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Existence of positive solutions for a class of p-Laplacian type generalized quasilinear Schrödinger equations with critical growth and potential vanishing at infinity

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Abstract. In this paper, we study the existence of positive solutions for the following generalized quasilinear Schrödinger equation

$$\begin{aligned} -\operatorname{div}(g^p(u)|\nabla u|^{p-2}\nabla u) + g^{p-1}(u)g'(u)|\nabla u|^p + V(x)|u|^{p-2}u \\ = K(x)f(u) + Q(x)g(u)|G(u)|^{p^*-2}G(u), \qquad x \in \mathbb{R}^N, \end{aligned}$$

where $N \geq 3$, $1 < p \leq N$, $p^* = \frac{Np}{N-p}$, $g \in C^1(\mathbb{R}, \mathbb{R}^+)$, $V(x)$ and $K(x)$ are positive continuous functions and $G(u) = \int_0^u g(t)dt$. By using a change of variable, we obtain the existence of positive solutions for this problem by using the Mountain Pass Theorem. Our results generalize some existing results.

Keywords: generalized quasilinear Schrödinger equation, positive solutions, critical growth; p-Laplacian.

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Introduction

This article is concerned with a class of generalized quasilinear Schrödinger equation

$$\begin{aligned} -\operatorname{div}(g^p(u)|\nabla u|^{p-2}\nabla u) + g^{p-1}(u)g'(u)|\nabla u|^p + V(x)|u|^{p-2}u \\ = K(x)f(u) + Q(x)g(u)|G(u)|^{p^*-2}G(u), \qquad x \in \mathbb{R}^N, \quad (1.1) \end{aligned}$$

where $N \geq 3$, $1 < p \leq N$, $p^* = \frac{pN}{N-p}$, $g \in C^1(\mathbb{R}, \mathbb{R}^+)$, $V(x)$ and $K(x)$ are positive continuous functions, $Q(x) \geq 0$ is a bounded continuous function and $G(u) = \int_0^u g(t)dt$.

If $p = 2$, then (1.1) will be reduced to the following generalized quasilinear Schrödinger equation

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = K(x)f(u) + Q(x)g(u)|G(u)|^{2^*-2}G(u), \quad x \in \mathbb{R}^N.$$

In nonlinear analysis, the existence of solitary wave solutions for the following quasi-linear Schrödinger equation has been widely considered

$$i\partial_t z = -\Delta z + W(x)z - k(x,|z|) - \Delta l(|z|^2)l'(|z|^2)z \tag{1.2}$$

where $z : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, $l : \mathbb{R} \to \mathbb{R}$ and $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are suitable functions. When l is different, the quasilinear equation of the form (1.2) can express several physical phenomenon. Especially, $l(s) = s$ was used for the superfluid film [26,27] equation in fluid mechanics by Kurihara [26]. For more physical background, we can refer to [5,6,11,25,28,36,38,39] and references therein. In addition, many conclusions about the equation (1.2) with $l(t) = t^\alpha$ for some $\alpha \geq 1$ have been studied, see [33–35,37] and the references therein. However, to our knowledge, only in the recent papers [20] and [40], the equation (1.2) with a general l has been studied.

If we let $z(t,x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a real function, then (1.2) can be reduce to (see [15]):

$$-\Delta u + V(x)u - \Delta l(u^2)l'(u^2)u = h(x,u), \qquad x \in \mathbb{R}^N. \tag{1.3}$$

If we take

$$g^2(u) = 1 + \frac{[(l^2(u))']^2}{2},$$

then (1.3) turns into quasilinear elliptic equations (see [40])

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(x,u), \qquad x \in \mathbb{R}^N. \tag{1.4}$$

Moreover, if we let

$$g^p(u) = 1 + \frac{[(l^2(u))']^p}{p},$$

the (1.1) turns to the following (see [45])

$$-\Delta_p + V(x)|u|^{p-2}u - \Delta_p(l(u^2))l'(u^2)\frac{2u}{p} = h(x,u), \qquad x \in \mathbb{R}^N.$$

For (1.4), in [20,21], Deng et al. proved the existence of positive solutions with critical exponents. In [20,21], they established the critical exponents, which are 2^* and $\alpha 2^*$, respectively. In [18,19], Deng et al. established the existence of nodal solutions. Especially, in [18], the authors gave some existence results about under critical growth condition. Moreover, in [29], Li et al. proved the existence of ground state solutions and geometrically distinct solutions via Nehari manifold method. In [30], the authors studied the existence of a positive solution, a negative solution and infinitely many solutions via symmetric mountain theorem. In [9], Chen et al. considered the existence and concentration behavior of ground state solutions for (1.4) with subcritical growth. Afterwards, Chen et al. [10] proved the existence and concentration behavior of ground state solutions for (1.4) with critical exponential 22^* growth. For more results, the readers can refer to [13,14,31,40–43]. In 2016, Li et al. [31,46] established the existence of sign-changing solutions and ground state solutions with potential vanishing at infinity as follows:

- (g) $g \in \mathcal{C}^1(\mathbb{R},\mathbb{R}^+)$ is even with $g'(t) \geq 0$ for all $t \in \mathbb{R}^+$ and $g(0) = 1$.
- (V) The potential function V is positive on $\mathbb{R}^N$ and belongs to $L^\infty(\mathbb{R}^N) \cap \mathcal{C}^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0,1)$.

(K) $K \in L^{\infty}(\mathbb{R}^N) \cap \mathcal{C}^{\alpha}(\mathbb{R}^N)$ is positive.

(K_1) If $\{A_n\} \subset \mathbb{R}^N$ is a sequence of Borel sets such that $|A_n| \leq M$, for all n and some $M > 0$, then we have

$$\lim_{r \to +\infty} \int_{A_n \cap B_r^c(0)} K(x)dx = 0, \quad \text{uniformly in } n \in \mathbb{N},$$

where $B_r^c(0) = \{x \in \mathbb{R}^N : |x| \geq r\}$

(K_2) The following condition holds:

$$\frac{K(x)}{V(x)} \in L^{\infty}(\mathbb{R}^N). \tag{1.5}$$

Note that conditions (V)–(K_2) are called potential vanishing at infinity. By using potential vanishing at infinity, there are many papers (see [1,4,12,23,24,31,32,43,44,46]) to study the existence of solutions for different equations. Especially in [22], Deng et al. proved the existence of positive solutions with critical growth and potential vanishing at infinity by making the change of variables $v = r^{-1}(u)$, where r is defined by

$$r'(t) = \frac{1}{(1+2r^2(t))^{1/2}} \quad \text{on } [0, +\infty),$$
$$r(-t) = r(t) \quad \text{on } (-\infty, 0].$$

However, conditions (V)–(K_2) are weaker than the following well-known condition:

(VK) $V, K : \mathbb{R}^N \to \mathbb{R}_+$ are smooth and there exist positive numbers α, β, a, b, and c such that

$$\frac{a}{1+|x|^{\alpha}} \leq V(x) \leq b, \qquad 0 < K(x) \leq \frac{c}{1+|x|^{\beta}}, \qquad x \in \mathbb{R}^3,$$

which was firstly introduced in [2].

Before stating our results, let us recall some basic notions. Let

$$D^{1,p}(\mathbb{R}^N) = \left\{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N)\right\}$$

with the norm

$$\|u\|_{D^{1,p}} = \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

Since the potential may vanish at infinity, it is natural to use the following working space:

$$E = \left\{v \in D^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^3} V(x)|v|^p dx < \infty\right\}$$

endowed with the norm

$$\|v\| = \left(\int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|v|^p)dx\right)^{\frac{1}{2}}, \qquad v \in E.$$

Moreover, we define the weighted Lebesgue space

$$L_K^q(\mathbb{R}^N) = \left\{u : u \text{ is measurable on } \mathbb{R}^N \text{ and } \int_{\mathbb{R}^N} K(x)|u|^q dx < \infty\right\}$$

endowed with the norm

$$\|u\|_{L^q_K} = \left(\int_{\mathbb{R}^N} (K(x)|u|^q) dx \right)^{\frac{1}{q}},$$

for some $q \in (p, p^*)$.

By the conditions (V)–(K_2), in [17], the authors got the following proposition.

Proposition 1.1 (see [17, Lemma 2.2]). *Suppose that (V)–(K_2) are satisfied. Then E is compactly embedded in $L^q_K(\mathbb{R}^N)$ for all $q \in (p, p^*)$ if (1.5) holds.*

To resolve the equation (1.1), due to the appearance of the nonlocal term $\int_{\mathbb{R}^N} g^p(u)|\nabla u|^p dx$, the right working space seems to be

$$E_0 = \left\{ u \in E : \int_{\mathbb{R}^N} g^p(u)|\nabla u|^p dx < \infty \right\}.$$

But it is easy to see that E_0 is not a linear space under the assumption of (g). To overcome this difficulty, a variable substitution as follows: for any $v \in E$, Shen and Wang [40] make a change of variable as

$$u = G^{-1}(v) \quad \text{and} \quad G(u) = \int_0^u g(t) dt,$$

then

$$\int_{\mathbb{R}^N} g^p(u)|\nabla u|^p dx = \int_{\mathbb{R}^N} g^p(G^{-1}(v))|\nabla G^{-1}(v)|^p dx := |\nabla v|_p^p < +\infty, \qquad v \in E.$$

In such a case, we can deduce formally that the Euler–Lagrange functional associated with the equation (1.1) is

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} [g^p(u)|\nabla u|^p + V(x)u^p] dx - \int_{\mathbb{R}^N} K(x)F(u) dx - \frac{1}{p^*} \int_{\mathbb{R}^N} Q(x)|G(u)|^{p^*} dx.$$

Therefore, by this change of variables E can be used as the working space and the equation (1.1) in form can be transformed into

$$\begin{aligned} \mathcal{J}(v) = & \frac{1}{p} \int_{\mathbb{R}^N} (|\nabla v|^p + V(x)|G^{-1}(v)|^p) dx \\ & - \int_{\mathbb{R}^N} K(x)F(G^{-1}(v)) dx - \frac{1}{p^*} \int_{\mathbb{R}^N} Q(x)|v|^{p^*} dx, \qquad x \in \mathbb{R}^N. \end{aligned} \tag{1.6}$$

By the fact of g is a nondecreasing positive function that $|G^{-1}(v)| \leq |v|$. From this and our hypotheses, it is clear that $\mathcal{J}$ is well defined in E and $\mathcal{J} \in \mathcal{C}^1$.

Furthermore, one can easily derive that if $v \in \mathcal{C}^2(\mathbb{R}^N)$ is a critical point of (1.6), then $u = G^{-1}(v) \in \mathcal{C}^2(\mathbb{R}^N)$ is a classical solution to the equation (1.1). To obtain a critical point of (1.6), we only need to seek for the weak solution to the following equation

$$-\Delta_p v + V(x) \frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} = K(x) \frac{f(G^{-1}(v))}{g(G^{-1}(v))} + Q(x)|v|^{p^*-2}v, \qquad x \in \mathbb{R}^N. \tag{1.7}$$

Here, we say that $v \in E$ is a weak solution to the equation (1.7) if it holds that

$$\begin{aligned} \langle \mathcal{J}'(v), \varphi \rangle = & \int_{\mathbb{R}^N} |\nabla v|^{p-2} \nabla v \nabla \varphi + \int_{\mathbb{R}^N} V(x) \frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))} \varphi \\ & - \int_{\mathbb{R}^N} K(x) \frac{f(G^{-1}(v))}{g(G^{-1}(v))} \varphi - \int_{\mathbb{R}^N} Q(x)|v|^{p^*-2} v \varphi, \qquad \varphi \in E. \end{aligned}$$

Then it is standard to obtain that $v \in E$ is a weak solution to the equation (1.7) if and only if v is a critical point of the functional $\mathcal{J}$ in E. To sum up, it is sufficient to find a critical point of the functional $\mathcal{J}$ in E to achieve a classical solution to the equation (1.1).

Very recently, Song and Chen [45] studied the existence of weak solutions for (1.1) when V is a positive potential bounded away from zero and $h(x,u) = h(u)$ is a nonlinear term of subcritical type. Now, it is natural to ask whether problem (1.1) has the existence of positive solutions in the case where h satisfies critical growth? To the best of our knowledge, there are few results on such above questions in current literature. Actually, this is one of the motivations for us to study the existence of positive solutions of (1.1) with critical growth. Motivated by the above works, in this paper, our goal is to deal with critical growth case and give the existence of positive solutions of (1.1) with potential vanishing at infinity.

Now, we answer the question in the affirmative, which is given in the front of the article. Before stating our results, we need to give the following assumptions on f:

(f) $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$, $f(t) = 0$ for $t \le 0$ and f has a "quasicritical" growth, namely

$$\lim_{|t|\to\infty} \frac{f(t)}{g(t)|G(t)|^{p^*-1}} = 0.$$

(f_1) $\lim_{t\to 0^+} \frac{f(t)}{g(t)|G(t)|^{p-1}} = 0$ if (1.5) holds.

(f_2) There exists a $\mu \in (p, p^*)$ such that for any $t > 0$

$$0 < \mu g(t)F(t) \le G(t)f(t) \quad \text{for all } s \in \mathbb{R},$$

where $F(u) = \int_0^u f(t)dt$.

In addition, we also assume that

(Q_1) There is a point x_0, such that

$$Q(x_0) = \sup_{x\in\mathbb{R}^N} Q(x).$$

(Q_2) For x close to x_0, we have

$$Q(x) = Q(x_0) + O(|x - x_0|^p) \quad \text{as } x \to x_0.$$

Now, we state our main results by the following theorems.

Theorem 1.2. *Suppose that (g), (V)–(K_2), (Q_1)–(Q_2) and (f)–(f_2) are satisfied. Then problem (1.1) has at least one positive solution if either $N \ge p^2$ or $p < N < p^*$ and $\mu > p^* - \frac{p}{p-1}$.*

Applying Theorem 1.2 to the case when $Q(x) = 1$ and $p = 2$, we can get the following corollary.

Corollary 1.3. *Suppose that (g), (V)–(K_2) and (f)–(f_2) are satisfied. Then the following problem*

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = K(x)f(u) + g(u)|G(u)|^{2^*-2}G(u), \qquad x \in \mathbb{R}^N$$

has at least one positive solution if either $N \ge 4$ or $N = 3$ and $\mu > 2^* - 2$.

The paper is organized as follows. In Section 2, we prove a solution of (1.1) with critical growth and potential vanishing at infinity. In Appendix A, we give some useful lemmas, respectively.

In the following, we denote by $L^p(\mathbb{R}^N)$ the usual Lebesgue space with norms $\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{\frac{1}{p}}$, where $1 \le p < \infty$; for any $z \in \mathbb{R}^2$ and $R > 0$, $B_R(z) := \{x \in \mathbb{R}^2 : |x - z| < R\}$; C possibly denotes the different constants in different place.

Main results

In this section, we present some useful lemmas and corollaries. Now, let us recall the following lemma which has been proved in [30].

Lemma 2.1 ([30]). *For the function g, G, and G^{-1}, the following properties hold:*

(1) *the functions $G(\cdot)$ and $G^{-1}(\cdot)$ are strictly increasing and odd;*

(2) *$G(s) \le g(s)s$ for all $s \ge 0$; $G(s) \ge g(s)s$ for all $s \le 0$;*

(3) *$g(G^{-1}(s)) \ge g(0) = 1$ for all $s \in \mathbb{R}$;*

(4) *$\frac{G^{-1}(s)}{s}$ is decreasing on $(0, +\infty)$ and increasing on $(-\infty, 0)$;*

(5) *$|G^{-1}(s)| \le \frac{1}{g(0)}|s| = |s|$ for all $s \in \mathbb{R}$;*

(6) *$\frac{|G^{-1}(s)|}{g(G^{-1}(s))} \le \frac{1}{g^2(0)}|s| = |s|$ for all $s \in \mathbb{R}$;*

(7) *$\frac{G^{-1}(s)s}{g(G^{-1}(s))} \le |G^{-1}(s)|^2$ for all $s \in \mathbb{R}$;*

(8) *$\lim_{|s|\to 0} \frac{G^{-1}(s)}{s} = \frac{1}{g(0)} = 1$ and*

$$\lim_{|s|\to+\infty} \frac{G^{-1}(s)}{s} = \begin{cases} \frac{1}{g(\infty)}, & \text{if } g \text{ is bounded,} \\ 0, & \text{if } g \text{ is unbounded.} \end{cases}$$

The next two lemmas show that the functional $\mathcal{J}$ verifies the mountain pass geometry.

Lemma 2.2. *Suppose that (V)–(K_2), (Q_1)–(Q_2), and (f)–$(f_2)$are satisfied. Then there exist $\alpha, \rho > 0$ such that $\mathcal{J}(v) \ge \alpha$ for all $\|v\| = \rho$.*

Proof. It follows from (1.6) that

$$\begin{aligned} \mathcal{J}(v) &= \frac{1}{p}\int_{\mathbb{R}^N} [|\nabla v|^p + V(x)|G^{-1}(v)|^p]dx - \int_{\mathbb{R}^N} K(x)F(G^{-1}(v))dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|v^+|^{p^*}dx \\ &= \frac{1}{p}\|\nabla v\|_p^p - \int_{\mathbb{R}^N}\left(-\frac{1}{p}V(x)|G^{-1}(v)|^p + K(x)F(G^{-1}(v))\right)dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|v^+|^{p^*}dx \\ &\ge \frac{1}{p}\|\nabla v\|_p^p - \int_{\mathbb{R}^N}\left(-\frac{1}{p}V(x)|G^{-1}(v)|^p + K(x)F(G^{-1}(v))\right)dx \\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|v|^{p^*}dx. \end{aligned} \tag{2.1}$$

On the one hand, if (1.5) holds and let $A(x,s) := -\frac{1}{p}|G^{-1}(s)|^p + \frac{K(x)}{V(x)}F(G^{-1}(s))$, then by Lemma 2.1-(8), we have

$$\lim_{s\to 0^+} \frac{A(x,s)}{|s|^p} = \lim_{s\to 0}\left[-\frac{1}{p}\left|\frac{G^{-1}(v)}{s}\right|^p + \frac{K(x)}{V(x)}\frac{F(G^{-1}(s))}{|s|^p}\right] = -\frac{1}{p} \tag{2.2}$$

and

$$\lim_{s\to+\infty}\frac{A(x,s)}{|s|^{p^*}}=\lim_{s\to+\infty}\left[-\frac{1}{p}\left|\frac{G^{-1}(v)}{s}\right|^p\left(\frac{1}{|s|^{p^*-p}}\right)+\frac{K(x)}{V(x)}\frac{F(G^{-1}(s))}{|s|^{p^*}}\right]=0, \tag{2.3}$$

since

$$\lim_{|s|\to+\infty}\frac{G^{-1}(s)}{s}=\begin{cases}\frac{1}{g(\infty)}, & \text{if } g \text{ is bounded,}\\ 0, & \text{if } g \text{ is unbounded.}\end{cases}$$

Thus, by (2.2) and (2.3), for $\varepsilon>0$ sufficiently small, there exists a constant $C_\varepsilon>0$ such that

$$V(x)A(x,s)\le\left(-\frac{1}{p}+\varepsilon\right)V(x)|s|^p+C_\varepsilon V(x)|s|^{p^*}. \tag{2.4}$$

Then by Proposition 1.1, (2.4), (2.1) and (Q_1), we have

$$\begin{aligned}\mathcal{J}(v)&\ge\frac{1}{p}|\nabla v|^p-\left(-\frac{1}{p}+\varepsilon\right)\int_{\mathbb{R}^N}V(x)|v|^pdx-C_\varepsilon\int_{\mathbb{R}^N}V(x)|v|^{p^*}dx-\frac{1}{p^*}Q(x_0)\int_{\mathbb{R}^N}|v|^{p^*}dx\\&\ge\frac{1}{p}\|v\|^p-C\int_{\mathbb{R}^N}|v|^{p^*}dx-\frac{1}{p^*}Q(x_0)\int_{\mathbb{R}^N}|v|^{p^*}dx\\&\ge\left(\frac{1}{p}-\varepsilon C\right)\|v\|^p-C\|v\|^{p^*},\end{aligned}$$

since there exists $C>0$ such that $0<K(x)\le C$ and $0<V(x)\le C$. It follows that

$$\mathcal{J}(v)\ge C\|v\|^p-C\|v\|^{p^*}, \tag{2.5}$$

if we choose sufficiently small $\rho>0$, which implies that

$$\mathcal{J}(v)\ge C\rho^p-C\rho^{p^*}=:\alpha>0.$$

This completes the proof. □

Lemma 2.3. *Suppose that (V)–(K_2), (Q_1)–(Q_2), and (f)–$(f_2)$are satisfied. Then there exists $e\in E$ such that $\mathcal{J}(e)<0$ and $\|e\|>\rho$.*

Proof. For any fixed $v_0\in E$ with $v_0\ge0$ and $v_0\not\equiv0$, by (1.6) and Lemma 2.1-(5) , we have

$$\begin{aligned}\mathcal{J}(tv_0)&=\frac{1}{p}\int_{\mathbb{R}^N}[|t\nabla v_0|^p+V(x)|G^{-1}(tv_0)|^p]dx-\int_{\mathbb{R}^N}K(x)F(G^{-1}(tv_0))dx\\&\quad-\frac{1}{p^*}\int_{\mathbb{R}^N}Q(x)|tv_0|^{p^*}dx\\&\le\frac{t^p}{p}\|v_0\|^p-\frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N}Q(x)|v_0|^{p^*}dx\\&\to-\infty,\quad\text{as } t\to+\infty,\end{aligned}$$

which gives that the results hold if we take $e=tv_0$ with t sufficiently large. This completes the proof. □

As a consequence of Lemma 2.2 and Lemma 2.3, for the constant

$$c_0=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\mathcal{J}(\gamma(t))>0,$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0,1],E),\ \gamma(0)=0,\ \gamma(1)\neq 0,\ \mathcal{J}(\gamma(1))<0\}.$$

Note that from Lemma 2.3, $\Gamma \neq \emptyset$. By the Mountain Pass Theorem in [3], then we have the existence of sequence $\{v_n\} \subset E$ satisfying

$$\mathcal{J}(v_n) \to c_0 \quad \text{and} \quad \mathcal{J}'(v_n) \to 0 \qquad n \to +\infty. \tag{2.6}$$

The above sequence is called a $(PS)_{c_0}$ sequence for $\mathcal{J}$.

Lemma 2.4. *The sequence $\{v_n\}$ in (2.6) are satisfied. Then $\{v_n\}$ is bounded in E.*

Proof. Since $\{v_n\} \subset E$ is a $(PS)_{c_0}$ sequence for $\mathcal{J}$, we have

$$\begin{aligned}\mathcal{J}(v_n) = &\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v_n|^p + V(x)|G^{-1}(v_n)|^p)dx - \int_{\mathbb{R}^N} K(x)F(G^{-1}(v_n))dx \\ &- \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|v_n^+|^{p^*}dx \to c_0\end{aligned} \tag{2.7}$$

and for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\langle \mathcal{J}'(v_n),\varphi\rangle = &\int_{\mathbb{R}^N}|\nabla v_n|^{p-2}\nabla v_n \nabla\varphi + \int_{\mathbb{R}^N} V(x)\frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}\varphi \\ &- \int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v_n))}\varphi - \int_{\mathbb{R}^N} Q(x)|v_n^+|^{p^*-2}v_n^+\varphi = o(1)\|\varphi\|,\end{aligned} \tag{2.8}$$

as $n \to \infty$. Since $\mathcal{C}_0^\infty(\mathbb{R}^N)$ is dense in E, by choosing $\varphi = v_n$ we deduce that

$$\begin{aligned}\langle \mathcal{J}'(v_n),v_n\rangle = &\int_{\mathbb{R}^N}|\nabla v_n|^p + \int_{\mathbb{R}^N} V(x)\frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n - \int_{\mathbb{R}^N} K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v_n))}v_n \\ &- \int_{\mathbb{R}^N} Q(x)|v_n^+|^{p^*-2}v_n^+ v_n = o(1)\|v_n\|,\end{aligned}$$

as $n \to \infty$. It follows from (2.7), (2.8) and Lemma 2.1 that

$$\begin{aligned}&\mu c_0 + o(1) - \langle \mathcal{J}'(v_n),v_n\rangle \\ &\quad \geq \mu\mathcal{J}(v_n) - \langle \mathcal{J}'(v_n),v_n\rangle \\ &\quad = \frac{\mu-p}{p}\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^{p-2}\left[\frac{1}{p}\mu|G^{-1}(v_n)|^2 - \frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\right]dx \\ &\qquad - \int_{\mathbb{R}^N} K(x)\left(\mu F(G^{-1}(v_n)) - \frac{f(G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n\right)dx - \left(\frac{\mu}{p^*}-1\right)\int_{\mathbb{R}^N} Q(x)|v_n^+|^{p^*}dx \\ &\quad \geq \frac{\mu-p}{p}\left[\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^p dx\right].\end{aligned} \tag{2.9}$$

By (f_2), we have $F(s) \geq CG(s)^\mu \geq CG(s)^p$ for all $s \geq 1$. Then

$$\begin{aligned}&\int_{\{x:||G^{-1}(v_n)|>1\}} V(x)|v_n|^p dx \\ &\quad \leq C\int_{\{x:|G^{-1}(v_n)>1\}} K(x)F(G^{-1}(v_n))dx \\ &\quad \leq C\int_{\mathbb{R}^N} K(x)F(G^{-1}(v_n))dx + \frac{C}{p^*}\int_{\mathbb{R}^N} Q(x)|v_n^+|^{p^*}dx \\ &\quad \leq C\left[\frac{1}{p}\left(\int_{\mathbb{R}^N}|\nabla v_n|^p dx + \int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^p dx\right) - c_0 + o_n(1)\right].\end{aligned} \tag{2.10}$$

On the other hand, for the case $x \in \{x : |G^{-1}(v_n)| \le 1\}$ we know that

$$\begin{aligned}\frac{1}{g^p(1)}\int_{\{x:||G^{-1}(v_n)|\le 1\}} V(x)|v_n|^p dx &\le C\int_{\{x:|G^{-1}(v_n)\le 1\}} V(x)|G^{-1}(v_n)|^p dx \\ &\le C\int_{\mathbb{R}^N} V(x)|G^{-1}(v_n)|^p dx.\end{aligned} \tag{2.11}$$

Since $g(s)$ is nondecreasing. Combining (2.9), (2.10) with (2.11), we deduce that $\{v_n\}$ is bounded in E. This completes the proof. □

We are going to verify that the level value c_0 is in an interval where the (PS) condition holds. To this end, by the method developed by [8], we also introduce a well-known fact that the minimization problem

$$S = \inf\{|\nabla v|_p^p : v \in D^{1,p}(\mathbb{R}^N), |v|_{p^*} = 1\}$$

has a solution given by

$$v_\epsilon(x) = \frac{c(N,p)\epsilon^{(N-p)/(p^2-p)}}{(\epsilon^{p/(p-1)} + |x-x_0|^{p/(p-1)})^{(N-p)/p}}$$

and

$$|\nabla v_\epsilon|_p^p = |v_\epsilon|_{p^*}^{p^*} = S^{N/p}.$$

For small enough $R > 0$, define a cut-off function $\psi(x) \in C_0^\infty(\mathbb{R}^N)$ such that $\psi(|x|) = 1$ for $|x - x_0| \le R$, $\psi(|x|) \in (0,1)$ for $R < |x - x_0| < 2R$ and $|\nabla\psi| \le \frac{C}{R}$, and $\psi(|x|) = 0$ for $|x - x_0| \ge 2R$. Define

$$w_\epsilon(x) = \psi(x)v_\epsilon(x) \tag{2.12}$$

and

$$\sigma_\epsilon(x) = w_\epsilon(x)\left[\int_{\mathbb{R}^N} Q(x)w_\epsilon^{p^*}(x)dx\right]^{-\frac{1}{p^*}}. \tag{2.13}$$

Denote

$$\begin{aligned}V_{max} &:= \max_{x\in B_{2R}(x_0)} V(x),\\ K_{min} &:= \min_{x\in B_{2R}(x_0)} K(x).\end{aligned}$$

Similar to the discussion of [17,22], by $\partial v_\epsilon/\partial\overrightarrow{n} \le 0$, we have that

$$\int_{B_R(x_0)} |\nabla w_\epsilon|^p dx = \int_{B_R(x_0)} |\nabla v_\epsilon|^p dx \le \int_{B_R(x_0)} |v_\epsilon|^{p^*} dx,$$

and by the assumption (Q_2) we also have

$$Q(x_0)\int_{B_R(x_0)} |\nabla v_\epsilon|^{p^*} dx \le Q(x)\int_{B_R(x_0)} |\nabla v_\epsilon|^{p^*} dx + O(\epsilon^p).$$

Simple calculations as [16] gives that

$$\int_{\mathbb{R}^N\setminus B_R(x_0)} |v_\epsilon|^{p^*} dx = O(\epsilon^{N/p-1}),$$

$$A_\epsilon := \int_{\mathbb{R}^N \setminus B_R(x_0)} |\nabla w_\epsilon|^{p^*} dx = O(\epsilon^{(N-p)/(p-1})$$

and

$$\int_{\mathbb{R}^N} |\sigma_\epsilon|^2 dx = \begin{cases} k\epsilon^p + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N > p^2, \\ k\epsilon^p |\ln \epsilon| + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N = p^2, \\ O(\epsilon^{(N-p)/(p-1)}), & \text{if } N < p^2, \end{cases} \tag{2.14}$$

as $\epsilon \to 0$, where k is a positive constant. Therefore, we can get

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_\epsilon|^p dx &= \int_{B_R(x_0)} |\nabla w_\epsilon|^p dx + A_\epsilon \\ &\le \int_{B_R(x_0)} |v_\epsilon|^{p^*} dx + A_\epsilon \\ &\le S \left[\int_{B_R(x_0)} |v_\epsilon|^{p^*} dx \right]^{\frac{p}{p^*}} + A_\epsilon \\ &\le S(\|Q\|_{L^\infty(\mathbb{R}^N)})^{-\frac{p}{p^*}} \left[\int_{B_R(x_0)} Q(x) |v_\epsilon|^{p^*} dx \right]^{\frac{p}{p^*}} + O(\epsilon^p) + O(\epsilon^{(N-p)/(p-1)}). \end{aligned}$$

Set $V_\epsilon \equiv \int_{\mathbb{R}^N} |\nabla \sigma_\epsilon|^p dx$, since for small $\epsilon > 0$, say $\epsilon \le \epsilon_0$, it is easy to see that

$$\int_{B_R(x_0)} Q(x) |w_\epsilon|^{p^*} dx \ge C_{\epsilon_0}$$

for some positive constant C_{ϵ_0}. The definition of V_ϵ and the last two inequalities imply that

$$V_\epsilon \le S(\|Q\|_{L^\infty(\mathbb{R}^N)})^{-\frac{p}{p^*}} + O(\epsilon^p) + O(\epsilon^{(N-p)/(p-1)}). \tag{2.15}$$

Lemma 2.5. *Suppose that (V)–(K_2), (Q_1)–(Q_2), and (f)–$(f_2)$are satisfied. Then there exists $v_0 \in E \setminus \{0\}$ such that*

$$0 < \sup_{t \ge 0} \mathcal{J}(tv_0) < \frac{1}{N} S^{N/p} [\|Q\|_{L^\infty(\mathbb{R}^N)}]^{\frac{p-N}{p}} \tag{2.16}$$

if either $N \ge p^2$ or $p < N < p^2$ and $\mu > p^* - \frac{p}{p-1}$.

Proof. Firstly, we claim that for $\epsilon > 0$ small enough, there exists a constant $t_\epsilon > 0$ such that

$$\mathcal{J}(t_\epsilon \sigma_\epsilon) = \max_{t \ge 0} \mathcal{J}(t\sigma_\epsilon)$$

and

$$0 < A_1 < t_\epsilon < A_2 < +\infty \quad \text{for all } \epsilon > 0 \text{ small enough,}$$

where A_1 and A_2 are positive constants independent of ϵ.

By (f)–(f_1), for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|f(t)| \le \delta g(t) |G(t)|^{p-1} + C_\delta g(t) |G(t)|^{p^*-1}. \tag{2.17}$$

Now, we consider

$$\begin{aligned} \mathcal{J}(t\sigma_\epsilon) &= \frac{1}{p} \int_{\mathbb{R}^N} [t^p |\nabla \sigma_\epsilon|^p + V(x) |G^{-1}(t\sigma_\epsilon)|^p] dx - \int_{\mathbb{R}^N} K(x) F(G^{-1}(t\sigma_\epsilon)) dx \\ &\quad - \frac{1}{p^*} \int_{\mathbb{R}^N} Q(x) |t\sigma_\epsilon^+|^{p^*} dx \\ &\le \frac{t^p}{p} \|\sigma_\epsilon\|^p - \int_{\mathbb{R}^N} K(x) F(G^{-1}(t\sigma_\epsilon)) dx - \frac{t^{p^*}}{p^*} \int_{\mathbb{R}^N} Q(x) |\sigma_\epsilon|^{p^*} dx \\ &\to -\infty, \quad \text{as } t \to \infty. \end{aligned}$$

Clearly, $\lim_{t\to+\infty}\mathcal{J}(t\sigma_\epsilon)=-\infty$ for all $\epsilon>0$. Since $\mathcal{J}(0)=0$ and $\mathcal{J}(t\sigma_\epsilon)=-\infty$, there exists $t_\epsilon>0$ such that

$$\mathcal{J}(t_\epsilon\sigma_\epsilon)=\max_{t\geq 0}\mathcal{J}(t\sigma_\epsilon)\quad\text{and}\quad \left.\frac{d\mathcal{J}(t\sigma_\epsilon)}{dt}\right|_{t=t_\epsilon}=0.$$

Thus we have

$$\begin{aligned} t_\epsilon^{p-1}\int_{B_{2R}(x_0)}|\nabla\sigma_\epsilon|^p dx+\int_{B_{2R}(x_0)}V(x)\frac{|G^{-1}(t_\epsilon\sigma_\epsilon)|^{p-2}G^{-1}(t_\epsilon\sigma_\epsilon)}{g(G^{-1}(t_\epsilon\sigma_\epsilon))}\sigma_\epsilon dx\\ =\int_{B_{2R}(x_0)}K(x)\frac{f(G^{-1}(t_\epsilon\sigma_\epsilon))}{g(G^{-1}(t_\epsilon\sigma_\epsilon))}\sigma_\epsilon dx+t_\epsilon^{p^*-1}\int_{B_{2R}(x_0)}Q(x)|\sigma_\epsilon|^{p^*}dx. \end{aligned}\tag{2.18}$$

On the one hand, if there is a sequence $t_{\epsilon_n}\to+\infty$, as $\epsilon_n\to 0^+$, by the above equality, we get

$$\begin{aligned} t_{\epsilon_n}^{p-1}\int_{B_{2R}(x_0)}|\nabla\sigma_{\epsilon_n}|^p dx+\int_{B_{2R}(x_0)}V(x)\frac{|G^{-1}(t_{\epsilon_n}\sigma_{\epsilon_n})|^{p-2}G^{-1}(t_{\epsilon_n}\sigma_{\epsilon_n})}{g(G^{-1}(t_\epsilon\sigma_{\epsilon_n}))}\sigma_{\epsilon_n}dx\\ \geq t_{\epsilon_n}^{p^*-1}\int_{B_{2R}(x_0)}Q(x)|\sigma_{\epsilon_n}|^{p^*}dx. \end{aligned}$$

Hence by Lemma 2.1-(7), we get

$$t_{\epsilon_n}^{p-1}\left[\int_{B_{2R}(x_0)}|\nabla\sigma_{\epsilon_n}|^p dx+\int_{B_{2R}(x_0)}V(x)|\sigma_{\epsilon_n}|^p dx\right]\geq t_{\epsilon_n}^{p^*-1}\int_{B_{2R}(x_0)}Q(x)|\sigma_{\epsilon_n}|^{p^*}dx,$$

which gives a contradiction since $p^*>p$.

On the other hand, we suppose there is a sequence $t'_{\epsilon_n}\to 0$ as $\epsilon_n\to 0^+$. If (1.5) holds, by (2.17), for any $\delta>0$ there exists $C_\delta>0$ such that

$$\begin{aligned} \int_{\mathbb{R}^N}K(x)\frac{f(G^{-1}(t_{\epsilon_n}\sigma_{\epsilon_n}))}{g(G^{-1}(t_{\epsilon_n}\sigma_{\epsilon_n}))}\sigma_{\epsilon_n}dx &\leq \delta {t'}_{\epsilon_n}^{p-1}\int_{\mathbb{R}^N}K(x)|\sigma_{\epsilon_n}|^p dx+C_\delta(t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N}K(x)|\sigma_{\epsilon_n}|^{p^*}dx\\ &\leq \delta C{t'}_{\epsilon_n}^{p-1}\int_{\mathbb{R}^N}(|\nabla\sigma_{\epsilon_n}|^p+V(x)|\sigma_{\epsilon_n}|^p)\,dx\\ &\quad +C_\delta(t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N}K(x)|\sigma_{\epsilon_n}|^{p^*}dx. \end{aligned}$$

By (2.18), we have

$$\begin{aligned} {t'}_{\epsilon_n}^{p-1}\left(\int_{\mathbb{R}^N}|\nabla\sigma_{\epsilon_n}|^p dx\right)+\int_{\mathbb{R}^N}V(x)\frac{|G^{-1}(t'_{\epsilon_n}\sigma_{\epsilon_n})|^{p-2}G^{-1}(t'_{\epsilon_n}\sigma_{\epsilon_n})}{(t'_{\epsilon_n}\sigma_{\epsilon_n})g(G^{-1}(t'_\epsilon\sigma_{\epsilon_n}))}\sigma_{\epsilon_n}^2 dx\\ \leq \delta C{t'}_{\epsilon_n}^{p-1}\int_{\mathbb{R}^N}(|\nabla\sigma_{\epsilon_n}|^p+V(x)|\sigma_{\epsilon_n}|^p)\,dx+(t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N}Q(x)|\sigma_{\epsilon_n}|^{p^*}dx\\ +C_\delta(t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N}K(x)|\sigma_{\epsilon_n}|^{p^*}dx. \end{aligned}$$

Thus taking $\delta=\frac{1}{2C}$, we have

$$\begin{aligned} {t'}_{\epsilon_n}^{p-1}\left(\frac{1}{p}\int_{\mathbb{R}^N}|\nabla\sigma_{\epsilon_n}|^p dx+\int_{\mathbb{R}^N}V(x)\left[\frac{|G^{-1}(t'_{\epsilon_n}\sigma_{\epsilon_n})|^{p-2}G^{-1}(t'_{\epsilon_n}\sigma_{\epsilon_n})}{|t'_{\epsilon_n}\sigma_{\epsilon_n}|^{p-1}g(G^{-1}(t'_\epsilon\sigma_{\epsilon_n}))}-\frac{1}{p}\right]|\sigma_{\epsilon_n}|^p dx\right)\\ \leq (t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N}Q(x)|\sigma_{\epsilon_n}|^{p^*}dx+C_\delta(t'_{\epsilon_n})^{p^*-1}\int_{\mathbb{R}^N}K(x)|\sigma_{\epsilon_n}|^{p^*}dx. \end{aligned}\tag{2.19}$$

When $t'_{\epsilon_n} \to 0$, we have

$$\frac{G^{-1}(t'_{\epsilon_n}\sigma_{\epsilon_n})}{|t'_{\epsilon_n}\sigma_{\epsilon_n}|^{p-1}g(G^{-1}(t'_{\epsilon}\sigma_{\epsilon_n}))} > \frac{1}{p}.$$

Therefore (2.19) is also impossible because of $p^* > p$. So we complete the proof of our claim.

Since $0 < A_1 < t_\epsilon < A_2 < +\infty$ for ϵ small enough, together with the definition of $V_{\max}$ and $K_{\min}$, we know that

$$\begin{aligned}
\mathcal{J}(t\sigma_\epsilon) &= \frac{1}{p}\int_{\mathbb{R}^N}(t^p|\nabla\sigma_\epsilon|^p + V(x)|G^{-1}(t\sigma_\epsilon)|^p)dx - \int_{\mathbb{R}^N}K(x)F(G^{-1}(t\sigma_\epsilon))dx \\
&\quad - \frac{t^{p^*}}{p^*}\int_{\mathbb{R}^N}Q(x)|\sigma_\epsilon|^{p^*}dx \\
&= \frac{t^p}{p}V_\epsilon + \frac{1}{p}\int_{B_{2R}(x_0)}V(x)|G^{-1}(t\sigma_\epsilon)|^p dx - \int_{B_{2R}(x_0)}K(x)F(G^{-1}(t\sigma_\epsilon))dx - \frac{t^{p^*}}{p^*} \\
&\le \frac{t_\epsilon^p}{p}V_\epsilon + \frac{1}{p}\int_{B_{2R}(x_0)}V(x)|G^{-1}(t_\epsilon\sigma_\epsilon)|^p dx - \int_{B_{2R}(x_0)}K(x)F(G^{-1}(t_\epsilon\sigma_\epsilon))dx - \frac{t_\epsilon^{p^*}}{p^*} \\
&\le \frac{t_\epsilon^p}{p}V_\epsilon + \frac{1}{p}V_{\max}\int_{B_{2R}(x_0)}|G^{-1}(t_\epsilon\sigma_\epsilon)|^p dx - K_{\min}\int_{B_{2R}(x_0)}F(G^{-1}(t_\epsilon\sigma_\epsilon))dx - \frac{t_\epsilon^{p^*}}{p^*} \\
&\le \frac{t_\epsilon^p}{p}V_\epsilon + \frac{t_\epsilon^p}{p}V_{\max}\int_{B_{2R}(x_0)}|\sigma_\epsilon|^p dx - K_{\min}\int_{B_{2R}(x_0)}F(G^{-1}(t_\epsilon\sigma_\epsilon))dx - \frac{t_\epsilon^{p^*}}{p^*}.
\end{aligned}$$

By virtue of $\frac{t^p}{p}V_\epsilon - \frac{t^{p^*}}{p^*} \le \frac{1}{N}V_\epsilon^{N/p}$ for all $t \ge 0$, the estimate (2.15) on V_ϵ and the above inequality imply that

$$\begin{aligned}
\sup_{t\ge0}\mathcal{J}(t\sigma_\epsilon) &= \mathcal{J}(t_\epsilon\sigma_\epsilon) \\
&\le \frac{1}{N}S^{N/p}\left[\|Q\|_{L^\infty(\mathbb{R}^N)}\right]^{-\frac{N-p}{p}} + O(\epsilon^p) + O(\epsilon^{(N-p)/(p-1)}) \\
&\quad + \frac{t_\epsilon^p}{p}V_{\max}\int_{B_{2R}(x_0)}|\sigma_\epsilon|^p dx - K_{\min}\int_{B_{2R}(x_0)}F(G^{-1}(t_\epsilon\sigma_\epsilon))dx \\
&\le \frac{1}{N}S^{N/p}\left[\|Q\|_{L^\infty(\mathbb{R}^N)}\right]^{-\frac{N-p}{p}} - K_{\min}\int_{B_{2R}(x_0)}F(G^{-1}(t_\epsilon\sigma_\epsilon))dx + O(\epsilon^p) \\
&\quad + O(\epsilon^{(N-p)/(p-1)}) + \begin{cases} k\epsilon^p + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N > p^2, \\ k\epsilon^p|\ln\epsilon| + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N = p^2, \\ O(\epsilon^(N-p)/(p-1)), & \text{if } N < p^2. \end{cases}
\end{aligned} \tag{2.20}$$

By (f_2), we have $F(s) \ge CG(s)^\mu$ for all $s > 0$. Therefore

$$\int_{B_{2R}(x_0)}F(G^{-1}(t_\epsilon\sigma_\epsilon))dx \ge C\int_{B_{2R}(x_0)}(t_\epsilon\sigma_\epsilon)^\mu dx \ge CA_1^\mu\int_{B_R(x_0)}(\sigma_\epsilon)^\mu dx.$$

It follows from (2.20), the above inequality and the definition of σ_ϵ that

$$\begin{aligned}\sup_{t\geq 0}\mathcal{J}(t\sigma_\epsilon) \leq & \frac{1}{N}S^{N/p}\left[\|Q\|_{L^\infty(\mathbb{R}^N)}\right]^{-\frac{N-p}{p}} - CA_1^\mu\int_{B_R(x_0)}(\sigma_\epsilon)^\mu dx + O(\epsilon^p)\\ & + O(\epsilon^{(N-p)/(p-1)}) + \begin{cases} k\epsilon^p + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N > p^2,\\ k\epsilon^p|\ln\epsilon| + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N = p^2,\\ O(\epsilon^(N-p)/(p-1)), & \text{if } N < p^2,\end{cases}\\ \leq & \frac{1}{N}S^{N/p}\left[\|Q\|_{L^\infty(\mathbb{R}^N)}\right]^{-\frac{N-p}{p}} - C\epsilon^{N-\frac{N-p}{p}\mu}\int_0^{\frac{R}{\epsilon}}\frac{r^{N-1}}{(1+r^{p/(p-1)})^{\frac{\mu(N-p)}{p}}}dr\\ & + O(\epsilon^p) + O(\epsilon^{(N-p)/(p-1)}) + \begin{cases} k\epsilon^p + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N > p^2,\\ k\epsilon^p|\ln\epsilon| + O(\epsilon^{(N-p)/(p-1)}), & \text{if } N = p^2,\\ O(\epsilon^(N-p)/(p-1)), & \text{if } N < p^2.\end{cases}\end{aligned} \tag{2.21}$$

For $N \geq p^2$ and $\mu \in (p, p^*)$, there exists a constant $C > 0$ such that

$$\int_0^\infty \frac{r^{N-1}}{(1+r^p/(p-1))^{\frac{\mu(N-p)}{p}}}dr \geq C > 0.$$

If $N \geq p^2$ and $\mu \in (p, p^*)$, then we have

$$N - \frac{N-p}{p}\mu < p \leq N - p. \tag{2.22}$$

Combined (2.21) with (2.22), when $\epsilon \to 0$, we have

$$\sup_{t\geq 0}\mathcal{J}(t\sigma_\epsilon) < \frac{1}{N}S^{N/p}\left[\|Q\|_{L^\infty(\mathbb{R}^N)}\right]^{-\frac{N-p}{p}}. \tag{2.23}$$

If $p < N < p^2$ and $\mu \in (p^* - p/(p-1), p^*)$, then we know that (2.23) also holds. Then we can get the following inequality

$$N - \frac{N-p}{p}\mu < N - p < p.$$

Hence inequality (2.23) also follows from (2.21) if we choose ϵ small enough. Thus we can imply that the inequality (2.16) holds by taking $u_0 = \sigma_\epsilon$ for sufficiently small ϵ. □

Next, we will prove the main results in this paper.

Proof of Theorem 1.2. By Lemma 2.2 and Lemma 2.3, all conditions of Mountain Pass Lemma in [3] are satisfied. Let $\{v_n\}$ be a $(PS)_{c_0}$ sequence of $\mathcal{J}$. Then

$$\begin{aligned}\mathcal{J}(v_n) = & \frac{1}{p}\int_{\mathbb{R}^N}[|\nabla v_n|^p + V(x)|G^{-1}(v_n)|^p]dx - \int_{\mathbb{R}^N}K(x)F(G^{-1}(v_n))dx\\ & - \frac{1}{p^*}\int_{\mathbb{R}^N}Q(x)|v_n^+|^{p^*}dx = c_0 + o_n(1)\end{aligned} \tag{2.24}$$

and

$$\begin{aligned}\langle\mathcal{J}'(v_n), v_n\rangle = & \int_{\mathbb{R}^N}\left[|\nabla v_n|^p + V(x)\frac{|G^{-1}(v_n)|^{p-2}G^{-1}(v_n)}{g(G^{-1}(v_n))}v_n\right] - \int_{\mathbb{R}^N}K(x)\frac{f(G^{-1}(v))}{g(G^{-1}(v_n))}v_n\\ & - \int_{\mathbb{R}^N}Q(x)|v_n^+|^{p^*-2}v_n^+ v_n dx = o_n(1)\|v_n\|.\end{aligned}$$

From Lemma 2.4, we know that $\{v_n\}$ is bounded in E. Passing to sequence, there exists a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{in } E \\ v_n &\to v \quad \text{in } L^q_K(\mathbb{R}^N) \text{ for } p<q<p^*, \\ v_n &\to v \quad a.e. \text{ in } \mathbb{R}^N. \end{aligned} \tag{2.25}$$

Let

$$\tilde{f}(x,v)=\frac{f(G^{-1}(v))}{g(G^{-1}(v))}+\frac{V(x)}{K(x)}|v|^{p-2}v-\frac{V(x)}{K(x)}\frac{|G^{-1}(v)|^{p-2}G^{-1}(v)}{g(G^{-1}(v))},$$

and

$$\tilde{F}(x,v)=\int_0^v\tilde{f}(x,v)dx=F(G^{-1}(v))+\frac{1}{p}\frac{V(x)}{K(x)}|v|^p-\frac{1}{p}\frac{V(x)}{K(x)}|G^{-1}(v)|^p,$$

then

$$\mathcal{J}(v)=\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v|^p+V(x)|v|^p)dx-\int_{\mathbb{R}^N}K(x)\tilde{F}(x,v)dx-\frac{1}{p^*}\int_{\mathbb{R}^N}Q(x)|v^+|^{p^*}dx.$$

Similar to [43], we can verify that

$$\lim_{s\to0}\frac{\tilde{F}(x,s)}{|s|^p}=0,\quad \lim_{s\to\infty}\frac{\tilde{F}(x,s)}{|s|^{p^*}}=0,\quad \lim_{s\to0}\frac{\tilde{f}(x,s)}{|s|^{p-1}}=0,\quad \lim_{s\to\infty}\frac{\tilde{f}(x,s)}{|s|^{p^*-1}}=0. \tag{2.26}$$

By Corollary A.3, we can get

$$\begin{aligned} \lim_{n\to\infty}\int_{\mathbb{R}^N}K(x)\tilde{F}(x,G^{-1}(v_n))&=\int_{\mathbb{R}^N}K(x)\tilde{F}(x,G^{-1}(v)),\\ \lim_{n\to\infty}\int_{\mathbb{R}^N}K(x)\frac{\tilde{f}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n&=\int_{\mathbb{R}^N}K(x)\frac{\tilde{f}(x,G^{-1}(v))}{g(G^{-1}(v))}v. \end{aligned} \tag{2.27}$$

Since $\mathcal{J}'(v_n)\to0$, by (2.27), we can get

$$\int_{\mathbb{R}^N}(|\nabla v|^p+V(x)|v|^p)dx-\int_{\mathbb{R}^N}K(x)\tilde{f}(x,v)vdx-\int_{\mathbb{R}^N}Q(x)|v^+|^{p^*}dx=0.$$

Denote $\vartheta_n=v_n-v$, then by (2.6) and the Brézis–Lieb Lemma in [7], we have

$$\mathcal{J}(v)+\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla\vartheta_n|^p+V(x)|\vartheta_n|^p)dx-\frac{1}{p^*}\int_{\mathbb{R}^N}Q(x)|\vartheta_n|^{p^*}dx=c_0+o(1) \tag{2.28}$$

and

$$\int_{\mathbb{R}^N}(|\nabla\vartheta_n|^p+V(x)|\vartheta_n|^p)dx-\int_{\mathbb{R}^N}Q(x)|\vartheta_n|^{p^*}dx=o(1).$$

Without loss of generality we can suppose

$$\int_{\mathbb{R}^N}(|\nabla\vartheta_n|^p+V(x)|\vartheta_n|^p)dx\to l\quad\text{as } n\to\infty \tag{2.29}$$

and then we have

$$\int_{\mathbb{R}^N}Q(x)|\vartheta_n|^{p^*}dx\to l,\qquad n\to\infty. \tag{2.30}$$

Moreover, by Sobolev's inequality, we know that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla \vartheta_n|^p dx &\geq S\left(\int_{\mathbb{R}^N} |\vartheta_n|^{p^*} dx\right)^{p/p^*}\\ &\geq S\left[\|Q\|_{L^\infty(\mathbb{R}^N)}\right]^{-p/p^*}\left(\int_{\mathbb{R}^N} Q(x)|\vartheta_n|^{p^*} dx\right)^{p/p^*}.\end{aligned} \tag{2.31}$$

Using (2.29), (2.30), (2.31), if $l > 0$, then we have

$$l \geq S^{N/p}\left[\|Q\|_{L^\infty(\mathbb{R}^N)}\right]^{\frac{p-N}{p}}.$$

By (2.28), we have

$$\mathcal{J}(v) = \left(c_0 - \frac{1}{p} - \frac{1}{p^*}\right) l \leq c_0 - \frac{1}{N} S^{N/p}\left[\|Q\|_{L^\infty(\mathbb{R}^N)}\right]^{\frac{p-N}{p}} < 0.$$

On the other hand, by (f_2), we have

$$\begin{aligned}\mathcal{J}(v) &= \frac{1}{p}\int_{\mathbb{R}^N} [|\nabla v|^p + V(x)|G^{-1}(v)|^p] dx - \int_{\mathbb{R}^N} K(x) F(G^{-1}(v)) dx\\ &\quad - \frac{1}{p^*}\int_{\mathbb{R}^N} Q(x)|v|^{p^*} dx\\ &= \frac{1}{p}\int_{\mathbb{R}^N} V(x)|G^{-1}(v)|^{p-2}\left[|G^{-1}(v)|^2 - \frac{G^{-1}(v)v}{g(G^{-1}(v))}\right] dx + \left(\frac{1}{p} - \frac{1}{p^*}\right)\int_{\mathbb{R}^N} Q(x)|v|^{p^*} dx\\ &\quad - \int_{\mathbb{R}^N} K(x)\left[F(G^{-1}(v)) - \frac{f(G^{-1}(v))}{g(G^{-1}(v))} v\right] dx\\ &\geq 0,\end{aligned}$$

which is a contradiction. It shows that $l = 0$. By the definition of ϑ_n we conclude that $\mathcal{J}$ satisfies $(PS)_{c_0}$ condition and thus

$$\mathcal{J}(v) = c_0 > 0 \quad \text{and} \quad \mathcal{J}'(v) = 0.$$

which gives that $u = G^{-1}(v)$ is a positive solution of (1.1). This completes the proof. □

Appendix A

In this part, we want to give some very useful lemmas.

Lemma A.1 ([17, Lemma 2.3]). *Suppose that (V)–(K_2) hold, and $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function, which satisfies the following conditions:*

(h_1) *h has a quasicritical growth, that is,* $\lim_{|s|\to+\infty} \frac{h(x,s)}{|s|^{p^*-1}} = 0$;

(h_2) *if (1.5) holds, then h satisfies* $\lim_{s\to 0} \frac{h(x,s)}{|s|^p} = 0$.

If a sequence $\{v_n\}$ converges weakly to v in E, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} KH(x, v_n) = \int_{\mathbb{R}^N} KH(x, v),$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^N} Kh(x, v_n)v_n = \int_{\mathbb{R}^N} Kh(x, v)v,$$

where $H(x,s) = \int_0^s h(x,t)dt$ for all $s \in \mathbb{R}$.

Lemma A.2. *Under the assumptions of Lemma A.1, if $v_n \rightharpoonup v$ in E, then for each $\phi \in E$ it holds that*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K\left[h(x,v_n)-h(x,v)\right]\phi dx = 0. \tag{A.1}$$

Proof. Motivated by [1,31,46], since $v_n \rightharpoonup v$ in E and $E \hookrightarrow L^{p^*}(\mathbb{R}^N)$, then there exists $M > 0$ such that

$$\|v_n\|, \|v\| \le M \qquad |v|_{p^*}^{p^*} \le M, \qquad n \in \mathbb{N}.$$

Now, we consider the case that (V)–(K_1), (1.5), (h_1) and (h_2) hold. it follows from (h_1) and (h_2) that for any $\varepsilon > 0$ and $q \in (p, p^*)$ there exists $C_\varepsilon > 0$ such that

$$h(x,s) \le \varepsilon(|s|^{p-1} + |s|^{p^*-1}) + C_\varepsilon |s|^{q-1}, \qquad s \in \mathbb{R}. \tag{A.2}$$

By (1.5), we have that

$$K(x)h(x,s) \le \varepsilon(|K/V|_\infty V(x)|s|^p + |K|_\infty |s|^{p^*}) + C_\varepsilon K(x)|s|^{q-1}, \qquad x \in \mathbb{R}^N \text{ and } s \in \mathbb{R}. \tag{A.3}$$

According to Proposition 1.1, it holds that $\int_{\mathbb{R}^N} K|v_n|^q \to \int_{\mathbb{R}^N} K|v|^q$ as $n \to \infty$. Then there exists $R = R_\varepsilon$ large enough such that

$$\int_{B_R^c} K|v_n|^q, \int_{B_R^c} K|v|^q \le \left(\frac{\varepsilon}{C_\varepsilon}\right)^{q/(q-1)}, \qquad n \in \mathbb{N}. \tag{A.4}$$

where $B_R^c = \{x \in \mathbb{R}^N : |x| \ge R\}$. Hence, we can derive from (A.3), the Hölder inequality, (A.2) and (A.4) that

$$\begin{aligned}\int_{B_R^c} K|h(x,v_n)\phi| &\le \int_{B_R^c} \varepsilon(|K/V|_\infty V(x)|v_n|^{p-1} + |K|_\infty |v_n|^{p^*-1}) + C_\varepsilon \int_{B_R^c} K(x)|v_n|^{q-1}|\phi| \\ &\le \varepsilon\left[|K/V|_\infty \|v_n\|\|\phi\| + |K|_\infty |v_n|_{p^*}^{p^*-1}|\phi|_{p^*}\right] + C_\varepsilon \left(\int_{B_R^c} K(x)|v_n|^q\right)^{(q-1)/q} |\phi|_{L_K^q} \\ &\le C\varepsilon. \end{aligned} \tag{A.5}$$

where C is independent of ε. Similarly, it holds that for some constant C_2 independent of ε,

$$\int_{B_R^c} Kh(x,v)\phi \le C\varepsilon. \tag{A.6}$$

Next, we only need to prove that

$$\lim_{n\to\infty}\int_{B_R} Kh(x,v_n)\phi = \int_{B_R} Kh(x,v)\phi. \tag{A.7}$$

In fact, since $v_n \rightharpoonup v$ in E, then exists a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) such that $v_n(x) \to v(x)$ for a.e. $x \in \mathbb{R}^N$. Thus $h(x,v_n) \to h(x,v)$ for a.e. $x \in \mathbb{R}^N$. Moreover, it follows from (A.3) that $\{h(x,v_n)\}$ is bounded in $L^{p^*/(p^*-p)}(B_R)$. Hence $h(x,v_n) \rightharpoonup h(v)$ in $L^{p^*/(p^*-p)}(B_R)$ as $n \to \infty$, and (A.7) holds as a consequence of the fact that $K\phi \in L^{p^*}(\mathbb{R}^N)$. Thus we can get that

$$\lim_{n\to\infty}\int_{B_R} Kh(x,v_n)\phi = \int_{B_R} Kh(x,v)\phi.$$

Combining (A.5), (A.6) with (A.7), (A.1) holds. This completes the proof. □

Corollary A.3. *Under the assumptions of Lemma A.1, if $v_n \rightharpoonup v$ in E, then it holds that*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K\tilde{F}(x,G^{-1}(v_n)) = \int_{\mathbb{R}^N} K\tilde{F}(x,G^{-1}(v)), \tag{A.8}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K\frac{\tilde{f}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}v_n = \int_{\mathbb{R}^N} K\frac{\tilde{f}(x,G^{-1}(v))}{g(G^{-1}(v))}v, \tag{A.9}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} K\frac{\tilde{f}(x,G^{-1}(v_n))}{g(G^{-1}(v_n))}\phi = \int_{\mathbb{R}^N} K\frac{\tilde{f}(x,G^{-1}(v))}{g(G^{-1}(v))}\phi, \qquad \phi \in E. \tag{A.10}$$

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