On the existence and multiplicity of solutions for nonlinear Klein–Gordon–Maxwell systems

Lixia Wang, **Chunlian Xiong**✉ and **Pingping Zhao**

Tianjin Chengjian University, 26 Jinjing Road, Tianjin, 300384, China

Abstract. In this paper, we study the existence and multiplicity solutions for the following Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $\omega > 0$ is a constant and the nonlinearity $f(x,u)$ is either asymptotically linear in u at infinity or the primitive of $f(x,u)$ is of 4-superlinear growth in u at infinity. Under some suitable assumptions, the existence and multiplicity of solutions are proved by using the Mountain Pass theorem and the fountain theorem, respectively.

Keywords: Klein–Gordon–Maxwell system, sign-changing potential, 4-superlinear, asymptotically linear.

2020 Mathematics Subject Classification: 35B33, 35J65, 35Q55.

1 Introduction and main results

In this paper we consider the following nonlinear Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x,u), & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{KGM}$$

where $\omega > 0$ is a constant. We are interested in the existence and multiplicity solutions of system (KGM) when the nonlinearity $f(x,u)$ is either asymptotically linear in u at infinity or the primitive of $f(x,u)$ is of 4-superlinear growth at infinity.

Such system has been firstly studied by Benci and Fortunato [6] as a model which describes nonlinear Klein–Gordon fields in three dimensional space interacting with the electrostatic field. For more details on the physical aspects of the problem we refer the readers to see [7] and the references therein.

✉Corresponding author. Email: xiongcltg@126.com

In 2002, Benci and Fortunato [7] first studied the following Klein–Gordon–Maxwell system

$$\begin{cases} -\Delta u + [m^2 - (\omega + \phi)^2]\phi u = f(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi + \phi u^2 = -\omega u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

with the pure power type nonlinearity, i.e. $f(x, u) = |u|^{q-2}u$, where ω and m are constants. By using a version of the mountain pass theorem, they proved that system (1.1) has infinitely many radially symmetric solutions under $|m| > |\omega|$ and $4 < q < 6$. It was complemented and improved by [3] and [19]. Azzollini and Pomponio [2] obtained the existence of a ground state solution for (1.1) under one of the conditions

(i) $4 \leq q < 6$ and $m > \omega$;

(ii) $2 < q < 4$ and $m\sqrt{q-2} > \omega\sqrt{6-q}$.

Soon afterwards, it is improved by Wang [33]. Motivated by the methods of [7], Cassani [9] considered (1.1) for the critical case by adding a lower order perturbation:

$$\begin{cases} -\Delta u + [m^2 - (\omega + \phi)^2]\phi u = \mu|u|^{q-2}u + |u|^{2^*-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.2}$$

where $\mu > 0$ and $2^* = 6$. He showed that (1.2) has at least a radially symmetric solution under one of the following conditions:

(i) $4 < q < 6, |m| > |\omega|$ and $\mu > 0$;

(ii) $q = 4, |m| > |\omega|$ and μ is sufficiently large.

It is improved and generalized by the results in [10] and [32]. Recently, the authors in [11,17,37] proved the existence of positive ground state solutions for the problem (1.2) with a periodic potential V or V is a constant:

$$\begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = \mu|u|^{q-2}u + |u|^{2^*-2}u, & x \in \mathbb{R}^3, \\ \Delta\phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3. \end{cases}$$

In [23], Georgiev and Visciglia introduced a system like (1.1) with potentials, however they considered a small external Coulomb potential in the corresponding Lagrangian density. Inspired by these works, He [24] first considered the existence of infinitely many solutions for system (*KGM*). The nonlinearity f satisfied (*AR*) condition:

(*AR*) There exists $\theta > 4$ such that $\theta F(x, t) \leq tf(x, t)$, for all $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$, where $F(x, t) = \int_0^t f(x, s)ds$.

Very recently, Ding and Li [21] obtained the existence of infinitely many solutions for (*KGM*) under the following condition:

(V) $V \in C(\mathbb{R}^3, \mathbb{R})$ is bounded below and, for every $C > 0$, meas$\{x \in \mathbb{R}^3 : V(x) \leq C\} < +\infty$, where meas denotes the Lebesgue measures;

(F_1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $|f(x, t)| \leq C_1|t| + C_2|t|^{p-1}$ for $4 \leq p < 2^*$, where C_1, C_2 are positive constants, $f(x, t)t \geq 0$ for $t \geq 0$;

(F_2) $\frac{F(x,t)}{t^4} \to +\infty$ as $|t| \to +\infty$;

(F_3) Let $\mathcal{F}(x,t) := \frac{1}{4}f(x,t)t - F(x,t)$, there exists $r_0 > 0$ such that if $|t| \geq r_0$, then $\mathcal{F}(x,t) \geq 0$ uniformly for $x \in \mathbb{R}^3$;

(F_4) $f(x,-t) = -f(x,t)$ for any $x \in \mathbb{R}^3, t \in \mathbb{R}$.

Cunha [18] considered the existence of positive and ground state solutions for (*KGM*) with periodic potential $V(x)$. By the Ekeland variational principle and the Mountain Pass Theorem, Li, A. Boucherif and N. D. Merzagui [27] obtained the existence of two different solutions for (*KGM*). Other related results about Klein–Gordon–Maxwell system on $\mathbb{R}^3$ can be found in [16,20,26,28,35]. By the way, we recall that Klein–Gordon–Maxwell system with nonhomogeneous nonlinearity is studied in [14,22,36,39] and the existence of infinitely many radial solitary waves solutions are studied in [12].

Before giving our main results, we give some notations. Let $H^1(\mathbb{R}^3)$ be the usual Sobolev space endowed with the standard scalar and norm

$$(u,v)_H = \int_{\mathbb{R}^3}(\nabla u \nabla v + uv)dx; \qquad \|u\|_H^2 = \int_{\mathbb{R}^3}(|\nabla u|^2 + |u|^2)dx.$$

$D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_D^2 := \|u\|_{D^{1,2}(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3}|\nabla u|^2 dx.$$

The norm on $L^s = L^s(\mathbb{R}^3)$ with $1 < s < \infty$ is given by $|u|_s^s = \int_{\mathbb{R}^3}|u|^s dx$.

System (*KGM*) has a variational structure. Indeed, we consider the functional $\mathcal{J} : H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\mathcal{J}(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx - \frac{1}{2}\int_{\mathbb{R}^3}(2\omega + \phi)\phi u^2 dx - \int_{\mathbb{R}^3}F(x,u)dx.$$

The solutions $(u,\phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ of system (*KGM*) are critical points of $\mathcal{J}$. However, the functional $\mathcal{J}$ is strongly indefinite and is difficult to investigate. Fortunately, this indefiniteness can be removed by using the reduction method described in [8]. Then we are led to the study of a new functional $I(u)$ which does not present such strongly indefinite nature.

Motivated by the above works, in the present paper we first consider system (*KGM*) with the superlinear case, and hence make the following assumptions:

(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and there exist $C > 0$ and $p \in (4,6)$ such that

$$|f(x,t)| \leq C(1 + |t|^{p-1});$$

(f_2) $f(x,t) = o(t)$ uniformly in x as $t \to 0$;

(f_3) $\frac{F(x,t)}{t^4} \to +\infty$ uniformly in x as $|t| \to +\infty$;

(f_4) There exists a positive constant b such that $\mathcal{F}(x,t) := \frac{1}{4}f(x,t)t - F(x,t) \geq -bt^2$.

Remark 1.1. We emphasize that unlike all previous results about system (*KGM*), see e.g. [18,24,26], we have not assume that the potential V is positive. This means that we allow the potential V be sign changing.

Remark 1.2. It is well known that the condition (AR) is widely used in the studies of elliptic problem by variational methods. The condition (AR) is used not only to prove that the Euler-Lagrange function associated has a mountain pass geometry, but also to guarantee that the Palais–Smale sequences, or Cerami sequences are bounded. Obviously, we can observe that the condition (AR) implies the following condition:

(A_1) There exist $\theta > 4$ and $C_1, C_2 > 0$ such that $F(x,t) \geq C_1|t|^\theta - C_2$, for every t sufficiently large.

Moreover, the condition (A_1) implies our condition (f_3).

Another widely employed condition is the following condition, which is first introduced by Jeanjean [25].

(Je) There exist $\theta \geq 1$ such that $\theta \mathcal{F}(x,t) \geq \mathcal{F}(x,st)$ for all $s \in [0,1]$ and $t \in \mathbb{R}$, where $\mathcal{F}(x,t)$ is given in (f_4).

We can observe that when $s = 0$, then $\mathcal{F}(x,t) \geq 0$, but for our condition (f_4), $\mathcal{F}(x,t)$ may assume negative values. Therefore, it is interesting to consider 4-superlinear problems under the conditions (f_3) and (f_4).

The condition (f_4) is motivated by Alves, Soares and Souto [1]. Supposing in addition

$$\alpha = \inf_{x \in \mathbb{R}^3} V(x) > 0 \tag{1.3}$$

and $b \in [0, \alpha)$, they proved that all Cerami sequences are bounded. In 2015, Chen and Liu [13] also used conditions (f_3) and (f_4) to show the existence of infinitely many solutions for Schrödinger–Maxwell systems. In our case, however, many technical difficulties arise to the presence of a non-local term ϕ, which is not homogeneous as it is in the Schrödinger–Maxwell systems. Hence, a more careful analysis of the interaction between the couple (u, ϕ) is required.

By (V), we know that V is bounded from below, hence we may choose $V_0 > 0$ such that

$$\tilde{V}(x) := V(x) + V_0 > 1, \quad \forall x \in \mathbb{R}^3$$

and define a new Hilbert space

$$E := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 dx < \infty \right\}$$

with the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^3} \left(\nabla u \cdot \nabla v + \tilde{V}(x)uv \right) dx$$

and the norm $\|u\| = \langle u, u \rangle^{1/2}$. Obviously, the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, for any $s \in [2, 2^*]$. The norm on $L^s = L^s(\mathbb{R}^3)$ with $1 < s < \infty$ is given by $|u|_s^s = \int_{\mathbb{R}^3} |u|^s dx$. Consequently, for each $s \in [2, 6]$, there exists a constant $d_s > 0$ such that

$$|u|_s \leq d_s \|u\|, \quad \forall u \in E. \tag{1.4}$$

Furthermore, we have that under the condition (V), the embedding $E \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2, 6)$ (see [4]). By the compact embedding $E \hookrightarrow L^2(\mathbb{R}^3)$ and the standard elliptic theory [40], it is easy to see that the eigenvalue problem

$$-\Delta u + V(x)u = \lambda u, \quad u \in E \tag{1.5}$$

possesses a complete sequence of eigenvalues

$$-\infty < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots, \quad \lambda_j \to +\infty.$$

Each λ_j has finite multiplicity and $|\lambda_j|_2 = 1$. Denote e_j be the eigenfunction of λ_j. E^- is spanned by the eigenfunctions corresponding to negative eigenvalues. Note that the negative space E^- of the quadratic part of I is nontrivial if and only if some λ_j is negative.

Now we can state our first result.

Theorem 1.3. *Suppose* (V), (f_1)–(f_4) *are satisfied, and f is odd in t. If 0 is not an eigenvalue of* (1.5), *then* (KGM) *has a sequence of solutions* $(u_n, \phi_n) \in E \times D^{1,2}(\mathbb{R}^3)$ *such that the energy* $\mathcal{J}(u_n, \phi_n) \to +\infty$.

Remark 1.4. *If u is a critical point of I, then $I(u) = \mathcal{J}(u, \phi_u)$ (see (2.1)). Therefore, in order to prove Theorem 1.1, we only need to find a sequence of critical points $\{u_n\}$ of I such that $I(u_n) \to +\infty$.*

Remark 1.5. Theorem 1.3 improves the recent results in [24]. In that paper, the author assumed in addition (1.3), and (AR) or (Je). When V is positive, the quadratic part of the functional I (see (2.1)) is positively definite, and I has a mountain pass geometry. Therefore, the mountain pass lemma [30] can be applied. In our case, the quadratic part may possesses a nontrivial negative space E^-, so I no longer possesses the mountain pass geometry. Therefore the methods in [21, 24] cannot be applied. To obtain our result, we adopt a technique developed in [13].

In the second part of this paper, we deal with the system (KGM) when the nonlinearity $f(x,t)$ is asymptotically linear at infinity in the second variable t. Set

$$\Omega = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)dx}{\int_{\mathbb{R}^3} u^2 dx},$$

i.e. Ω is the infimum of the spectrum of the Schrödinger operator $-\Delta + V$.

We make the following assumptions:

(H_1) $V(x) \in C(\mathbb{R}^3, \mathbb{R})$ satisfies $V(x) \geq D_0 > 0$ for all $x \in \mathbb{R}^3$;

(H_2) $\lim_{|x| \to +\infty} V(x) = V_\infty \in (0, +\infty)$;

(H_3) $f(x,t) \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $\lim_{t \to 0} \frac{f(x,t)}{t} = 0$ uniformly in x;

(H_4) There exists $A \in (\Omega, V_\infty)$ such that $\lim_{t \to +\infty} \frac{f(x,t)}{t} = A$ uniformly in x and $0 \leq \frac{f(x,t)}{t} \leq A$ for all $t \neq 0$.

Theorem 1.6. *Assume (H_1)–(H_4) hold, then there exists a constant $\omega^* > 0$ such that* (KGM) *has a positive solution for any $\omega \in (0, \omega^*)$.*

Theorem 1.7. *Assume (H_1)–(H_4) hold, then there exists a constant $\omega^\sharp > 0$ such that* (KGM) *has no nontrival solution for any $\omega > \omega^\sharp$.*

Remark 1.8.

(a) It follows from the condition $\Omega < A < V_\infty$ that $V(x)$ is not a constant.

(b) By Theorem 1.7, it is easy to know that ω^* is finite.

Remark 1.9. To our best knowledge, it seems that there are few results for system (*KGM*) in this case: the nonlinear term $f(x, t)$ in t is asymptotically linear at infinity. In order to get our results, we have to solve some difficulties. The first difficult is how to prove the variational function satisfies the assumptions of the Mountain Pass Theorem. The second difficult is how to check the (*PS*) condition, i.e., how to verify the boundedness and compactness of a (*PS*) sequence. To overcome these difficulties we use some techniques used in [29], [31] and [34]. However, it seems difficult to use this method to the case $f(x, t)$ is superlinear in t at infinity.

We denote by " $\rightharpoonup$ " weak convergence and by " $\rightarrow$ " strong convergence. Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again $\{u_n\}$.

The paper is organized as follows. In Section 2, we will introduce the variational setting for the problem, give some related preliminaries and prove Theorem 1.3. We give the proofs of Theorem 1.6 and Theorem 1.7 in Section 3.

2   Proof of Theorem 1.3

By [3], we know that the signs of ω is not relevant for the existence of solutions, so we can assume that $\omega > 0$. Evidently, the properties of ϕ_u plays an important role in the study of $\mathcal{J}$. So we need the following technical results.

Proposition 2.1. *For any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ which satisfies*

$$\Delta \phi = (\phi + \omega)u^2 \quad \text{in } \mathbb{R}^3.$$

Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto \phi_u \in D^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and

(i) $-\omega \leq \phi_u \leq 0$ *on the set $\{x \in \mathbb{R}^3 | u(x) \neq 0\}$;*

(ii) $\|\phi_u\|_D^2 \leq C\|u\|^2$ *and* $\int_{\mathbb{R}^3} \phi_u u^2 dx \leq C|u|_{12/5}^4 \leq C\|u\|^4$.

The proof is similar to Proposition 2.1 in [24] by using the fact $E \hookrightarrow L^s(\mathbb{R}^3)$, for any $s \in [2, 6]$ is continuous.

By Proposition 2.1, we can consider the functional $I : H^1(\mathbb{R}^3) \mapsto \mathbb{R}$ defined by $I(u) = \mathcal{J}(u, \phi_u)$.

Multiplying $-\Delta \phi_u + \phi_u u^2 = -\omega u^2$ by ϕ_u and integration by parts, we obtain

$$\int_{\mathbb{R}^3} (|\nabla \phi_u|^2 + \phi_u^2 u^2) dx = - \int_{\mathbb{R}^3} \omega \phi_u u^2 dx.$$

By the above equality and the definition of $\mathcal{J}$, we obtain a C^1 functional $I : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx \tag{2.1}$$

and its Gateaux derivative is

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv) dx - \int_{\mathbb{R}^3} (2\omega + \phi_u) \phi_u uv dx - \int_{\mathbb{R}^3} f(x, u) v dx$$

for all $v \in H^1(\mathbb{R}^3)$. Here we use the fact that $(\Delta - u^2)^{-1}[\omega u^2] = \phi_u$.

If $\lambda_1 > 0$, we can easy to prove that I has the mountain pass geometry, so we omit this case. Since 0 is not an eigenvalue of (1.5), we assume that there exists $l \geq 1$ such that $0 \in (\lambda_l, \lambda_{l+1})$. Set

$$E^- = \text{span}\{e_1, \ldots, e_l\}, \quad E^+ = (E^-)^\perp. \tag{2.2}$$

Then E^- and E^+ are the negative space and positive space of the quadratic form

$$N(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx$$

respectively, and $\dim E^- < \infty$. Moreover, there is a positive constant B such that

$$\pm N(u) \geq B\|u\|^2, \quad u \in E^\pm. \tag{2.3}$$

In order to prove Theorem 1.3, we shall use the fountain theorem of Bartsch [5], see also Theorem 3.6 in [38]. For $k = 1, 2, \ldots$, set

$$Y_k = \text{span}\{e_1, \ldots, e_k\}, \quad Z_k = \overline{\text{span}\{e_{k+1}, \ldots, \}}. \tag{2.4}$$

Proposition 2.2 (Fountain Theorem). *Assume that the even functional $I \in C^1(E, \mathbb{R})$ satisfies the* (PS) *condition. If there is a positive constant K such that for any $k \geq K$ there exist $\rho_k > r_k > 0$ such that*

(i) $a_k = \max_{u \in Y_k, \|u\|=\rho_k} I(u) \leq 0$,

(ii) $b_k = \inf_{u \in Z_k, \|u\|=r_k} I(u) \to +\infty$ *as* $k \to +\infty$,

then I has a sequence of critical points $\{u_k\}$ such that $I(u_k) \to +\infty$.

In order to study the functional I, we will write the functional I in a form in which the quadratic part is $\|u\|^2$. Let $g(x,t) = f(x,t) + V_0 t$. Then, by an computation, we obtain that

$$G(x,t) := \int_0^t g(x,s) ds \leq \frac{t}{4} g(x,t) + \frac{\tilde{V}_0}{4} t^2, \quad \tilde{V}_0 := 4b + V_0 > 0. \tag{2.5}$$

By (f_3) we have

$$\lim_{|t| \to \infty} \frac{g(x,t)t}{t^4} = +\infty. \tag{2.6}$$

Furthermore, by (f_2) we obtain

$$\lim_{|t| \to 0} \frac{g(x,t)t}{t^4} = \lim_{|t| \to 0} \left(\frac{t^2}{t^4} \cdot \frac{f(x,t)t + V_0 t^2}{t^2} \right) = +\infty.$$

Hence there exists $M > 0$ such that

$$g(x,t)t \geq -Mt^4, \quad \forall t \in \mathbb{R}. \tag{2.7}$$

With the modified nonlinearity g, the functional $I : E \to \mathbb{R}$ can be rewritten in the following

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{\omega}{2} \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} G(x,u) dx \tag{2.8}$$

with the derivative

$$\langle I'(u), v \rangle = \langle u, v \rangle - \int_{\mathbb{R}^3} (2\omega + \phi_u)\phi_u uv dx - \int_{\mathbb{R}^3} g(x,u)v dx.$$

Lemma 2.3. *Suppose* (V), (f_1)–(f_4) *are satisfied, then the function I satisfies the (PS) condition.*

Proof. It follows from $\frac{1}{4}tf(x,t) - F(x,t) \geq -bt^2$ that the condition (f_3) is equivalent to

$$\lim_{|t|\to+\infty} \frac{G(x,t)}{t^4} = +\infty.$$

Let $\{u_n\}$ be a (PS) sequence, i.e.,

$$\sup_n |I(u_n)| < \infty, \quad I'(u_n) \to 0.$$

We first prove that $\{u_n\}$ is bounded in E. Arguing by contradiction, suppose that $\{u_n\}$ is unbounded, passing to a subsequence, by (2.5), we obtian

$$
\begin{aligned}
4\sup_n I(u_n) + \|u_n\| &\geq 4I(u_n) - \langle I'(u_n), u_n \rangle \\
&= \|u_n\|^2 + \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx + \int_{\mathbb{R}^3} (g(x,u_n)u_n - 4G(x,u_n))dx \\
&\geq \|u_n\|^2 - \tilde{V}_0 \int_{\mathbb{R}^3} u_n^2 dx.
\end{aligned}
\tag{2.9}
$$

Let $v_n = \frac{u_n}{\|u_n\|}$. Then, going if necessary to a subsequence, by the compact embedding $E \hookrightarrow L^2(\mathbb{R}^3)$ we may assume that

$$
\begin{aligned}
v_n &\rightharpoonup v_0 && \text{in } E; \\
v_n &\to v_0 && \text{in } L^2(\mathbb{R}^3); \\
v_n(x) &\to v_0(x) && \text{a.e. in } \mathbb{R}^3.
\end{aligned}
$$

Dividing both sides of (2.9) by $\|u_n\|^2$, we have

$$\tilde{V}_0 \int_{\mathbb{R}^3} v_0^2 dx \geq 1 \quad \text{as } n \to \infty.$$

Consequently, we have that $v_0 \neq 0$.

By (1.4) and (2.7), we have

$$
\begin{aligned}
\int_{v_0=0} \frac{g(x,u_n)u_n}{\|u_n\|^4} dx &= \int_{v_0=0} \frac{g(x,u_n)u_n}{u_n^4} v_n^4 dx \\
&\geq -M \int_{v_0=0} v_n^4 dx \geq -M \int_{\mathbb{R}^3} v_n^4 dx \\
&= -M|v_n|_4^4 \geq -Md_4^4 > -\infty.
\end{aligned}
\tag{2.10}
$$

For $x \in \{x \in \mathbb{R}^3 | v_0 \neq 0\}$, we have $|u_n(x)| \to +\infty$ as $n \to \infty$. By (2.6) we have

$$\frac{g(x,u_n(x))u_n(x)}{\|u_n\|^4} = \frac{g(x,u_n(x))u_n(x)}{u_n^4(x)} v_n^4(x) \to +\infty. \tag{2.11}$$

Hence, by (2.10) and (2.11) and Fatou's lemma we obtain

$$\int_{\mathbb{R}^3} \frac{g(x,u_n)u_n}{\|u_n\|^4} dx \geq \int_{v_0\neq0} \frac{g(x,u_n)u_n}{u_n^4} v_n^4(x)dx - Md_4^4 \to +\infty. \tag{2.12}$$

Hence we obtain that

$$\int_{\mathbb{R}^3} \frac{G(x, u_n)}{\|u_n\|^4} dx \to +\infty. \tag{2.13}$$

Since $\{u_n\}$ is a (PS) sequence, using Proposition 2.1 and (2.12), for n large enough, we obtain

$$c\omega + 1 \geq \frac{1}{\|u_n\|^4} \left(\frac{1}{2} \|u_n\|^2 - \frac{\omega}{2} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - I(u_n) \right)$$
$$= \int_{\mathbb{R}^3} \frac{G(x, u_n)}{\|u_n\|^4} dx \to +\infty, \tag{2.14}$$

which is a contradiction.

Now we have proved that $\{u_n\}$ is bounded in E. By a similar argument in [15], the compact embedding $E \hookrightarrow L^2(\mathbb{R}^3)$ and

$$E = \overline{\bigcup_{n \in \mathbb{N}} E_n},$$

we can show that $\{u_n\}$ has a subsequence converging to a critical point of I. $\qquad \square$

Lemma 2.4. *Let X be a finite dimensional subspace of E, then I is anti-coercive on X, i.e.*

$$I(u) \to -\infty, \quad as \ \|u\| \to \infty, \ u \in X.$$

Proof. If it is not true, we can choose a sequence $\{u_n\} \subset X$ and ξ is a real number such that

$$\|u_n\| \to \infty, \quad I(u_n) \geq \xi. \tag{2.15}$$

Let $v_n = \frac{u_n}{\|u_n\|}$. Since $\dim X < \infty$, going if necessary to a subsequence we have

$$\|v_n - v_0\| \to 0, \quad v_n(x) \to v_0(x) \quad \text{a.e. in } \mathbb{R}^3$$

for every $v_0 \in X$, with $\|v_0\| = 1$. Since $v_0 \neq 0$, similar to (2.13) we obtain that

$$\int_{\mathbb{R}^3} \frac{G(x, u_n)}{\|u_n\|^4} dx \to +\infty.$$

And arguing similar to (2.14), it follows from $\sup_n |I(u_n)| < \infty$ that

$$I(u_n) = \|u_n\|^4 \left(\frac{1}{2\|u_n\|^2} - \frac{\omega}{2\|u_n\|^4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx - \int_{\mathbb{R}^3} \frac{G(x, u_n)}{\|u_n\|^4} dx \right) \to -\infty,$$

which is contradict with $I(u_n) \geq \xi$. The proof is complete. $\qquad \square$

Now, we are ready to prove our main result.

Proof of Theorem 1.3. We will find a sequence of critical points $\{u_n\}$ of I such that $I(u_n) \to +\infty$.

Since $f(x, t)$ is odd in t, I is an even function. It follows from Lemma 2.3 that I satisfies (PS) condition. Therefore, it suffices to verify (i) and (ii) of Proposition 2.2.

(i) Since $\dim Y_k < \infty$, by Lemma 2.4, we get the conclusion of (i).

(ii) By $(f_1), (f_2)$, we have

$$|f(x, t)| \leq \epsilon |t| + C_\epsilon |t|^{p-1}, \quad |F(x, t)| \leq \frac{\epsilon}{2} t^2 + \frac{C_\epsilon}{p} |t|^p,$$

where $\epsilon > 0$ is very small. Then we have

$$|F(x,t)| \leq \frac{B}{2d_2^2}t^2 + \frac{CB}{p}|t|^p, \tag{2.16}$$

where B is defined in (2.3). We assume that $0 \in [\lambda_l, \lambda_{l+1})$. Then if $k > l$, we have that $Z_k \subset E^+$, where E^+ is defined in (2.2). Now we have

$$N(u) \geq B\|u\|^2, \quad u \in Z_k \tag{2.17}$$

and, as proof of Lemma 3.8 in [38],

$$\beta_p(k) = \sup_{u \in Z_k, \|u\|=1} |u|_p \to 0, \quad \text{as } k \to \infty.$$

Let $r_k = (Cp\beta_p(k))^{1/(2-p)}$, where C is chosen as in (2.16). For $u \in Z_k \subset E^+$ with $\|u\| = r_k$, $\phi_u \leq 0$, by (2.17) we deduce that

$$\begin{aligned}
I(u) &= N(u) - \frac{1}{2}\omega \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx \\
&\geq B\|u\|^2 - \frac{B}{2d_2^2}|u|_2^2 - CB|u|_p^p \\
&\geq B\left(\frac{1}{2}\|u\|^2 - C\beta_p^p\|u\|^p\right) \\
&= B\left(\frac{1}{2} - \frac{1}{p}\right)(Cp\beta_p^p)^{2/(2-p)},
\end{aligned}$$

where $\beta_p^p := (\beta_p(k))^p$. Since $\beta_p(k) \to 0$ and $p > 2$, it follows that

$$b_k = \inf_{u \in Z_k, \|u\|=r_k} I(u) \to +\infty.$$

We get the conclusion of (ii). The proof is complete.   □

3   Proofs of Theorem 1.6 and Theorem 1.7

Under the condition (H_1), we define a new Hilbert space

$$F := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx < \infty\right\}.$$

with the inner product

$$(u,v)_F = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x)uv)\, dx$$

and the norm $\|u\|_F = (u,u)_F^{1/2}$, which is equivalent to the usual Sobolev norm on $H^1(\mathbb{R}^3)$. Obviously, the embedding $F \hookrightarrow L^s(\mathbb{R}^3)$ is continuous, for any $s \in [2, 2^*]$. Consequently, for each $s \in [2,6]$, there exists a constant $v_s > 0$ such that

$$|u|_s \leq v_s\|u\|_F, \quad \forall u \in F. \tag{3.1}$$

Furthermore, we know that under assumption (H_1), the embedding $F \hookrightarrow L^s(\mathbb{R}^3)$ is compact for any $s \in [2, 2^*)$ (see [40]).

By Proposition 2.1, we can consider the functional I_ω on $(F, \|\cdot\|_F)$:

$$I_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2)dx - \int_{\mathbb{R}^3}F(x,u)dx,$$

with its Gateaux derivative is

$$\langle I'_\omega(u), v\rangle = \int_{\mathbb{R}^3}[\nabla u\nabla v + V(x)uv - (2\omega + \phi_u)\phi_u uv - f(x,u)v]dx.$$

Lemma 3.1. *Suppose (H_1)–(H_4) hold. Then there exist some positive constants ρ_0, α_0 such that $I_\omega(u)|_{\|u\|_F = \rho_0} \geq \alpha_0$ for all $u \in F$. Moreover, there exists a function $u_0 \in F$ with $\|u_0\|_F > \rho_0$ and $\omega^* > 0$ such that $I_\omega(u_0) < 0$ for $0 < \omega < \omega^*$.*

Proof. By $(H_3), (H_4)$, for any $\varepsilon > 0$, there exists q with $1 < q < 5$ and $M_1 = M_1(\varepsilon, p) > 0$ such that

$$|F(x,t)| \leq \frac{\varepsilon}{2}t^2 + M_1 t^{q+1}, \quad \text{for all } t > 0. \tag{3.2}$$

By $\phi_u \leq 0$ and the Sobolev inequality, we get that

$$\begin{aligned}
I_\omega(u) &= \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2 - \omega\phi_u u^2)dx - \int_{\mathbb{R}^3}F(x,u)dx \\
&\geq \frac{1}{2}\|u\|_F^2 - \frac{\varepsilon}{2}v_2^2\|u\|_F^2 - M_1 v_{q+1}^{q+1}\|u\|_F^{q+1} \\
&= \left(\frac{1}{2} - \frac{\varepsilon}{2}v_2^2\right)\|u\|_F^2 - M_1 v_{q+1}^{q+1}\|u\|_F^{q+1}.
\end{aligned}$$

Since $1 < q < 5$, let $\varepsilon = \frac{1}{2v_2^2}$ and $\|u\|_F = \rho_0 > 0$ small enough, then we can obtain $I_\omega(u)|_{\|u\|_F = \rho} \geq \alpha_0$ for all $u \in F$.

By (H_4), we have $A > \Omega$. From the definition of Ω, there exists a nonnegative function $u_1 \in H^1(\mathbb{R}^3)$ such that

$$\|u_1\|_F^2 = \int_{\mathbb{R}^3}(|\nabla u_1|^2 + V(x)u_1^2)dx < A\int_{\mathbb{R}^3}u_1^2 dx = A|u_1|_2^2.$$

Hence, by (H_4) and Fatou's lemma we obtain that

$$\begin{aligned}
\lim_{t\to+\infty}\frac{I_0(tu_1)}{t^2} &= \frac{1}{2}\|u_1\|_F^2 - \lim_{t\to+\infty}\int_{\mathbb{R}^3}\frac{F(x,tu_1)}{t^2 u_1^2}u_1^2 dx \\
&\leq \frac{1}{2}\|u_1\|_F^2 - \frac{A}{2}\int_{\mathbb{R}^3}u_1^2 dx \\
&= \frac{1}{2}(\|u_1\|_F^2 - A|u_1|_2^2) < 0.
\end{aligned}$$

If $I_0(tu_1) \to -\infty$ as $t \to +\infty$, then there is $u_0 \in F$ with $\|u_0\|_F > \rho_0$ such that $I_0(u_0) < 0$. Since $I_\omega(u_0) \to I_0(u_0)$ as $\omega \to 0^+$. We have that there is a positive constant $\omega^* > 0$ such that $I_\omega(u_0) < 0$ for all $0 < \omega < \omega^*$. The proof is complete. $\qquad\square$

Lemma 3.2. *Suppose (H_1)–(H_4) hold. Then any sequence $\{u_n\} \subset F$ satisfying*

$$I_\omega(u_n) \to c > 0, \quad \langle I'_\omega(u_n), u_n\rangle \to 0$$

is bounded in F. Moreover, $\{u_n\}$ has a strongly convergent subsequence in F.

Proof. (i) We first to prove that $\{u_n\}$ is bounded. For any fixed $L > 0$, let $\eta_L \in C^\infty(\mathbb{R}^3, \mathbb{R})$ be a cut-off function such that

$$\eta_L = \begin{cases} 0, & \text{for } |x| \leq L/2, \\ 1, & \text{for } |x| \geq L, \end{cases}$$

and $|\nabla \eta_L| \leq \frac{C}{L}$ for all $x \in \mathbb{R}^3$ and C is a positive constant. For any $u \in F$ and all $L \geq 1$, there exists a constant $C_0 > 0$, which is independent of L, such that $\|\eta_L u\|_F \leq C_0 \|u\|_F$.

Since $I'_\omega(u_n) \to 0$ as $n \to +\infty$ in $H^{-1}(\mathbb{R}^3)$, for n large enough we have that

$$\langle I'_\omega(u_n), \eta_L u_n \rangle \leq \|I'_\omega(u_n)\|_{F^{-1}} \|\eta_L u_n\|_F \leq \|u_n\|_F, \tag{3.3}$$

and

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2)\eta_L dx + \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \eta_L dx - \int_{\mathbb{R}^3} (2\omega + \phi_{u_n})\phi_{u_n}\eta_L u_n^2 dx$$
$$\leq \int_{\mathbb{R}^3} f(x, u_n)u_n \eta_L dx + \|u_n\|_F, \tag{3.4}$$

where F^{-1} is the dual space of F.

By assumptions (H_2) and (H_4), there exist $\gamma > 0$ and $L_1 > 0$ such that $V(x) \geq A + \gamma$ for all $|x| \geq L_1$. Choosing $L > 2L_1$, since $|\nabla \eta_L(x)| \leq \frac{C}{L}$ for all $x \in \mathbb{R}^3$, $2\omega + \phi_{u_n} \geq 0$ and $f(x, u_n(x))u_n(x) \leq A u_n^2(x)$ for all $x \in \mathbb{R}^3$ by (H_4). Following from (3.4) we get that

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + \gamma u_n^2)\eta_L dx \leq \frac{C}{L}\left(\int_{\mathbb{R}^3} u_n^2 dx + \int_{\mathbb{R}^3} \nabla u_n^2 dx\right) + \|u_n\|_F. \tag{3.5}$$

Similar to (3.3), we have that $\langle I'_\omega(u_n), u_n \rangle \leq \|u_n\|_F$, that is

$$\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2 - 2\omega\phi_{u_n}u_n^2 - \phi_{u_n}^2 u_n^2 - f(x, u_n)u_n)dx \leq \|u_n\|_F. \tag{3.6}$$

Motivated by [31] (see also [39]), we give an inequality by using the second equality of system (*KGM*). Multiplying both sides of $\Delta\phi_{u_n} = (\omega + \phi_{u_n})u_n^2$ by $|u_n|$, integrating by parts and using the Young's inequality, we have

$$\sqrt{\frac{3}{4}} \int_{\mathbb{R}^3} (\omega + \phi_{u_n})|u_n|^3 dx \leq \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{3}{4}\int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 dx. \tag{3.7}$$

Then by Proposition 2.1, one has

$$\sqrt{3}\int_{\mathbb{R}^3} (\omega + \phi_{u_n})|u_n|^3 dx \leq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{3}{2}\int_{\mathbb{R}^3} |\nabla \phi_{u_n}|^2 dx$$
$$\leq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} \omega\phi_{u_n}u_n^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx$$
$$\leq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} \omega\phi_{u_n}u_n^2 dx - \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx. \tag{3.8}$$

By (3.6), (3.7), (3.8) and $V(x) > 0$, $\phi_{u_n} \leq 0$, $f(x, u_n(x))u_n(x) \leq A u_n^2(x)$ for all $x \in \mathbb{R}^3$ we have

that

$$
\begin{aligned}
\frac{1}{2} &\int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2)dx + \int_{\mathbb{R}^3} (\sqrt{3}(\omega + \phi_{u_n})|u_n|^3 - Au_n^2)dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2)dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx - \frac{3}{2} \int_{\mathbb{R}^3} \omega\phi_{u_n}u_n^2 dx \\
&\quad - \int_{\mathbb{R}^3} \phi_{u_n}^2 u_n^2 dx - \int_{\mathbb{R}^3} f(x,u_n)u_n dx \\
&= \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V(x)u_n^2 - 2\omega\phi_{u_n}u_n^2 - \phi_{u_n}^2 u_n^2 - f(x,u_n)u_n)dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^3} V(x)u_n^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} \omega\phi_{u_n}u_n^2 dx \\
&\leq \|u_n\|_F,
\end{aligned}
$$

that is

$$
\frac{1}{2}\|u_n\|_F^2 + \int_{\mathbb{R}^3} h(u_n)dx \leq \|u_n\|_F, \tag{3.9}
$$

where $h(u_n) = \sqrt{3}(\omega + \phi_{u_n})|u_n|^3 - Au_n^2$.

By (3.5), there is a positive constant $C_1 > 0$ (independent of L) such that

$$
\int_{|x| \geq L} u_n^2 dx \leq \frac{C_1}{L}\|u_n\|_F^2 + C_1\|u_n\|_F.
$$

Let $\delta = \inf_{t \in R} h(t)$. Then $\delta \in (-\infty, 0)$ and by above inequality we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} h(u_n)dx &\geq \int_{|x| \leq L} \delta dx + \int_{|x| \geq L} (-Au_n^2)dx \\
&\geq \delta|B_L(0)| - \frac{AC_1}{L}\|u_n\|_F^2 - AC_1\|u_n\|_F, \tag{3.10}
\end{aligned}
$$

where $|B_L(0)|$ denotes the volume of $B_L(0)$. It follows from (3.9) and (3.10) that

$$
\frac{1}{2}\|u_n\|_F^2 \leq |\delta||B_L(0)| + \frac{AC_1}{L}\|u_n\|_F^2 + AC_1\|u_n\|_F + \|u_n\|_F.
$$

Since C_1 is a constant independent of L, we can choose L large enough such that $\frac{AC_1}{L} < \frac{1}{2}$. Then we obtain that $\{u_n\}$ is bounded in F by above inequality.

(ii) Now we shall show that $\{u_n\}$ has a strongly convergent subsequence in F. From case (i), $\{u_n\}$ is bounded in F. Then (3.3) and (3.5) become

$$
\langle I'_\omega(u_n), \eta_L u_n \rangle = \circ(1)
$$

and

$$
\int_{|x| \geq L} (|\nabla u_n|^2 + u_n^2)dx \leq \frac{C}{L}\|u_n\|_F^2 + \circ(1), \tag{3.11}
$$

respectively. Therefore, for any $\varepsilon > 0$, there exists $L > 0$ such that for n large enough,

$$
\int_{|x| \geq L} (|\nabla u_n|^2 + u_n^2)dx \leq \varepsilon. \tag{3.12}
$$

Since $\{u_n\}$ is bounded in F, passing to a subsequence if necessary, there exists $u \in F$ such that $u_n \rightharpoonup u$ in F. In view of the embedding $F \hookrightarrow L^s(\mathbb{R}^3)$ are compact for any $s \in [2, 6)$, $u_n \to u$

in $L^s(\mathbb{R}^3)$ for $1 < s < 6$ and $u_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^3$. Hence it follows from assumptions of Lemma 3.2 and the derivative of I_ω, we easily obtain

$$\|u_n - u\|_F^2 = \langle I'_\omega(u_n) - I'_\omega(u), u_n - u \rangle + \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u)dx$$

$$+ 2\omega \int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)dx + \int_{\mathbb{R}^3} (\phi_{u_n}^2 u_n - \phi_u^2 u)(u_n - u)dx.$$

It is clear that

$$\langle I'_\omega(u_n) - I'_\omega(u), u_n - u \rangle \to 0 \quad \text{as } n \to \infty.$$

By Proposition 2.1, the Hölder inequality and the Sobolev inequality, we have

$$\left| \int_{\mathbb{R}^3} \phi_{u_n} u_n (u_n - u)dx \right| \leq |\phi_{u_n}|_6 |u_n|_{12/5} |u_n - u|_{12/5}$$

$$\leq C_1 \|\phi_{u_n}\|_D |u_n|_{12/5} |u_n - u|_{12/5}$$

$$\leq C_2 |u_n|_{12/5}^3 |u_n - u|_{12/5} \to 0.$$

Since $u_n \to u$ in $L^s(\mathbb{R}^3)$ for any $s \in [2, 2^*)$. We obtain

$$\int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) u_n (u_n - u)dx \to 0 \quad \text{as } n \to \infty$$

and

$$\int_{\mathbb{R}^3} \phi_u (u_n - u)^2 dx \leq |\phi_u|_6 |u_n - u|_3 |u_n - u|_2 \to 0 \quad \text{as } n \to \infty.$$

Thus, we get

$$\int_{\mathbb{R}^3} (\phi_{u_n} u_n - \phi_u u)(u_n - u)dx = \int_{\mathbb{R}^3} (\phi_{u_n} - \phi_u) u_n (u_n - u)dx + \int_{\mathbb{R}^3} \phi_u (u_n - u)^2 dx$$

$$\to 0, \quad \text{as } n \to \infty.$$

Now, we shall prove

$$\int_{\mathbb{R}^3} f(x, u_n)(u_n - u)dx = \circ(1) \quad \text{and} \quad \int_{\mathbb{R}^3} f(x, u)(u_n - u) = \circ(1). \tag{3.13}$$

We only to prove the first one and the second one is similar. Since $|f(x, u_n)| \leq A|u_n|$ and $\|u_n\|_F$ is bounded, by (3.12), the Hölder inequality and the Sobolev inequality, we have

$$\left| \int_{\mathbb{R}^3} f(x, u_n)(u_n - u)dx \right| \leq \left| \int_{|x| \leq L} f(x, u_n)(u_n - u)dx \right| + \left| \int_{|x| \geq L} f(x, u_n)(u_n - u)dx \right|$$

$$\leq C|u_n - u|_{L^2(B_L(0))} + C \left(\int_{|x| \geq L} u_n^2 dx \right)^{1/2}$$

$$\to 0 \quad \text{as } n \to \infty \text{ and } L \to +\infty.$$

So (3.13) hold. Therefore, $\|u_n - u\|_F \to 0$ as $n \to \infty$. The proof is complete. $\square$

Now we can prove our main results Theorem 1.6 and Theorem 1.7.

Proof of Theorem 1.6. By Lemma 3.1 and Lemma 3.2 we can obtain that u is a solution of system (KGM). And by using bootstrap arguments and the maximum principle, we can conclude that u is positive. The proof is complete. $\square$

Proof of Theorem 1.7. Let $(u, \phi_u) \in F \times D^{1,2}(\mathbb{R}^3)$ be a solution of (*KGM*). Then $\langle I'_\omega(u), u \rangle = 0$, i.e.

$$\langle I'_\omega(u), u \rangle = \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2 - (2\omega + \phi_u)\phi_u u^2 - f(x, u)u] dx = 0. \tag{3.14}$$

Similar to (3.8), we deduce that

$$\sqrt{3} \int_{\mathbb{R}^3} (\omega + \phi_u)|u|^3 dx \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{3}{2} \int_{\mathbb{R}^3} \omega \phi_{u_n} u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx. \tag{3.15}$$

By (H_3) and (H_4), there exists $C = C(D_0)$ such that

$$f(x, u)u \leq D_0 u^2 + C|u|^3. \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14), we obtain that

$$\begin{aligned}
0 &= \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2 - (2\omega + \phi_u)\phi_u u^2 - f(x, u)u] dx \\
&\geq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} \omega \phi_u u^2 dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 dx \\
&\quad + \int_{\mathbb{R}^3} (V(x) - D_0)u^2 dx - \int_{\mathbb{R}^3} C|u|^3 dx \\
&\geq \int_{\mathbb{R}^3} [\sqrt{3}(\omega + \phi_u) - C]|u|^3 dx + \int_{\mathbb{R}^3} (V(x) - D_0)u^2 dx \\
&\geq \int_{\mathbb{R}^3} [\sqrt{3}(\omega + \phi_u) - C]|u|^3 dx.
\end{aligned}$$

Therefore, if ω is large enough such that $\omega + \phi_u > \frac{\sqrt{3}}{3}C$, system (*KGM*) only has the trivial solution $u \equiv 0$. The proof is complete. □

References

[1] C. O. Alves, M. A. S. Souto, S. H. M. Soares, Schrödinger–Poisson equations without Ambrosetti–Rabinowitz condition, *J. Math. Anal. Appl.* **377**(2011), 584–592. https://doi.org/10.1016/j.jmaa.2010.11.031; MR2769159; Zbl 1211.35249

[2] A. Azzollini, A. Pomponio, Ground state solutions for the nonlinear Klein–Gordon–Maxwell equations, *Topol. Methods Nonlinear Anal.* **35**(2010), 33–42. https://doi.org/10.1007/s11721-009-0037-5; MR2677428; Zbl 1203.35274

[3] A. Azzollini, L. Pisani, A. Pomponio, Improved estimates and a limit case for the electrostatic Klein–Gordon–Maxwell system, *Proc. Roy. Soc. Edinburgh Sect. A* **141**(2011), 449–403. https://doi.org/10.1017/S0308210509001814; MR2805612; Zbl 1231.35244

[4] T. Bartsch, Z. Q. Wang, Existence and multiplicity results for some superlinear elliptic problem on $\mathbb{R}^N$, *Comm. Partial Differential Equations* **20**(1995), 1725–1741. https://doi.org/10.1080/03605309508821149; MR1349229; Zbl 0837.35043

[5] T. Bartsch, Infinitely many solutions of a symmetric Dirichlet problem, *Nonlinear Anal.* **20**(1993), 1205–1216. https://doi.org/10.1016/0362-546X(93)90151-H; MR1219237; Zbl 0799.35071

[6] V. Benci, D. Fortunato, The nonlinear Klein–Gordon equation coupled with the Maxwell equations, *Nonlinear Anal.* **47**(2001), 6065–6072. `https://doi.org/10.1016/S0362-546X(01)00688-5`; MR1970778; Zbl 1042.78500

[7] V. Benci, D. Fortunato, Solitary waves of the nonlinear Klein–Gordon equation coupled with the Maxwell equations, *Rev. Math. Phys.* **14**(2002), 409–420. `https://doi.org/10.1142/S0129055X02001168`; MR1901222; Zbl 1037.35075

[8] V. Benci, D. Fortunato, A. Masiello, L. Pisani, Solitons and the electromagnetic field, *Math. Z.* **232**(1999), No. 1, 73–102. MRR1714281; Zbl 0930.35168

[9] D. Cassani, Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell's equations, *Nonlinear Anal.* **58**(2004), 733–747. `https://doi.org/10.1016/j.na.2003.05.001`; MR2085333; Zbl 1057.35041

[10] P. C. Carrião, P. L. Cunha, O. H. Miyagaki, Existence results for the Klein–Gordon–Maxwell equations in higher dimensions with critical exponents, *Commun. Pure Appl. Anal.* **10**(2011), No. 2, 709–718. `https://doi.org/10.3934/cpaa.2011.10.709`; MR2754298; Zbl 1231.35247

[11] P. C. Carrião, P. L. Cunha, O. H. Miyagaki, Positive ground state solutions for the critical Klein–Gordon–Maxwell system with potentials, *Nonlinear Anal.* **75**(2012), No. 10, 4068–4078. `https://doi.org/10.1016/j.na.2012.02.023`; MR2914593; Zbl 1238.35109

[12] S. J. Chen, L. Li, Infinitely many solutions for Klein–Gordon–Maxwell system with potentials vanishing at infinity, *Z. Anal. Anwend.* **37**(2018), 39–50. `https://doi.org/10.4171/ZAA/1601`; MR3746496; Zbl 1388.35058

[13] H. Y. Chen, S. B. Liu, Standing waves with large frequency for 4-superlinear Schrödinger–Poisson systems, *Ann. Mat. Pura Appl. (4)* **194**(2015), 43–53. `https://doi.org/10.48550/arXiv.1211.3475`; MR3303004; Zbl 1309.35088

[14] S. J. Chen, S. Z. Song, Multiple solutions for nonhomogeneous Klein–Gordon–Maxwell equations on $\mathbb{R}^3$, *Nonlinear Anal. Real World Appl.* **22**(2015), 259–271. `https://doi.org/10.1016/j.nonrwa.2014.09.006`; MR3280832; Zbl 1306.35024

[15] S. J. Chen, C. L. Tang, High energy solutions for the superlinear Schrödinger–Maxwell equations, *Nonlinear Anal.* **71**(2009), 4927–4934. `https://doi.org/10.1016/j.na.2009.03.050`

[16] T. S. Chen, H. X. Tang, Improved results for Klein–Gordon–Maxwell systems with general nonlinearity, *Discrete Contin. Dyn. Syst.* **38**(2018), 2333–2348. `https://doi.org/10.3934/dcds.2018096`; MR3809038; Zbl 1398.35026

[17] Z. Chen, H. X. Tang, L. Qin, D. D. Qin, Improved results for Klein–Gordon–Maxwell systems with critical growth, *Appl. Math. Lett.* **91**(2019), 158–164. `https://doi.org/10.1016/j.aml.2018.12.015`; MR3895092; Zbl 1411.35080

[18] P. Cunha, Subcritical and supercritical Klein–Gordon–Maxwell equtions without Ambrosetti–Rabinowitz condition, *Differential Integral Equations* **27**(2014), No. 3–4, 387–399. `https://doi.org/10.1007/s10589-013-9591-2`; MR3161609; Zbl 1324.35021

[19] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations, *Proc. Roy. Soc. Edinb. Sect. A* **134**(2004), 893–906. https://doi.org/10.1017/S030821050000353X; MR2099569; Zbl 1064.35182

[20] P. d'Avenia, L. Pisani, G. Siciliano, Dirichlet and Neumann problems for Klein–Gordon–Maxwell systems, *Nonlinear Anal.* **71**(2009), No. 12, 1985–1995. https://doi.org/10.1134/S1061934809030010; MR2671970; Zbl 1238.35115

[21] L. Ding, L. Li, Infinitely many standing wave solutions for the nonlinear Klein–Gordon–Maxwell system with sign-changing potential, *Comput. Math. Appl.* **69**(2014), 5895–595. https://doi.org/10.1016/j.camwa.2014.07.001; MR3245804; Zbl 1362.35258

[22] L. G. Gan, T. Xiao, Q. F. Zhang, Improved results of nontrivial solutions for a nonlinear nonhomogeneous Klein–Gordon–Maxwell system involving sign-changing potential, *Adv. Difference Equ.* **2020**, Paper No. 167, 16 pp. https://doi.org/10.1186/s13662-020-02634-9; MR4087360; Zbl 1482.35224

[23] V. Georgiev, N. Visciglia, Solitary waves for Klein–Gordon–Maxwell system with external Coulomb potential, *J. Math. Pures Appl.* **9**(2005), 957–983. https://doi.org/10.1016/j.matpur.2004.09.016; MR2144648; Zbl 1078.35098

[24] X. M. He, Multiplicity of solutions for a nonlinear Klein–Gordon–Maxwell system, *Acta Appl. Math.* **130**(2014), 237–250. https://doi.org/10.1007/s10440-013-9845-0; MR3180946; Zbl 1301.35175

[25] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A* **129**(1999), 787–809. https://doi.org/10.1017/S0308210500013147; MR1718530; Zbl 0935.35044

[26] L. Li, C. L. Tang, Infinitely many solutions for a nonlinear Klein–Gordon–Maxwell system, *Nonlinear Anal.* **110**(2014), 157–169. https://doi.org/10.1016/j.na.2014.07.019; MR3259740; ZB1301.35136

[27] L. Li, A. Boucherif, N. D. Merzagui, Multiple solutions for 4-superlinear Klein–Gordon–Maxwell system without odd nonlinearity, *Taiwanese J. Math.* **21**(2017), No. 1, 151–165. https://doi.org/10.11650/tjm.21.2017.7680; MR3613978; Zbl 1360.35225

[28] X. Q. Liu, S. J. Chen, C. L. Tang, Infinitely many solutions and concentration of ground state solutions for the Klein–Gordon–Maxwell system, *J. Math. Anal. Appl.* **505**(2022), 1–18. https://doi.org/10.1016/j.jmaa.2021.125521; MR4295171; ZB1479.35344

[29] M. del Pino, P. Felmer, Local mountain passes for semilinear elliptic problems in unbounded domains, *Calc. Var. Partial Differential Equations* **4**(1996), 121–137. https://doi.org/10.1007/BF01189950; MR1379196; Zbl 0844.35032

[30] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. Math. Vol. 65, American Mathematical Society, Providence, 1986. https://doi.org/10.1007/978-1-4612-9892-2; MR0845785; Zbl 0609.58002

[31] D. Ruiz, The Schrödinger–Poisson equation under the effect of a nonlinear local term, *J. Funct. Anal.* **237**(2006), 655–674. https://doi.org/10.1016/j.jfa.2006.04.005; MR2230354; Zbl 1136.35037

[32] F. Z. WANG, Solitary waves for the Klein–Gordon–Maxwell system with critical exponent, *Nonlinear Anal.* **74**(2011), No.3, 827–835. https://doi.org/10.1016/j.na.2010.09.033; MR2738634; Zbl 1204.35159

[33] F. Z. WANG, Ground-state solutions for the electrostatic nonlinear Klein–Gordon–Maxwell system, *Nonlinear Anal.* **74**(2011), No. 14, 4796–4803. https://doi.org/10.1016/j.na.2011.04.050; MR2810718; Zbl 1225.35232

[34] Z. P. WANG, H. S. ZHOU, Positive solutions for a nonlinear stationary Schrödinger–Poisson system in $\mathbb{R}^3$, *Discrete Contin. Dyn. Syst.* **18**(2007), 809–816. https://doi.org/10.3934/dcds.2007.18.809; MR2318269; Zbl 1133.35427

[35] L. X. WANG, S. J. CHEN, Two solutions for nonhomogeneous Klein–Gordon–Maxwell system with sign-change potentail, *Electron. J. Differential Equations* **2018**, No. 124, 1–21. MR3831870; Zbl 1398.35218

[36] L. X. WANG, Two solutions for nonhomogeneous Klein–Gordon–Maxwell system, *Electron. J. Qual. Theory Differ. Equ.* **2019**, No. 40, 1–12. https://doi.org/10.14232/ejqtde.2019.1.40; MR3962461; Zbl 1438.35124

[37] L. X. WANG, X. M. WANG, L. Y. ZHANG, Ground state solutions for the critical Klein–Gordon–Maxwell system, *Acta Math. Sci. Ser. B (Engl. Ed.)* **39**(2019), No. 5, 1451–1460. https://doi.org/10.1007/s10473-019-0521-y; MR4068831; Zbl 1499.35065

[38] M. WILLEM, *Minimax theorems*, Progress in nonlinear differential equations and their applications, Vol. 24, Birkhäuser, Boston Inc., Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-4146-1; MR1400007; Zbl 0856.49001

[39] L. P. XU, H. B. CHEN, Existence and multiplicity of solutions for nonhomogenous Klein–Gordon–Maxwell equations, *Electron. J. Differential Equations* **2015**, No. 102, 1–12. MR3358474; Zbl 1315.35073

[40] W. M. ZOU, M. SCHECHTER, *Critical point theory and its applications*, Springer, New York, 2006. https://doi.org/10.1007/0-387-32968-4; MR2232879; Zbl 1125.58004