
\begin{aligned}
&-\mu\int_0^1[(u'_{\lambda,\mu}(x))^3]'\sin(\pi x)dx+\lambda\int_0^1[u_{\lambda,\mu}(x)]^2\sin(\pi x)dx\\
&\quad=-\mu\left[\underbrace{\sin(\pi x)[u'_{\lambda,\mu}(x)]^3\Big|_0^1}_{=0}-\pi\int_0^1\cos(\pi x)[u'_{\lambda,\mu}(x)]^3dx\right]+\lambda\int_0^1[u_{\lambda,\mu}(x)]^2\sin(\pi x)dx\\
&\quad=\mu\pi\left[\int_0^{1/2}\underbrace{\cos(\pi x)[u'_{\lambda,\mu}(x)]^3}_{>0}dx+\int_{1/2}^1\underbrace{\cos(\pi x)[u'_{\lambda,\mu}(x)]^3}_{>0}dx\right]+\underbrace{\lambda\int_0^1[u_{\lambda,\mu}(x)]^2\sin(\pi x)dx}_{>0}\\
&\quad>0.
\end{aligned}
$$

This contradicts (3.3). Hence (1.1) has no positive solution for $\lambda\le\pi^2$.

Claim: Existence of a positive solution $u_{\lambda,\mu}$ for $\lambda>\pi^2$.

Consider $\psi(x)=\varepsilon\sin(\pi x)$ with $\varepsilon>0$. Then $\psi''(x)=-\varepsilon\pi^2\sin(\pi x)$ and $(\psi'(x))^3=\varepsilon^3\pi^3[\cos(\pi x)]^3$. Hence

$$
\begin{aligned}
&-\mu[(\psi'(x))^3]'-\psi''(x)-\lambda\psi(x)(1-\psi(x))\\
&\quad=-\mu\Big(-3\varepsilon^3\pi^4\cos^2(\pi x)\sin(\pi x)\Big)-\Big(-\varepsilon\pi^2\sin(\pi x)\Big)-\lambda\varepsilon\sin(\pi x)[1-\varepsilon\sin(\pi x)]\\
&\quad=\varepsilon\sin(\pi x)\Big(3\mu\varepsilon^2\pi^4\cos^2(\pi x)+\pi^2-\lambda+\lambda\varepsilon\sin(\pi x)\Big)\\
&\quad<0;\qquad x\in(0,1)
\end{aligned}
$$

for $\varepsilon\approx0$ when $\lambda>\pi^2$. Clearly the boundary conditions are satisfied by ψ. Thus, ψ is a subsolution of (1.1) for $\lambda>\pi^2$. Now $Z\equiv1$ is a supersolution of (1.1) and $\psi<Z$ for $\varepsilon\approx0$. Hence by Lemma 2.1, (1.1) has a positive solution $u_{\lambda,\mu}\in[\psi,Z]$ for all $\lambda>\pi^2$.

Claim: Existence of a unique positive solution $u_{\lambda,\mu}$ such that $\|u_{\lambda,\mu}\|_\infty\to0$ as $\lambda\to\pi^2$ and $\|u_{\lambda,\mu}\|_\infty\to1$ as $\lambda\to\infty$.

Recall $G(\lambda,\rho)$ from (2.3). Note that

$$
G(\lambda,\rho)=\int_0^\rho\frac{ds}{\sqrt{\sqrt{12\mu\lambda[F(\rho)-F(s)]+1}-1}}=\int_0^1\frac{\rho}{\sqrt{\sqrt{12\mu\lambda[F(\rho)-F(\rho v)]+1}-1}}dv.\quad(3.4)
$$

Now, using (3.4) we have

$$
G_\rho(\lambda,\rho)=\int_0^1\frac{N(v)}{\sqrt{2\lambda\mu\rho^2(2\rho(v^3-1)-3v^2+3)+1}\Big(\sqrt{2\lambda\mu\rho^2(2\rho(v^3-1)-3v^2+3)+1}-1\Big)^{3/2}}dv,\tag{3.5}
$$

where

$$
N(v)=\lambda\mu\rho^2\Big(\rho(v^3-1)-3v^2+3\Big)-\sqrt{2\lambda\mu\rho^2\Big(2\rho(v^3-1)-3v^2+3\Big)+1}+1.
$$

Clearly the denominator of (3.5) is positive. Further, $N(1) = 0$. Hence, if we prove that $N'(v) < 0$, then $N(v)$ has to be positive on $[0,1)$. Now,

$$N'(v) = 3\lambda\mu\rho^2 v(\rho v - 2) - \frac{6\lambda\mu\rho^2 v(\rho v - 1)}{\sqrt{2\lambda\mu\rho^2\big(2\rho(v^3-1) - 3v^2 + 3\big) + 1}},$$

so $N'(v) < 0$ provided that $2 - \rho v > \frac{2(1-\rho v)}{\sqrt{2\lambda\mu\rho^2\sigma(v)+1}}$, where $\sigma(v) = \big(2\rho(v^3-1) - 3v^2 + 3\big)$. But since $\sigma'(v) = 6v(\rho v - 1) < 0$ and $\sigma(1) = 0$, we must have $\sigma(v) \geq 0$; $v \in (0,1)$. Hence, $N'(v) < 0$ provided $2 - \rho v > 2(1 - \rho v)$, which is clearly true. So $G_\rho(\lambda,\rho) > 0$ for $\lambda > 0$ and $\rho \in (0,1)$. Now combining this with our existence of a positive solution for $\lambda > \pi^2$, we see that there exists a unique $\rho \in (0,1)$ such that $G(\lambda,\rho) = \frac{1}{2\sqrt{3\mu}}$. Further, from (2.3) it is easy to see that $G_\lambda(\lambda,\rho) < 0$ for $\lambda > 0$ and $\rho \in (0,1)$ (See Figure 3.1). Thus, by the Implicit Function Theorem, there exists a unique function $\lambda : (0,1) \to (\pi^2,\infty)$ satisfying $G(\lambda(\rho),\rho) = \frac{1}{2\sqrt{3\mu}}$ and

$$\frac{d\lambda}{d\rho} = -\frac{G_\rho(\lambda,\rho)}{G_\lambda(\lambda,\rho)} > 0. \tag{3.6}$$

Recall that we already established a positive solution for $\lambda > \pi^2$. Combining this result with (3.6) we now have a unique positive solution $u_{\lambda,\mu}$ for $\lambda > \pi^2$. Further, combining with our nonexistence result for $\lambda \leq \pi^2$, we have the following:

$$\lim_{\rho\to 0}\lambda(\rho) = \pi^2 \qquad \Big(\lim_{\lambda\to\pi^2}\|u_{\lambda,\mu}\|_\infty = 0\Big)$$
$$\lim_{\rho\to 1}\lambda(\rho) = \infty \qquad \Big(\lim_{\lambda\to\infty}\|u_{\lambda,\mu}\|_\infty = 1\Big).$$

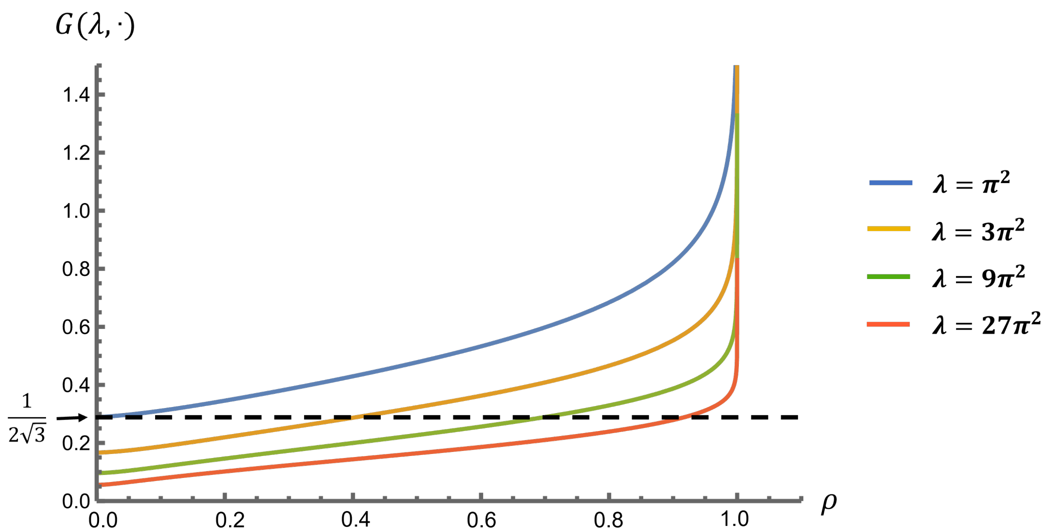

Figure 3.1: Plots of $G(\lambda,\cdot)$ for various λ. Observe their intersections with the level $\frac{1}{2\sqrt{3\mu}}$ when $\mu = 1$.

Claim: For $\mu_2 > \mu_1$, $u_{\lambda,\mu_1}(x) \geq u_{\lambda,\mu_2}(x)$ for all $x \in [0,1]$.

Let $\mu_2 > \mu_1$ and $\lambda > \pi^2$ be fixed. Now, let u_{λ,μ_1} be a positive solution to (1.1) with $\mu = \mu_1$. Then u_{λ,μ_1} satisfies $-\mu_1[(u'_{\lambda,\mu_1}(x))^3]' - u''_{\lambda,\mu_1}(x) = \lambda f(u_{\lambda,\mu_1}(x))$; $x \in (0,1)$. We proceed by showing that u_{λ,μ_1} is a supersolution to (1.1) with $\mu = \mu_2$. Observe that

$$\begin{aligned}
-\mu_2[(u'_{\lambda,\mu_1}(x))^3]' - u''_{\lambda,\mu_1}(x) &= -\mu_2\left(-\frac{1}{\mu_1}\Big(\lambda f(u_{\lambda,\mu_1}(x)) + u''_{\lambda,\mu_1}(x)\Big)\right) - u''_{\lambda,\mu_1}(x)\\
&= \frac{\mu_2}{\mu_1}\Big(\lambda f(u_{\lambda,\mu_1}(x)) + u''_{\lambda,\mu_1}(x)\Big) - u''_{\lambda,\mu_1}(x)\\
&= \frac{\mu_2}{\mu_1}\lambda f(u_{\lambda,\mu_1}(x)) + \Big(\frac{\mu_2}{\mu_1} - 1\Big)u''_{\lambda,\mu_1}(x)\\
&\geq \lambda f(u_{\lambda,\mu_1}(x)); \qquad x \in (0,1)
\end{aligned}$$

provided that

$$\Big(\frac{\mu_2}{\mu_1} - 1\Big)\Big(\lambda f(u_{\lambda,\mu_1}(x)) + u''_{\lambda,\mu_1}(x)\Big) \geq 0; \qquad x \in (0,1). \tag{3.7}$$

Given our assumption that $\mu_2 > \mu_1$, we have $\frac{\mu_2}{\mu_1} - 1 > 0$. By (1.1) with $p = 4$ and $q = 2$, it is easy to see that $u''_{\lambda,\mu_1}(x) = \frac{-\lambda f(u_{\lambda,\mu_1}(x))}{1+3\mu_1(u'_{\lambda,\mu_1}(x))^2}$; $x \in (0,1)$. Hence

$$\lambda f(u_{\lambda,\mu_1}(x)) + u''_{\lambda,\mu_1}(x) = \lambda f(u_{\lambda,\mu_1}(x))\left(1 - \frac{1}{1+3\mu_1(u'_{\lambda,\mu_1}(x))^2}\right) \geq 0; \qquad x \in (0,1).$$

So (3.7) is satisfied and u_{λ,μ_1} is a supersolution to (1.1) with $\mu = \mu_2$. Recall that $\psi(x) = \varepsilon\sin(\pi x)$ with $\varepsilon > 0$ and $\varepsilon \approx 0$ is a subsolution to (1.1) for any $\mu > 0$ when $\lambda > \pi^2$ and clearly $\psi \leq u_{\lambda,\mu_1}$ when $\varepsilon \approx 0$. Thus, the unique positive solution u_{λ,μ_2} to (1.1) with μ_2 when $\lambda > \pi^2$ must be such that $u_{\lambda,\mu_2} \in [\psi, u_{\lambda,\mu_1}]$. Hence, $u_{\lambda,\mu_1}(x) \geq u_{\lambda,\mu_2}(x)$ for all $x \in [0,1]$. □

4 Computation of bifurcation diagrams as μ varies

The bifurcation diagrams for $\mu > 0$ in Figure 4.1 are computed using (2.3). In particular, for a sequence of values $\rho \in (0,1)$, we determine the corresponding sequence of $\lambda > 0$ such that (2.3) is satisfied using the *FindRoot* function in *Mathematica*. The bifurcation curves are generated using linear interpolation of the points $\{(\lambda,\rho)\}$. Similarly, for the $\mu = 0$ case, we apply (2.10).

In Figure 4.2, we generate profiles of positive solutions for $\lambda = 50$, $\mu_1 = 5$, and $\mu_2 = 30$ using (2.8) for $x \in [0, \frac{1}{2})$ and appealing to the symmetry established in Lemma 2.2. This illustrates that $u_{50,5}(x) \geq u_{50,30}(x)$ for all $x \in [0,1]$ as described in Theorem 1.1 for particular choices of μ_1 and μ_2. By considering a uniform sequence of x-values lying in $[0,1]$ and solving (2.8) with corresponding λ, ρ, μ values within a specified tolerance using *FindRoot*, then linearly interpolating the points $\{(x, u_{\lambda,\mu}(x))\}$, we obtain the solution profiles.

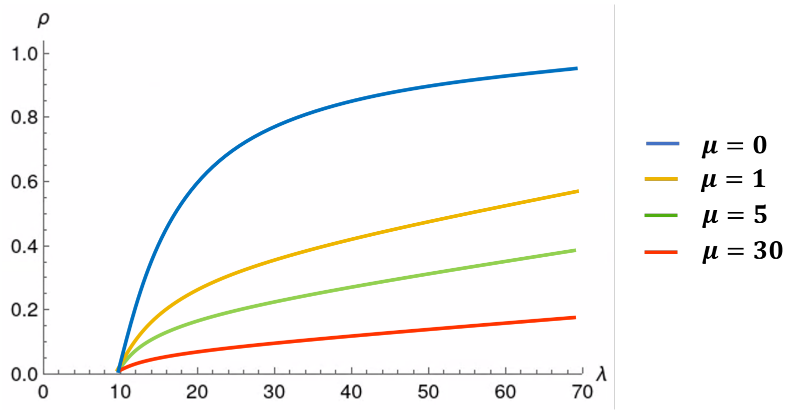

Figure 4.1: Evolution of exact bifurcation diagrams of positive solutions to (1.1) as $\mu \geq 0$ varies.

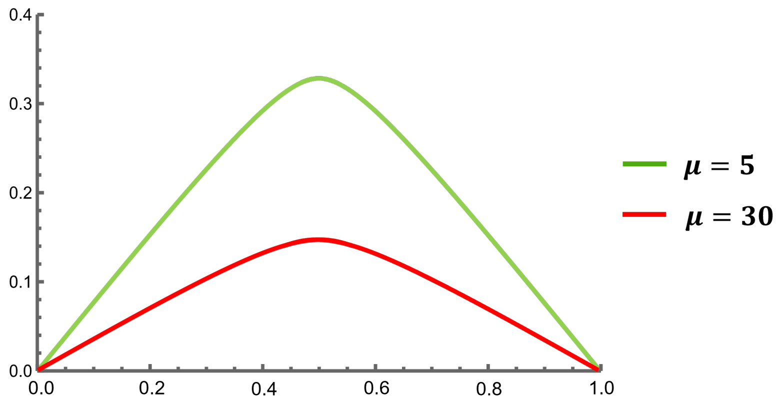

Figure 4.2: Profiles of positive solutions u_{λ,μ_1} and u_{λ,μ_2} to (1.1) for $\lambda = 50$, $\mu_1 = 5$, and $\mu_2 = 30$.

Acknowledgements

We thank the referees for their suggestions which aided us in improving the presentation of this paper.

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