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Sharp results for oscillation of second-order neutral delay differential equations

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Abstract. The aim of the present paper is to continue earlier works by the authors on the oscillation problem of second-order half-linear neutral delay differential equations. By revising the set method, we present new oscillation criteria which essentially improve a number of related ones from the literature. A couple of examples illustrate the value of the results obtained.

Keywords: half-linear neutral differential equation, delay, second-order, oscillation.

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1 Introduction

In the paper, we consider the second-order half-linear neutral delay differential equation

$$\left(r\left(z'\right)^{\alpha}\right)'(t)+q(t)x^{\alpha}(\sigma(t))=0, \quad t\geq t_0>0, \tag{1.1}$$

where $z(t)=x(t)+p(t)x(\tau(t))$. As in [10], we will assume

(H$_1$) $\alpha>0$ is a quotient of odd positive integers;

(H$_2$) $r\in\mathcal{C}([t_0,\infty),(0,\infty))$ satisfies

$$\pi(t_0):=\int_{t_0}^{\infty}r^{-1/\alpha}(s)\mathrm{d}s<\infty;$$

(H$_3$) σ, $\tau\in\mathcal{C}([t_0,\infty),\mathbb{R})$, $\sigma(t)\leq t$, and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\sigma(t)=\infty$;

(H$_4$) $p\in\mathcal{C}([t_0,\infty),[0,\infty))$ and $q\in\mathcal{C}([t_0,\infty),(0,\infty))$;

(H$_5$) there exists a constant $p_0\in[0,1)$ such that

$$p_0\geq\begin{cases}p(t)\frac{\pi(\tau(t))}{\pi(t)} & \text{for } \tau(t)\leq t\\ p(t) & \text{for } \tau(t)\geq t.\end{cases} \tag{1.2}$$

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Under a solution of (1.1), we mean a function $x \in \mathcal{C}([t_a,\infty),\mathbb{R})$ with $t_a = \min\{\tau(t_b),\sigma(t_b)\}$, for some $t_b \geq t_0$, which has the property $z \in \mathcal{C}^1([t_a,\infty),\mathbb{R})$, $r\,(z')^\alpha \in \mathcal{C}^1([t_a,\infty),\mathbb{R})$ and satisfies (1.1) on $[t_b,\infty)$. We only consider those solutions of (1.1) which exist on some half-line $[t_b,\infty)$ and satisfy the condition $\sup\{|x(t)| : t_c \leq t < \infty\} > 0$ for any $t_c \geq t_b$. Oscillation and nonoscillation of such solutions is defined in the usual way.

Oscillation theory of second-order differential equations has gained much research interest in the past decades, and we refer the reader to the monographs by Agarwal et al. [1,3,4], Berezansky et al. [7], and Saker [33] for recent developments and summaries of known results. Due to the importance of second-order neutral differential equations in the modeling of various phenomena in natural sciences and engineering [12,18,33], the qualitative behavior of solutions such equations has been intensively studied through different techniques.

This paper is the second continuation of our earlier work [9] from 2017, followed by [10] in 2020. To start with, let us summarize briefly the two main ideas employed therein. Let x be a nonoscillatory, say positive solution of (1.1) subject to (H_1)–(H_5). Then z is also positive and either strictly increasing or strictly decreasing. These two possible classes of nonoscillatory solutions were treated independently in the literature, see, e.g., [2,5,19,22–24,26,36–39]. In [9], we pointed out that conditions eliminating positive solutions x with z decreasing are sufficient for the nonexistence of those with z increasing. This observation allowed us to remove a redundant but commonly imposed condition and formulate, in contrast with existing works, single-condition oscillation criteria.

To eliminate the important class of positive solutions with z decreasing, the second main idea in [9] was to sharpen the lower bound 1 of the quantity $z(\sigma(t))/z(t)$ using equation (1.1) itself, which, within the Riccati transformation technique, led to qualitatively stronger results. However, such a lower bound strongly depended on properties of first-order delay differential equations and required σ to be nondecreasing.

The ideas from [9] have been extended and applied in investigation of various classes of equations, e.g., half-linear neutral differential equations with: damping term [28,35], sublinear term [13,15,34], several delay arguments [30]; generalized Emden–Fowler neutral differential equations [25,27,32], half-linear neutral difference equations [8,11,16], neutral dynamic equations on time scales [17,31,40,41], and others.

In [10], we continued our work [9] by removing the restrictions (see [9, (H_3)]) $\tau(t) \leq t$ and $\sigma'(t) \geq 0$. For the reader's convenience, we recall the main results from [10], formulated in terms of the following couple of limit inferiors:

$$\beta_* := \frac{1}{\alpha} \liminf_{t\to\infty} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t) \quad \text{and} \quad \lambda_* := \liminf_{t\to\infty} \frac{\pi(\sigma(t))}{\pi(t)}. \tag{1.3}$$

Theorem A (See [10, Theorem 1, Theorem 2]). *If*

$$\beta_* > \begin{cases} 0 & \text{for } \lambda_* = \infty, \\ \dfrac{\max\{b^\alpha(1-b)\lambda_*^{-\alpha b} : 0 < b < 1\}}{(1-p_0)^\alpha} & \text{for } \lambda_* < \infty, \end{cases}$$

then (1.1) *is oscillatory.*

Although the obtained results can be seen as sharp in the sense that they are unimprovable in a nonneutral case, it is easy to observe that Theorem A does not take the influence of $\tau(t) \geq t$ into account and becomes inefficient as p_0 is close to 1. The aim of this paper is to address these issues and to improve Theorem A when $\lambda_* < \infty$ and $p(t) \neq 0$. As in [10], we employ a recent method of sequentially improved monotonicities of nonoscillatory solutions of binomial differential equations, which has been successfully applied in the investigation of second-order half-linear functional differential equations and as well as linear differential and difference equations of higher order. For a discussion on the results already achieved by the method so far, we refer the reader to [21, Section 4].

For the sake of completeness, let us recall the three main steps of the method we used in [10]: firstly, we showed that the positivity of β_* is sufficient for the nonexistence of positive solutions x with z positive and increasing; secondly, we provided, for x positive with z decreasing, bounds of the ratio x/z, i.e.,

$$1 - p_0 \leq \frac{x}{z} \leq 1. \tag{1.4}$$

The third step was intended to improve the lower bound 1 of the quantity $z(\sigma(t))/z(t)$ so that it was, unlike the one we used in [9], independent of the properties of first-order delay differential equations and the monotone growth of σ. We related this problem to that of finding an optimal value $a > 0$ such that

$$a \leq \frac{-r^{1/\alpha} z' \pi}{z},$$

which corresponds to the monotonicity

$$\left(\frac{z}{\pi^a}\right)' < 0,$$

and tackled it by building an appropriate sequence defined in terms of β_* and λ_*. It turned out that the convergence of the given sequence was necessary for the existence of a nonoscillatory solution of (1.1), and Theorem A emerged as a simple consequence of this fact.

In this work, we revise the set method as follows. Firstly, we provide a sharper lower bound of the quantity x/z than in (1.4). Secondly, we sequentially improve both lower and upper bounds of the ratio $-\pi r^{1/\alpha} z'/z$ up to their limit values by building two iteration processes represented by the sequences $\{\beta_{k,n}\}_{n\in\mathbb{N}_0}$ and $\{\gamma_{k,n}\}_{n\in\mathbb{N}_0}$ (see Section 2) such that

$$\beta_{k,n} \leq \frac{-r^{1/\alpha} z' \pi}{z} \leq 1 - \gamma_{k,n},$$

which correspond to the monotonicities

$$\left(\frac{z}{\pi^{\beta_{k,n}}}\right)' < 0 \quad \text{and} \quad \left(\frac{z}{\pi^{1-\gamma_{k,n}}}\right)' \geq 0,$$

allowing us to improve the lower bound of x/z in each iteration step. Finally, we state the main results – sufficient conditions for (1.1) to be oscillatory – as a direct consequence of these obtained bounds. To illustrate the applicability of the results, two examples are given.

2 Notation and preliminary results

In this section, we list all constants and functions used in the paper. For any $k \in \mathbb{N}_0$, we set

$$\beta_k^* := \frac{1}{\alpha} \liminf_{t\to\infty} r^{1/\alpha}(t) \pi^{\alpha+1}(t) q(t) \left(1 + H_k(\sigma(t))\right)^{\alpha}, \tag{2.1}$$

where

$$H_k(t) = \begin{cases} 0 & \text{for } k = 0, \\ \sum_{i=1}^{k} \prod_{j=0}^{2i-1} p(\tau^j(t)) & \text{for } \tau(t) \le t \text{ and } k \in \mathbb{N}, \\ \sum_{i=1}^{k} \frac{\pi(\tau^{2i}(t))}{\pi(t)} \prod_{j=0}^{2i-1} p(\tau^j(t)) & \text{for } \tau(t) \ge t \text{ and } k \in \mathbb{N}, \end{cases}$$

where $\tau^0(t) = t$ and $\tau^j(t) = \tau(\tau^{j-1}(t))$ for all $j \in \mathbb{N}$. As in [10], we set

$$\lambda_* := \liminf_{t\to\infty} \frac{\pi(\sigma(t))}{\pi(t)}$$

and, in addition, we put

$$\psi_* := \liminf_{t\to\infty} \frac{\pi(\tau(t))}{\pi(t)} \quad \text{for } \tau(t) \le t,$$
$$\omega_* := \liminf_{t\to\infty} \frac{\pi(t)}{\pi(\tau(t))} \quad \text{for } \tau(t) \ge t.$$

By virtue of (H_2) and (H_3), it is immediate to see that $\{\lambda_*, \omega_*, \psi_*\} \in [1, \infty)$. Our reasoning will often rely on the obvious fact that there is a $t_1 \ge t_0$ large enough such that, for arbitrary fixed $\beta_k \in (0, \beta_k^*)$, $\lambda \in [1, \lambda_*)$, $\psi \in [1, \psi_*)$, and $\omega \in [1, \omega_*)$, we have

$$\begin{aligned} r^{1/\alpha}(t)\pi^{\alpha+1}(t)q(t)\left(1 + H_k(\sigma(t))\right)^\alpha &\ge \alpha\beta_k, \\ \frac{\pi(\sigma(t))}{\pi(t)} &\ge \lambda, \\ \frac{\pi(\tau(t))}{\pi(t)} &\ge \psi \quad \text{for } \tau(t) \le t, \\ \frac{\pi(t)}{\pi(\tau(t))} &\ge \omega \quad \text{for } \tau(t) \ge t, \end{aligned} \tag{2.2}$$

on $[t_1, \infty)$.

Remark 2.1. In our previous work [10], we formulated the results in terms of $\beta_0^* = \beta_*$ (see (1.3)), which we required to be positive. Clearly, for any $k \in \mathbb{N}$, the positivity of β_0^* is sufficient for that of β_k^*.

Lemma 2.2. *If $\tau(t) \le t$ and $\psi_* = \infty$, or $\tau(t) \ge t$ and $\omega_* = \infty$, then*

$$\liminf_{t\to\infty} H_k(t) = 0 \quad \text{for any } k \in \mathbb{N}$$

and so $\beta_k^* = \beta_0^*$ for any $k \in \mathbb{N}$.

Proof. Using (H_2) and (H_5), the proof is obvious and hence omitted. □

The method used in this paper is based on the properties of the sequences $\{\beta_{k,n}\}_{n\in\mathbb{N}_0}$ and $\{\gamma_{k,n}\}_{n\in\mathbb{N}_0}$, which we define (as long as they exist) as follows. For positive and finite β_k^*, λ_*, ψ_*, and ω_*, we set, for any $k \in \mathbb{N}_0$ fixed,

$$\beta_{k,0} := (1 - p_0)\sqrt[\alpha]{\beta_k^*},$$
$$\gamma_{k,0} := (1 - p_0)^\alpha \beta_k^* = \beta_{k,0}^\alpha,$$

and for $n \in \mathbb{N}_0$, we put

1. for $\tau(t) \le t$ and $\psi_* = \infty$ or $\tau(t) \ge t$ and $\omega_* = \infty$:

$$\beta_{k,n+1} := \lambda_*^{\beta_{k,n}} \sqrt[\alpha]{\frac{\beta_k^*}{1-\beta_{k,n}}} = \lambda_*^{\beta_{0,n}} \sqrt[\alpha]{\frac{\beta_0^*}{1-\beta_{0,n}}},$$
$$\gamma_{k,n+1} := \beta_k^* \left(\frac{\lambda_*^{\beta_{k,n}}}{1-\gamma_{k,n}}\right)^\alpha = \beta_0^* \left(\frac{\lambda_*^{\beta_{0,n}}}{1-\gamma_{0,n}}\right)^\alpha$$

2. for $\tau(t) \le t$ and $\psi_* < \infty$:

$$\beta_{k,n+1} := \frac{\beta_{k,0}\lambda_*^{\beta_{k,n}}}{\sqrt[\alpha]{1-\beta_{k,n}}} \left(\frac{1-p_0\psi_*^{-\gamma_{k,n}}}{1-p_0}\right) = \lambda_*^{\beta_{k,n}} \sqrt[\alpha]{\frac{\beta_k^*}{1-\beta_{k,n}}}(1-p_0\psi_*^{-\gamma_{k,n}}),$$
$$\gamma_{k,n+1} := \frac{\gamma_{k,0}\lambda_*^{\alpha\beta_{k,n}}}{(1-\gamma_{k,n})^\alpha} \left(\frac{1-p_0\psi_*^{-\gamma_{k,n}}}{1-p_0}\right)^\alpha = \beta_k^* \left(\frac{\lambda_*^{\beta_{k,n}}}{1-\gamma_{k,n}}\right)^\alpha (1-p_0\psi_*^{-\gamma_{k,n}})^\alpha$$

3. for $\tau(t) \ge t$ and $\omega_* < \infty$:

$$\beta_{k,n+1} := \frac{\beta_{k,0}\lambda_*^{\beta_{k,n}}}{\sqrt[\alpha]{1-\beta_{k,n}}} \left(\frac{1-p_0\omega_*^{-\beta_{k,n}}}{1-p_0}\right) = \lambda_*^{\beta_{k,n}} \sqrt[\alpha]{\frac{\beta_k^*}{1-\beta_{k,n}}}(1-p_0\omega_*^{-\beta_{k,n}}),$$
$$\gamma_{k,n+1} := \frac{\gamma_{k,0}\lambda_*^{\alpha\beta_{k,n}}}{(1-\gamma_{k,n})^\alpha} \left(\frac{1-p_0\omega_*^{-\beta_{k,n}}}{1-p_0}\right)^\alpha = \beta_k^* \left(\frac{\lambda_*^{\beta_{k,n}}}{1-\gamma_{k,n}}\right)^\alpha (1-p_0\omega_*^{-\beta_{k,n}})^\alpha.$$

It can be easily verified by induction that if for some $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ fixed, $\beta_{k,i} < 1$ and $\gamma_{k,i} < 1$, $i = 0, 1, \dots, n$, then $\beta_{k,n+1}$ and $\gamma_{k,n+1}$ exist and

$$\begin{aligned} \beta_{k,n+1} &= \ell_{k,n}\beta_{k,n} > \beta_{k,n}, \\ \gamma_{k,n+1} &= h_{k,n}\gamma_{k,n} > \gamma_{k,n}, \end{aligned} \tag{2.3}$$

where $\ell_{k,n}$ and $h_{k,n}$ are defined as follows:

1. for $\tau(t) \le t$ and $\psi_* = \infty$ or $\tau(t) \ge t$ and $\omega_* = \infty$:

$$\ell_{k,0} := \frac{\lambda_*^{\beta_{k,0}}}{(1-p_0)\sqrt[\alpha]{1-\beta_{k,0}}},$$
$$\ell_{k,n+1} := \lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \sqrt[\alpha]{\frac{1-\beta_{k,n}}{1-\ell_{k,n}\beta_{k,n}}}$$

and

$$h_{k,0} := \left[\frac{\lambda_*^{\beta_{k,0}}}{(1-\gamma_{k,0})(1-p_0)}\right]^\alpha,$$
$$h_{k,n+1} := \left[\lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \left(\frac{1-\gamma_{k,n}}{1-h_{k,n}\gamma_{k,n}}\right)\right]^\alpha$$

2. for $\tau(t) \le t$ and $\psi_* < \infty$:

$$\ell_{k,0} := \frac{\lambda_*^{\beta_{k,0}}}{\sqrt[\alpha]{1-\beta_{k,0}}} \left(\frac{1-p_0\psi_*^{-\gamma_{k,0}}}{1-p_0}\right),$$
$$\ell_{k,n+1} := \lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \sqrt[\alpha]{\frac{1-\beta_{k,n}}{1-\ell_{k,n}\beta_{k,n}}} \left(\frac{1-p_0\psi_*^{-h_{k,n}\gamma_{k,n}}}{1-p_0\psi_*^{-\gamma_{k,n}}}\right)$$

and

$$h_{k,0} := \left[\frac{\lambda_*^{\beta_{k,0}}}{1-\gamma_{k,0}} \left(\frac{1-p_0\psi_*^{-\gamma_{k,0}}}{1-p_0} \right) \right]^{\alpha},$$
$$h_{k,n+1} := \left[\lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \left(\frac{1-\gamma_{k,n}}{1-h_{k,n}\gamma_{k,n}} \right) \left(\frac{1-p_0\psi_*^{-h_{k,n}\gamma_{k,n}}}{1-p_0\psi_*^{-\gamma_{k,n}}} \right) \right]^{\alpha}$$

3. for $\tau(t) \geq t$ and $\omega_* < \infty$:

$$\ell_{k,0} := \frac{\lambda_*^{\beta_{k,0}}}{\sqrt[\alpha]{1-\beta_{k,0}}} \left(\frac{1-p_0\omega_*^{-\beta_{k,0}}}{1-p_0} \right),$$
$$\ell_{k,n+1} := \lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \sqrt[\alpha]{\frac{1-\beta_{k,n}}{1-\ell_{k,n}\beta_{k,n}}} \left(\frac{1-p_0\omega_*^{-\ell_{k,n}\beta_{k,n}}}{1-p_0\omega_*^{-\beta_{k,n}}} \right)$$

and

$$h_{k,0} := \left[\frac{\lambda_*^{\beta_{k,0}}}{1-\gamma_{k,0}} \left(\frac{1-p_0\omega_*^{-\beta_{k,0}}}{1-p_0} \right) \right]^{\alpha},$$
$$h_{k,n+1} := \left[\lambda_*^{\beta_{k,n}(\ell_{k,n}-1)} \left(\frac{1-\gamma_{k,n}}{1-h_{k,n}\gamma_{k,n}} \right) \left(\frac{1-p_0\omega_*^{-\ell_{k,n}\beta_{k,n}}}{1-p_0\omega_*^{-\beta_{k,n}}} \right) \right]^{\alpha}.$$

The following simple statement, resulting from the definition of the sequences $\{\beta_{k,n}\}_{n\in\mathbb{N}_0}$ and $\{\gamma_{k,n}\}_{n\in\mathbb{N}_0}$ and (2.3), will play an important role in obtaining our main results. As a matter of fact, we will show (see Corollary 3.8) that all assumptions of Lemma 2.3 are necessary for the existence of a nonoscillatory solution of (1.1), i.e., if (1.1) possesses a nonoscillatory solution, then there exists a solution $\{b, g\} \in (0,1)$ of a particular limit system.

Lemma 2.3. *Let $\beta_0^* > 0$, $\lambda_* < \infty$, and the sequences $\{\beta_{k,n}\}_{n\in\mathbb{N}_0}$ and $\{\gamma_{k,n}\}_{n\in\mathbb{N}_0}$ be well-defined and bounded from above for some fixed $k \in \mathbb{N}_0$. Then*

$$\lim_{n\to\infty} \beta_{k,n} = b \in (0,1)$$

and

$$\lim_{n\to\infty} \gamma_{k,n} = g \in (0,1),$$

where $\{b, g\}$ is a solution of the system

1. *for $\tau(t) \leq t$ and $\psi_* = \infty$ or $\tau(t) \geq t$ and $\omega_* = \infty$:*

$$\begin{cases} \beta_0^* = b^{\alpha}(1-b)\lambda_*^{-\alpha b} \\ \beta_0^* = g(1-g)^{\alpha}\lambda_*^{-\alpha b} \end{cases} \tag{2.4}$$

2. *for $\tau(t) \leq t$ and $\psi_* < \infty$:*

$$\begin{cases} \beta_k^* = \dfrac{b^{\alpha}(1-b)\lambda_*^{-\alpha b}}{\left(1-p_0\psi_*^{-g}\right)^{\alpha}} \\[2ex] \beta_k^* = \dfrac{g(1-g)^{\alpha}\lambda_*^{-\alpha b}}{\left(1-p_0\psi_*^{-g}\right)^{\alpha}} \end{cases} \tag{2.5}$$

3. *for $\tau(t) \geq t$ and $\omega_* < \infty$:*

$$\begin{cases} \beta_k^* = \dfrac{b^\alpha(1-b)\lambda_*^{-\alpha b}}{\left(1-p_0\omega_*^{-b}\right)^\alpha} \\ \beta_k^* = \dfrac{g(1-g)^\alpha\lambda_*^{-\alpha b}}{\left(1-p_0\omega_*^{-b}\right)^\alpha}. \end{cases} \tag{2.6}$$

3 Main results

In the sequel, all occurring functional inequalities are assumed to hold eventually, that is, they are satisfied for all t large enough. As usual and without loss of generality, in the proofs of the main results, we only need to be concerned with positive solutions of (1.1) since the proofs for eventually negative solutions are similar.

We start by recalling an important result from our previous work.

Lemma 3.1 (See [10, Lemma 2]). *Let $\beta_0^* > 0$. If x is an eventually positive solution of* (1.1)*, then z eventually satisfies*

(i) $z > 0$, $\left(r\left(z'\right)^\alpha\right)' < 0$, *and* $x(t) \geq z(t) - p(t)z(\tau(t))$;

(ii) $z' < 0$;

(iii) $(z/\pi)' \geq 0$;

(iv) $x \geq (1-p_0)z$;

(v) $\lim_{t\to\infty} z(t) = 0$.

In order to improve the estimate (iv) between x and z, we need the following auxiliary result.

Lemma 3.2. *If x is an eventually positive solution of* (1.1)*, then z eventually satisfies*

$$\begin{aligned} x(t) &\geq z(t) - p(t)z(\tau(t)) \\ &\quad + \sum_{i=1}^{k}\left(\prod_{j=0}^{2i-1} p(\tau^j(t))\right)\left[z(\tau^{2i}(t)) - p(\tau^{2i}(t))z(\tau^{2i+1}(t))\right], \quad k \in \mathbb{N}. \end{aligned} \tag{3.1}$$

Proof. It follows from the definition of z that

$$\begin{aligned} x(t) &= z(t) - p(t)x(\tau(t)) \\ &= z(t) - p(t)\left[z(\tau(t)) - p(\tau(t))x(\tau^2(t))\right] \\ &= z(t) - p(t)z(\tau(t)) + p(t)p(\tau(t))x(\tau^2(t)). \end{aligned} \tag{3.2}$$

Evaluating (3.2) in $\tau^2(t)$, we get

$$x(\tau^2(t)) = z(\tau^2(t)) - p(\tau^2(t))z(\tau^3(t)) + p(\tau^2(t))p(\tau^3(t))x(\tau^4(t)). \tag{3.3}$$

Now using (3.3) in (3.2), we have

$$\begin{aligned} x(t) &= z(t) - p(t)z(\tau(t)) \\ &\quad + p(t)p(\tau(t))\left[z(\tau^2(t)) - p(\tau^2(t))z(\tau^3(t))\right] \\ &\quad + p(t)p(\tau(t))p(\tau^2(t))p(\tau^3(t))x(\tau^4(t)). \end{aligned}$$

Repeating the process, it is easy to show via induction that

$$\begin{aligned} x(t) &= z(t) - p(t)z(\tau(t)) \\ &+ \sum_{i=1}^{k} \left(\prod_{j=0}^{2i-1} p(\tau^j(t)) \right) \left[z(\tau^{2i}(t)) - p(\tau^{2i}(t)) z(\tau^{2i+1}(t)) \right] \\ &+ \left(\prod_{j=0}^{2k+1} p(\tau^j(t)) \right) x(\tau^{2k+2}(t)), \end{aligned}$$

which implies (3.1). The proof is complete. □

Lemma 3.3. *Let $\beta_0^* > 0$. If x is an eventually positive solution of* (1.1), *then z eventually satisfies*

$$x(t) \geq z(t)(1 - p_0) \left(1 + H_k(t)\right), \quad k \in \mathbb{N}_0. \tag{3.4}$$

Proof. First, let $\tau(t) \leq t$. Using the fact that z/π is nondecreasing (see Lemma 3.1 (iii)) and (H_5), we have

$$z(t) - p(t)z(\tau(t)) \geq z(t) - p(t)\frac{\pi(\tau(t))}{\pi(t)} z(t) \geq z(t)(1 - p_0). \tag{3.5}$$

Evaluating (3.5) in $\tau^{2i}(t)$ and using that z is decreasing (see Lemma 3.1 (ii)), we obtain

$$z(\tau^{2i}(t)) - p(\tau^{2i}(t)) z(\tau^{2i+1}(t)) \geq z(\tau^{2i}(t))(1 - p_0) \geq z(t)(1 - p_0). \tag{3.6}$$

Using (3.5) and (3.6) in (3.1), we get

$$x(t) \geq z(t)(1 - p_0) \left[1 + \sum_{i=1}^{k} \prod_{j=0}^{2i-1} p(\tau^j(t)) \right], \quad k \in \mathbb{N}.$$

and hence, (3.4) holds. Now, let $\tau(t) \geq t$. Again, by Lemma 3.1 (ii), (iii) and (H_5), we see that

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &\geq z(t) - p(t)z(t) \\ &\geq z(t)(1 - p_0) \end{aligned}$$

and

$$\begin{aligned} z(\tau^{2i}(t)) - p(\tau^{2i}(t)) z(\tau^{2i+1}(t)) &\geq z(\tau^{2i}(t)) \left(1 - p(\tau^{2i}(t)) \right) \\ &\geq z(\tau^{2i}(t))(1 - p_0) \\ &\geq z(t) \frac{\pi(\tau^{2i}(t))}{\pi(t)} (1 - p_0), \end{aligned}$$

which in view of (3.1) yields

$$x(t) \geq z(t)(1 - p_0) \left[1 + \sum_{i=1}^{k} \frac{\pi(\tau^{2i}(t))}{\pi(t)} \prod_{j=0}^{2i-1} p(\tau^j(t)) \right], \quad k \in \mathbb{N},$$

and hence, (3.4) holds in this case as well. The proof is complete. □

Remark 3.4. In [20], the authors investigated (1.1) with $p(t) \equiv p > 0$ and $\tau(t) < t$, and required, instead of (H_5), that

$$p_* = \sum_{k=0}^{(n-1)/2} p_0^{2k} \left(1 - p\frac{\pi(\tau^{2k+1})(t)}{\pi(\tau^{2k}(t))}\right) > 0, \quad n \in \mathbb{N}.$$

Then they proved that an eventually positive solution of (1.1) satisfies

$$x \geq (1 - p_*)z. \tag{3.7}$$

Note that (H_5) is sufficient for the positivity of p_* and consequently, (3.7) becomes a particular case of (3.4).

The next step of our approach lies in improving Lemma 3.1 (ii)–(iv) by using the equation (1.1) itself, which can be seen as an improved and extended variant of [10, Lemma 3]. While the improved decreasing monotonicity $(i)_0$ results from minor modification of the original proof, the opposite monotonicity $(ii)_0$, needed to sharpen the relation between x and z in $(iii)_0$, extends the original version of [10, Lemma 3].

Lemma 3.5. *Assume $\beta_0^* > 0$. If x is an eventually positive solution of (1.1), then, for any $\beta_k \in (0, \beta_k^*)$ with $k \in \mathbb{N}_0$ fixed,*

$(i)_0$ $(z/\pi^{\sqrt[\alpha]{\beta_k}(1-p_0)})' < 0$;

$(ii)_0$ $(z/\pi^{1-\beta_k(1-p_0)^\alpha})' \geq 0$;

$(iii)_0$ $x \geq a_k(1 + H_k)z$, *where*

$$a_k = \begin{cases} \varepsilon & \text{for } \tau(t) \leq t,\ \psi_* = \infty \text{ or } \tau(t) \geq t,\ \omega_* = \infty, \text{ and any } \varepsilon \in (0,1); \\ 1 - p_0\psi^{-\beta_k(1-p_0)^\alpha} & \text{for } \tau(t) \leq t, \psi_* < \infty, \text{ and any } \psi \in [1, \psi_*); \\ 1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)} & \text{for } \tau(t) \geq t, \omega_* < \infty, \text{ and any } \omega \in [1, \omega_*), \end{cases}$$

eventually.

Proof. Pick $t_1 \geq t_0$ such that

$$x(t) > 0, \quad x(\tau(t)) > 0, \quad \text{and } x(\sigma(t)) > 0,$$

z satisfies Lemma 3.1 with (iv) replaced by (3.4), and (2.2) holds for $t \geq t_1$. Using (3.4) in (1.1), we have

$$\left(r\left(z'\right)^\alpha\right)'(t) + (1 - p_0)^\alpha q(t)(1 + H_k(\sigma(t)))^\alpha z^\alpha(\sigma(t)) \leq 0, \quad t \geq t_1,$$

which in view of (2.2) implies

$$\left(r\left(z'\right)^\alpha\right)'(t) + \frac{\beta_k\alpha(1-p_0)^\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} z^\alpha(\sigma(t)) \leq 0. \tag{3.8}$$

Now using that z is decreasing (see Lemma 3.1 (ii)) and (H_3), we find

$$\frac{z(\sigma(t))}{z(t)} \geq 1. \tag{3.9}$$

Hence, (3.8) becomes

$$\left(r\left(z'\right)^\alpha\right)'(t) + \frac{\beta_k\alpha(1-p_0)^\alpha}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} z^\alpha(t) \leq 0. \tag{3.10}$$

(i)$_0$ Integrating (3.10) from t_1 to t and using again Lemma 3.1 (ii), we find

$$\begin{aligned}-r(t)\left(z'(t)\right)^{\alpha} &\geq -r(t_1)\left(z'(t_1)\right)^{\alpha}+\beta_k(1-p_0)^{\alpha}\int_{t_1}^{t}\frac{\alpha z^{\alpha}(s)}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)}\mathrm{d}s\\ &\geq -r(t_1)\left(z'(t_1)\right)^{\alpha}+\beta_k(1-p_0)^{\alpha}z^{\alpha}(t)\int_{t_1}^{t}\frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha+1}(s)}\mathrm{d}s\\ &= -r(t_1)\left(z'(t_1)\right)^{\alpha}+\beta_k(1-p_0)^{\alpha}z^{\alpha}(t)\left(\frac{1}{\pi^{\alpha}(t)}-\frac{1}{\pi^{\alpha}(t_1)}\right).\end{aligned}\tag{3.11}$$

Since $\lim_{t\to\infty}z(t)=0$ (see Lemma 3.1 (v)), there exists $t_2\geq t_1$ such that

$$-r(t_1)\left(z'(t_1)\right)^{\alpha}>\frac{\beta_k(1-p_0)^{\alpha}}{\pi^{\alpha}(t_1)}z^{\alpha}(t),\quad t\geq t_2.$$

Using this in (3.11) yields

$$-r^{1/\alpha}z'\pi>\sqrt[\alpha]{\beta_k}(1-p_0)z$$

and so (i)$_0$ holds.

(ii)$_0$ Set

$$Z:=z+r^{1/\alpha}z'\pi.\tag{3.12}$$

Since z/π is nondecreasing (see Lemma 3.1 (iii)), Z is clearly nonnegative. Differentiating Z and using the chain rule

$$\left(r(z')^{\alpha}\right)'=\alpha\left(r^{1/\alpha}z'\right)^{\alpha-1}\left(r^{1/\alpha}z'\right)'$$

along with (3.10), we get

$$\begin{aligned}Z'&=\left(r^{1/\alpha}z'\right)'\pi\\ &=\frac{\pi}{\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}\left(r\left(z'\right)^{\alpha}\right)'\\ &\leq -\frac{\pi}{\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}\frac{\beta_k\alpha(1-p_0)^{\alpha}}{r^{1/\alpha}\pi^{\alpha+1}}z^{\alpha}\\ &= -\frac{\beta_k(1-p_0)^{\alpha}}{r^{1/\alpha}\pi^{\alpha}}\left(r^{1/\alpha}z'\right)^{1-\alpha}z^{\alpha}<0.\end{aligned}\tag{3.13}$$

Using again Lemma 3.1 (iii) in (3.13), we obtain

$$Z'\leq -\frac{\beta_k(1-p_0)^{\alpha}}{r^{1/\alpha}\pi^{\alpha}}\left(r^{1/\alpha}z'\right)^{1-\alpha}(-r^{1/\alpha}z')^{\alpha}\pi^{\alpha}=\beta_k(1-p_0)^{\alpha}z'.$$

Integrating the above inequality from t to ∞ and using that z is decreasing and tending to zero eventually (see Lemma 3.1 (ii) and (v)), we have

$$Z(t)\geq Z(\infty)-\beta_k(1-p_0)^{\alpha}z(\infty)+\beta_k(1-p_0)^{\alpha}z(t)\geq\beta_k(1-p_0)^{\alpha}z(t),$$

which in view of the definition of Z gives

$$(1-\beta_k(1-p_0)^{\alpha})z\geq -r^{1/\alpha}z'\pi.$$

Hence, (ii)$_0$ holds.

(iii)$_0$ First, let $\tau(t) \leq t$. Using (ii)$_0$ and (H$_5$), we see that

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &\geq z(t) - p(t)\left(\frac{\pi(\tau(t))}{\pi(t)}\right)^{1-\beta_k(1-p_0)^\alpha} z(t) \\ &\geq z(t)\left(1 - p_0\left(\frac{\pi(t)}{\pi(\tau(t))}\right)^{\beta_k(1-p_0)^\alpha}\right) \\ &\geq z(t)\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right). \end{aligned} \tag{3.14}$$

Evaluating (3.14) in $\tau^{2i}(t)$ and using the decreasing nature of z (see Lemma 3.1 (ii)), we get

$$\begin{aligned} z(\tau^{2i}(t)) - p(\tau^{2i}(t))z(\tau^{2i+1}(t)) &\geq z(\tau^{2i}(t))\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right) \\ &\geq z(t)\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right). \end{aligned} \tag{3.15}$$

Using (3.14) and (3.15) in (3.1), we find

$$\begin{aligned} x(t) &\geq z(t)\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right)\left[1 + \sum_{i=1}^{k}\prod_{j=0}^{2i-1} p(\tau^j(t))\right] \\ &= z(t)\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right)(1 + H_k(t)). \end{aligned}$$

If $\tau(t) \geq t$, then similarly as before, we get

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &\geq z(t) - p(t)\left(\frac{\pi(\tau(t))}{\pi(t)}\right)^{\sqrt[\alpha]{\beta_k}(1-p_0)} z(t) \\ &\geq z(t)\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right), \end{aligned}$$

where we used (i)$_0$ and (H$_5$). Evaluating the above inequality in $\tau^{2i}(t)$ and using the nonincreasing nature of z/π (see Lemma 3.1 (iii)), we obtain

$$\begin{aligned} z(\tau^{2i}(t)) - p(\tau^{2i}(t))z(\tau^{2i+1}(t)) &\geq z(\tau^{2i}(t))\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right) \\ &\geq z(t)\frac{\pi(\tau^{2i}(t))}{\pi(t)}\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right). \end{aligned}$$

Then,

$$\begin{aligned} x(t) &\geq z(t)\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right)\left[1 + \sum_{i=1}^{k}\frac{\pi(\tau^{2i}(t))}{\pi(t)}\prod_{j=0}^{2i-1} p(\tau^j(t))\right] \\ &= z(t)\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right)(1 + H_k(t)). \end{aligned}$$

Finally, if $\tau(t) \leq t$ and $\psi_* = \infty$ [$\tau(t) \geq t$ and $\omega_* = \infty$], then it follows from Lemma 2.2 that for any $\varepsilon \in (0,1)$, there is t sufficiently large such that

$$\left(1 - p_0\psi^{-\beta_k(1-p_0)^\alpha}\right)(1 + H_k(t)) < \varepsilon \quad \left[\left(1 - p_0\omega^{-\sqrt[\alpha]{\beta_k}(1-p_0)}\right)(1 + H_k(t)) < \varepsilon\right].$$

The proof is complete. $\square$

The following result iteratively improves the previous one.

Lemma 3.6. *Assume $\beta_0^* > 0$. If x is an eventually positive solution of (1.1), then, for any k, $n \in \mathbb{N}_0$,*

(i)$_n$ $(z/\pi^{\beta_{k,n}})' < 0$;

(ii)$_n$ $(z/\pi^{1-\gamma_{k,n}})' > 0$;

(iii)$_n$ $x \geq a_{k,n}(1 + H_k)z$, *where*

$$a_k = \begin{cases} \varepsilon & \text{for } \tau(t) \leq t,\ \psi_* = \infty \text{ or } \tau(t) \geq t,\ \omega_* = \infty, \text{ and any } \varepsilon \in (0,1); \\ 1 - p_0\psi^{-\gamma_{k,n}} & \text{for } \tau(t) \leq t,\ \psi_* < \infty, \text{ and any } \psi \in [1, \psi_*); \\ 1 - p_0\omega^{-\beta_{k,n}} & \text{for } \tau(t) \geq t,\ \omega_* < \infty, \text{ and any } \omega \in [1, \omega_*), \end{cases}$$

eventually.

Proof. Pick $t_1 \geq t_0$ large enough such that

$$x(t) > 0, \quad x(\sigma(t)) > 0, \quad \text{and } x(\tau(t)) > 0,$$

z satisfies Lemma 3.1 with (iv) replaced by (3.4), and (2.2) holds for $t \geq t_1$. The proof will proceed in two steps.

1. First, we are going to show via induction on n that for arbitrary ${}_{\beta}\varepsilon_{k,n} \in (0,1)$ and ${}_{\gamma}\varepsilon_{k,n} \in (0,1)$ one can set

$$\tilde{\beta}_{k,n} = {}_{\beta}\varepsilon_{k,n}\beta_{k,n}$$
$$\tilde{\gamma}_{k,n} = {}_{\gamma}\varepsilon_{k,n}\gamma_{k,n}$$

so that

(I)$_n$

$$\left(\frac{z}{\pi^{\tilde{\beta}_{k,n}}}\right)' < 0,$$

(II)$_n$

$$\left(\frac{z}{\pi^{1-\tilde{\gamma}_{k,n}}}\right)' \geq 0,$$

and

(III)$_n$

$$x \geq \tilde{a}_{k,n}(1 + H_k)z,$$

where

$$\tilde{a}_{k,n} = \begin{cases} \varepsilon & \text{for } \tau(t) \leq t,\ \psi_* = \infty \text{ or } \tau(t) \geq t,\ \omega_* = \infty; \\ 1 - p_0\psi^{-\tilde{\gamma}_{k,n}} & \text{for } \tau(t) \leq t,\ \psi_* < \infty; \\ 1 - p_0\omega^{-\tilde{\beta}_{k,n}} & \text{for } \tau(t) \geq t,\ \omega_* < \infty. \end{cases}$$

For $n = 0$, the conclusion apparently follows from (i)$_0$–(iii)$_0$ with

$${}_{\beta}\varepsilon_{k,0}^{\alpha} = {}_{\gamma}\varepsilon_{k,0} = \frac{\beta_k}{\beta_k^*}.$$

Clearly,

$$\lim_{\beta_k \to \beta_k^*} {}_{\beta}\varepsilon_{k,0} = \lim_{\beta_k \to \beta_k^*} {}_{\gamma}\varepsilon_{k,0} = 1.$$

Now, assume that (I)$_n$–(III)$_n$ hold for some $n \geq 1$ and $t \geq t_n \geq t_1$, and we will show that they hold for $n + 1$, with ${}_{\beta}\varepsilon_{k,n+1}$ and ${}_{\gamma}\varepsilon_{k,n+1}$ defined by:

(a) for either $\tau(t) \le t$ and $\psi_* = \infty$ or $\tau(t) \ge t$ and $\omega_* = \infty$:

$$\begin{aligned} {}_{\beta}\varepsilon_{k,n} &= \sqrt[\alpha]{{}_{\beta}\varepsilon_{k,0}}\,\varepsilon \frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}} \sqrt[\alpha]{\frac{1-\beta_{k,n-1}}{1-\tilde{\beta}_{k,n-1}}}, \\ {}_{\gamma}\varepsilon_{k,n} &= {}_{\gamma}\varepsilon_{k,0}\varepsilon^{\alpha} \left[\frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}}\left(\frac{1-\gamma_{k,n-1}}{1-\tilde{\gamma}_{k,n-1}}\right)\right]^{\alpha} \end{aligned}$$

(b) for $\tau(t) \le t$ and $\psi_* < \infty$:

$$\begin{aligned} {}_{\beta}\varepsilon_{k,n} &= \sqrt[\alpha]{{}_{\beta}\varepsilon_{k,0}} \frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}} \sqrt[\alpha]{\frac{1-\beta_{k,n-1}}{1-\tilde{\beta}_{k,n-1}}} \left(\frac{1-p_0\psi^{-\tilde{\gamma}_{k,n-1}}}{1-p_0\psi_*^{-\gamma_{k,n-1}}}\right), \\ {}_{\gamma}\varepsilon_{k,n} &= {}_{\gamma}\varepsilon_{k,0} \left[\frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}}\left(\frac{1-\gamma_{k,n-1}}{1-\tilde{\gamma}_{k,n-1}}\right)\left(\frac{1-p_0\psi^{-\tilde{\gamma}_{k,n-1}}}{1-p_0\psi_*^{-\gamma_{k,n-1}}}\right)\right]^{\alpha} \end{aligned}$$

(c) for $\tau(t) \ge t$ and $\omega_* < \infty$:

$$\begin{aligned} {}_{\beta}\varepsilon_{k,n} &= \sqrt[\alpha]{{}_{\beta}\varepsilon_{k,0}} \frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}} \sqrt[\alpha]{\frac{1-\beta_{k,n-1}}{1-\tilde{\beta}_{k,n-1}}} \left(\frac{1-p_0\omega^{-\tilde{\beta}_{k,n-1}}}{1-p_0\omega_*^{-\beta_{k,n-1}}}\right), \\ {}_{\gamma}\varepsilon_{k,n} &= {}_{\gamma}\varepsilon_{k,0} \left[\frac{\lambda^{\tilde{\beta}_{k,n-1}}}{\lambda_*^{\beta_{k,n-1}}}\left(\frac{1-\gamma_{k,n-1}}{1-\tilde{\gamma}_{k,n-1}}\right)\left(\frac{1-p_0\omega^{-\tilde{\beta}_{k,n-1}}}{1-p_0\omega_*^{-\beta_{k,n-1}}}\right)\right]^{\alpha} \end{aligned}$$

for $n \in \mathbb{N}$. Clearly, in all three cases, we have

$$\begin{aligned} \lim_{(\beta_k,\lambda,\varepsilon)\to(\beta_k^*,\lambda_*,1)} {}_{\beta}\varepsilon_{k,n} &= \lim_{(\beta_k,\lambda,\varepsilon)\to(\beta_k^*,\lambda_*,1)} {}_{\gamma}\varepsilon_{k,n} = 1, \\ \lim_{(\beta_k,\lambda,\psi)\to(\beta_k^*,\lambda_*,\psi_*)} {}_{\beta}\varepsilon_{k,n} &= \lim_{(\beta,\lambda,\psi)\to(\beta_k^*,\lambda_*,\psi_*)} {}_{\gamma}\varepsilon_{k,n} = 1, \end{aligned}$$

and

$$\lim_{(\beta_k,\lambda,\omega)\to(\beta_k^*,\lambda_*,\omega_*)} {}_{\beta}\varepsilon_{k,n} = \lim_{(\beta_k,\lambda,\omega)\to(\beta_k^*,\lambda_*,\omega_*)} {}_{\gamma}\varepsilon_{k,n} = 1,$$

respectively.

Using $(\mathrm{III})_n$ in (1.1), we get

$$\left(r\left(z'\right)^{\alpha}\right)'(t) + q(t)\tilde{a}_{k,n}^{\alpha}(1+H_k(\sigma(t)))^{\alpha} z^{\alpha}(\sigma(t)) \le 0, \quad t \ge t_n,$$

which in view of (2.2) becomes

$$\left(r\left(z'\right)^{\alpha}\right)'(t) + \frac{\beta_k\alpha\tilde{a}_{k,n}^{\alpha}}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} z^{\alpha}(\sigma(t)) \le 0. \tag{3.16}$$

Now using that $z/\pi^{\tilde{\beta}_{k,n}}$ is decreasing (see $(\mathrm{I})_n$), (H_3) and (2.2), we find

$$\frac{z(\sigma(t))}{z(t)} \ge \left(\frac{\pi(\sigma(t))}{\pi(t)}\right)^{\tilde{\beta}_{k,n}} \ge \lambda^{\tilde{\beta}_{k,n}}.$$

Hence, (3.16) becomes

$$\left(r\left(z'\right)^{\alpha}\right)'(t) + \frac{\beta_k\alpha\tilde{a}_{k,n}^{\alpha}\lambda^{\alpha\tilde{\beta}_{k,n}}}{r^{1/\alpha}(t)\pi^{\alpha+1}(t)} z^{\alpha}(t) \le 0. \tag{3.17}$$

(I)$_{n+1}$ Integrating (3.17) from t_n to t and using (I)$_n$, we have

$$\begin{aligned}
-r(t)\left(z'(t)\right)^\alpha \geq & -r(t_n)\left(z'(t_n)\right)^\alpha \\
& +\beta_k\tilde{a}_{k,n}^\alpha\lambda^{\alpha\tilde{\beta}_{k,n}}\left(\frac{z}{\pi^{\tilde{\beta}_{k,n}}}\right)^\alpha(t)\int_{t_n}^t \frac{\alpha}{r^{1/\alpha}(s)\pi^{\alpha(1-\tilde{\beta}_{k,n})+1}(s)}\mathrm{d}s \\
= & -r(t_n)\left(z'(t_n)\right)^\alpha \\
& +\frac{\beta_k\tilde{a}_{k,n}^\alpha\lambda^{\alpha\tilde{\beta}_{k,n}}}{1-\tilde{\beta}_{k,n}}\left(\frac{z}{\pi^{\tilde{\beta}_{k,n}}}\right)^\alpha(t)\left(\frac{1}{\pi^{\alpha(1-\tilde{\beta}_{k,n})}(t)}-\frac{1}{\pi^{\alpha(1-\tilde{\beta}_{k,n})}(t_n)}\right).
\end{aligned} \tag{3.18}$$

Similarly as in the proof of [10, Lemma 4, pp. 8–9], it can be shown that

$$\lim_{t\to\infty}\frac{z(t)}{\pi^{\tilde{\beta}_{k,n}}(t)}=0$$

and so, there exists $t_n' \geq t_n$ such that

$$-r(t_n)\left(z'(t_n)\right)^\alpha > \frac{\beta_k\tilde{a}_{k,n}^\alpha\lambda^{\alpha\tilde{\beta}_{k,n}}}{1-\tilde{\beta}_{k,n}}\left(\frac{z}{\pi^{\tilde{\beta}_{k,n}}}\right)^\alpha(t)\frac{1}{\pi^{\alpha(1-\tilde{\beta}_{k,n})}(t_n)}, \quad t\geq t_n'. \tag{3.19}$$

Using (3.19) in (3.18) implies that

$$-\pi r^{1/\alpha}z' > \tilde{a}_{k,n}\lambda^{\tilde{\beta}_{k,n}}\sqrt[\alpha]{\frac{\beta_k}{1-\tilde{\beta}_{k,n}}}z=\tilde{\beta}_{k,n+1}z \tag{3.20}$$

and

$$\left(\frac{z}{\pi^{\tilde{\beta}_{k,n+1}}}\right)'<0,$$

which completes the induction step.

(II)$_{n+1}$ Differentiating as in (3.13) and using (3.17), we get

$$\begin{aligned}
Z' &= \left(r^{1/\alpha}z'\right)'\pi \\
&= \frac{\pi}{\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}\left(r\left(z'\right)^\alpha\right)' \\
&\leq -\frac{\pi}{\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}\frac{\beta_k\alpha\tilde{a}_{k,n}^\alpha\lambda^{\alpha\tilde{\beta}_{k,n}}}{r^{1/\alpha}\pi^{\alpha+1}}z^\alpha \\
&= -\frac{\beta_k\tilde{a}_{k,n}^\alpha\lambda^{\alpha\tilde{\beta}_{k,n}}}{r^{1/\alpha}\pi^\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}z^\alpha<0.
\end{aligned} \tag{3.21}$$

Using (II)$_n$, which corresponds to

$$(1-\tilde{\gamma}_{k,n})z \geq -r^{1/\alpha}z'\pi$$

in (3.21), we obtain

$$Z' \leq -\frac{\beta_k\tilde{a}_{k,n}^\alpha\lambda^{\alpha\tilde{\beta}_{k,n}}}{r^{1/\alpha}\pi^\alpha}\left(r^{1/\alpha}z'\right)^{1-\alpha}\frac{(-r^{1/\alpha}z'\pi)^\alpha}{(1-\tilde{\gamma}_{k,n})^\alpha}=\frac{\beta_k\tilde{a}_{k,n}^\alpha\lambda^{\alpha\tilde{\beta}_{k,n}}}{(1-\tilde{\gamma}_{k,n})^\alpha}z'.$$

Integrating the above inequality from t to ∞ and using that z is decreasing and tending to zero eventually (see Lemma 3.1 (ii) and (v)), we have

$$Z(t) \geq Z(\infty) - \frac{\beta_k \tilde{a}_{k,n}^{\alpha} \lambda^{\alpha \tilde{\beta}_{k,n}}}{(1-\tilde{\gamma}_{k,n})^{\alpha}} z(\infty) + \frac{\beta_k \tilde{a}_{k,n}^{\alpha} \lambda^{\alpha \tilde{\beta}_{k,n}}}{(1-\tilde{\gamma}_{k,n})^{\alpha}} z(t) \geq \frac{\beta_k \tilde{a}_{k,n}^{\alpha} \lambda^{\alpha \tilde{\beta}_{k,n}}}{(1-\tilde{\gamma}_{k,n})^{\alpha}} z(t),$$

which in view of the definition of Z (see (3.12)) gives

$$\left(1 - \frac{\beta_k \tilde{a}_{k,n}^{\alpha} \lambda^{\alpha \tilde{\beta}_{k,n}}}{(1-\tilde{\gamma}_{k,n})^{\alpha}}\right) z \geq -r^{1/\alpha} z' \pi$$

and

$$\left(\frac{z}{\pi^{1-\tilde{\gamma}_{k,n+1}}}\right)' \geq 0,$$

which completes the induction step.

$(\mathrm{III})_{n+1}$ The proof proceeds in the same way as in the case $n=0$ and hence is omitted.

2. To prove the statement, we claim that $(\mathrm{I})_n$ and $(\mathrm{II})_n$ implies $(\mathrm{i})_{n-1}$ and $(\mathrm{ii})_{n-1}$ for $n \in \mathbb{N}$. Clearly, $(\mathrm{I})_n$ and $(\mathrm{II})_n$ correspond to

$$\tilde{\beta}_{k,n} z < -r^{1/\alpha} z' \pi \tag{3.22}$$

and

$$(1-\tilde{\gamma}_{k,n}) z \geq -r^{1/\alpha} z' \pi \tag{3.23}$$

respectively. Then, by virtue of Lemma 3.1 (ii) and (iii), it is easy to see that

$$\tilde{\beta}_{k,n} < 1 \quad \text{and} \quad \tilde{\gamma}_{k,n} < 1.$$

Using this and (2.3), we have

$$1 > \tilde{\beta}_{k,n} = {}_{\beta}\varepsilon_{k,n} \ell_{k,n-1} \beta_{k,n-1} > \beta_{k,n-1} \tag{3.24}$$

and

$$1 > \tilde{\gamma}_{k,n} = {}_{\gamma}\varepsilon_{k,n} h_{k,n-1} \gamma_{k,n-1} > \gamma_{k,n-1}, \tag{3.25}$$

where we used that ${}_{\beta}\varepsilon_n \in (0,1)$ and ${}_{\gamma}\varepsilon_n \in (0,1)$ are arbitrary. Therefore, (3.22) and (3.23) become

$$\beta_{k,n-1} z \leq -r^{1/\alpha} z' \pi$$

and

$$(1-\gamma_{k,n-1}) z > -r^{1/\alpha} z' \pi,$$

for $n \in \mathbb{N}$, which proves our claim. Finally, $(\mathrm{iii})_{n-1}$ is just a consequence of $(\mathrm{i})_{n-1}$ and $(\mathrm{ii})_{n-1}$. □

In view of the newly obtained monotonicities $(\mathrm{i})_n$ and $(\mathrm{ii})_n$, our first main result follows immediately.

Theorem 3.7. *Let $\beta_0^* > 0$, $\lambda_* < \infty$, $\beta_{k,i} < 1$ and $\gamma_{k,i} < 1$ for $i = 0, 1, \ldots, n$ for some $k, n \in \mathbb{N}_0$. If*

$$\beta_{k,n+1} + \gamma_{k,n+1} > 1,$$

then (1.1) is oscillatory.

The second main result of this work results as a simple consequence of Lemma 3.6 (see (3.24) and (3.25)).

Corollary 3.8. *Let $\beta_0^* > 0$. If x is an eventually positive solution of (1.1), then, for some $k \in \mathbb{N}_0$, both sequences $\{\beta_{k,n}\}_{n\in\mathbb{N}_0}$ and $\{\gamma_{k,n}\}_{n\in\mathbb{N}_0}$ are well-defined and bounded from above.*

Now we are prepared to state the second main result of this paper, which is a straightforward consequence of Theorem A (condition (C_1)), Corollary 3.8 and Lemma 2.3 (conditions (C_2)–(C_4)).

Theorem 3.9. *If one of the conditions*

- (C_1) $\beta_0^* > 0$ *and* $\lambda_* = \infty$;
- (C_2) $\beta_0^* > 0$, $\lambda_* < \infty$, *either* $\tau(t) \le t$ *and* $\psi_* = \infty$ *or* $\tau(t) \ge t$ *and* $\omega_* = \infty$, *and the system* (2.4) *does not have a solution* $\{b, g\} \in (0, 1)$;
- (C_3) $\beta_0^* > 0$, $\lambda_* < \infty$, $\tau(t) \le t$, $\psi_* < \infty$, *and the system* (2.5) *does not have a solution* $\{b, g\} \in (0, 1)$;
- (C_4) $\beta_0^* > 0$, $\lambda_* < \infty$, $\tau(t) \ge t$, $\omega_* < \infty$, *and the system* (2.6) *does not have a solution* $\{b, g\} \in (0, 1)$

is satisfied for some $k \in \mathbb{N}_0$, then (1.1) is oscillatory.

By stating explicit conditions for the nonexistence of solutions $\{b, g\} \in (0, 1)$ of the systems (2.4)–(2.6), we get the following results.

Corollary 3.10. *If $\lambda_* < \infty$, either $\tau(t) \le t$ and $\psi_* = \infty$ or $\tau(t) \ge t$ and $\omega_* = \infty$, and*

$$\beta_k^* > \max\left\{ b^\alpha (1-b) \lambda_*^{-\alpha b} : 0 < b < 1 \right\},$$

then (1.1) is oscillatory.

Corollary 3.11. *If $\beta_0^* > 0$, $\lambda_* < \infty$, $\tau(t) \le t$, $\psi_* < \infty$, and*

$$\beta_k^* > \max\left\{ \frac{b^\alpha (1-b) \lambda_*^{-\alpha b}}{(1 - p_0 \psi_*^{-g})^\alpha} : 0 < g < 1, \ \text{where } b = -\frac{\ln \frac{\beta_k^* (1 - p_0 \psi_*^{-g})^\alpha}{g(1-g)^\alpha}}{\alpha \ln \lambda_*} \right\},$$

then (1.1) is oscillatory.

Corollary 3.12. *If $\lambda_* < \infty$, $\tau(t) \ge t$, $\omega_* < \infty$, and*

$$\beta_k^* > \max\left\{ \frac{b^\alpha (1-b) \lambda_*^{-\alpha b}}{(1 - p_0 \omega_*^{-b})^\alpha} : 0 < b < 1 \right\},$$

then (1.1) is oscillatory.

The method of iteratively improved monotonicity properties gives us useful information about the asymptotic behavior of solutions in case when (1.1) is nonoscillatory (i.e., it possesses a nonoscillatory solution). The following results, which are a direct consequence of Lemma 3.6, improve and complement our previous statement [10, Corollary 1], and also complement and extend the results from [6, 14] in nonneutral linear and half-linear case, respectively. It is worth to note that in the linear case $\alpha = 1$, we have $\beta_{k,n} = \gamma_{k,n}$, which is stated separately for the sake of future reference.

Theorem 3.13. *Let $\beta_0^* > 0$ and $\lambda_* < \infty$. If x is an eventually positive solution of* (1.1)*, then for any $c \in (0,1)$ and $k \in \mathbb{N}_0$,*

$$\frac{z(\sigma(t))}{z(t)} \geq c\lambda_*^{\beta_{k,n}},$$

eventually.

Theorem 3.14. *Let $\beta_0^* > 0$ and $\lambda_* < \infty$. If x is an eventually positive solution of* (1.1)*, then there exist $c_i > 0$, $i = 1, 2$, such that*

$$z \leq c_1 \pi^{\beta_{k,n}} \quad \text{and} \quad z \geq c_2 \pi^{1-\gamma_{k,n}}, \quad k \in \mathbb{N}_0,$$

eventually.

Corollary 3.15. *Let $\beta_0^* > 0$, $\lambda_* < \infty$, and $\alpha = 1$. If x is an eventually positive solution of* (1.1)*, then there exist $c_i > 0$, $i = 1, 2$, such that*

$$z \leq c_1 \pi^{\beta_{k,n}} \quad \text{and} \quad z \geq c_2 \pi^{1-\beta_{k,n}}, \quad k \in \mathbb{N}_0,$$

eventually.

4 Examples

Finally, we illustrate the importance of our results on two examples. The first one is intended to show the progress attained in case when p_0 from (H_5) is close to 1.

Example 4.1. Consider the Euler type differential equation

$$\left(t^{\alpha+1}\left(\left(x(t) + \frac{0.99}{t^{(1-\lambda_1)/\alpha}} x\left(t^{\lambda_1}\right)\right)'\right)^{\alpha}\right)' + q_0 x^{\alpha}(\lambda_2 t) = 0, \quad t \geq t_0 > 0, \tag{4.1}$$

where $\alpha > 0$ is a quotient of odd positive integers, $\lambda_1 \in (0,1)$, $\lambda_2 \in (0,1]$, $q_0 > 0$. Here,

$$\pi(t) = \frac{\alpha}{t^{1/\alpha}}, \quad \lambda_* = \frac{1}{\lambda_2^{1/\alpha}}, \quad \psi_* = \lim_{t\to\infty} t^{(1-\lambda_1)/\alpha} = \infty, \quad p_0 = p(t)\frac{\pi(\tau(t))}{\pi(t)} = 0.99,$$

and

$$\beta_k^* = \beta_0^* = q_0 \alpha^{\alpha}.$$

It follows from [29, Theorem 2.8] that

$$\beta_0^* > \frac{\alpha^{\alpha}}{(1-p_0)^{\alpha}(\alpha+1)^{\alpha+1}} = 100^{\alpha} \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} = 100^{\alpha} \max\{b^{\alpha}(1-b) : 0 < b < 1\} \tag{4.2}$$

is sufficient for (4.1) to be oscillatory. By [9, Theorem 2.4] proved by the present authors, the same conclusion is attained if

$$\rho := q_0^{1/\alpha}(1-p_0)^{\alpha} \ln \frac{1}{\lambda_2} > \frac{1}{\mathrm{e}}$$

or, if $\rho \leq 1/\mathrm{e}$ and

$$\beta_0^* > \frac{1}{(1-p_0)^{\alpha} f(\rho)} \cdot \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} = \frac{100^{\alpha}}{f(\rho)} \max\{b^{\alpha}(1-b) : 0 < b < 1\}, \tag{4.3}$$

where
$$f(\rho) = -\frac{W_0(-\rho)}{\rho}, \quad W_0 \text{ is a principal branch of the Lambert function.}$$
We have also showed in [9] that (4.3) simplifies and improves related results from [5,19,22–24, 26,36–39].

By Theorem A (see also [10, Theorem 2]), (4.1) is oscillatory if
$$\beta_0^* > \frac{\max\{b^\alpha(1-b)\lambda_2^b : 0 < b < 1\}}{(1-p_0)^\alpha} = 100^\alpha \max\{b^\alpha(1-b)\lambda_2^b : 0 < b < 1\}, \tag{4.4}$$
which improves (4.3). Finally, by the newly obtained Theorem 3.9, (4.1) is oscillatory if
$$\beta_0^* > \max\{b^\alpha(1-b)\lambda_2^b : 0 < b < 1\}. \tag{4.5}$$
It is obvious that (4.2) does not take λ_2 into account, which is already included in (4.3)–(4.5). Moreover, in Theorem 3.9, the impact of p_0 was removed by that of λ_1 and so (4.5) gives 100^α-times qualitatively better result than (4.4).

Example 4.2. As in [10, Example 1], we consider
$$\left(t^{\alpha+1}\left(\left(x(t)+p_0x(\lambda_1 t)\right)'\right)^\alpha\right)' + q_0x^\alpha(\lambda_2 t) = 0, \quad t \ge t_0 > 0, \tag{4.6}$$
where $\alpha > 0$ is a quotient of odd positive integers, $\lambda_1 > 0$, $\lambda_2 \in (0,1]$, $q_0 > 0$, and
$$p_0 < \begin{cases} \lambda_1^{1/\alpha} & \text{for } \lambda_1 \le 1, \\ 1 & \text{for } \lambda_1 > 1. \end{cases}$$
Here,
$$\pi(t) = \frac{\alpha}{t^{1/\alpha}}, \quad \lambda_* = \frac{1}{\lambda_2^{1/\alpha}}, \quad \psi_* = \frac{1}{\lambda_1^{1/\alpha}} \text{ (for } \lambda_1 \le 1), \quad \omega_* = \lambda_1^{1/\alpha} \text{ (for } \lambda_1 > 1),$$
and
$$\beta_0^* = \alpha^\alpha q_0$$
$$\beta_k^* = \begin{cases} \beta_0^*\left(\sum_{i=0}^k p_0^{2i}\right)^\alpha & \text{for } \lambda_1 \le 1,\ k \in \mathbb{N}, \\ \beta_0^*\left(\sum_{i=0}^k \left(\frac{p_0}{\lambda_1^{1/\alpha}}\right)^{2i}\right)^\alpha & \text{for } \lambda_1 > 1,\ k \in \mathbb{N}. \end{cases}$$
It is easy to compute the limit
$$\beta^* := \lim_{k\to\infty} \beta_k^* = \begin{cases} \frac{\beta_0^*}{(1-p_0^2)^\alpha} & \text{for } \lambda_1 \le 1, \\ \frac{\beta_0^*}{\left(1-\left(p_0\lambda_1^{-1/\alpha}\right)^2\right)^\alpha} & \text{for } \lambda_1 > 1. \end{cases}$$

First, assume $\lambda_1 \le 1$. By Theorem A, (4.6) is oscillatory if

$$\beta_0^* > \frac{\max\{b^\alpha(1-b)\lambda_2^b : 0 < b < 1\}}{\left(1 - p_0\lambda_1^{-1/\alpha}\right)^\alpha}. \tag{4.7}$$

Let us recall (see [10, Example 1]) that (4.6) has a nonoscillatory solution, if

$$\beta_0^* \le \max\left\{b^\alpha(1-b)\lambda_2^b\left(1 + p_0\lambda_1^{-b/\alpha}\right)^\alpha : 0 < b < 1\right\}. \tag{4.8}$$

In the nonneutral case $p_0 = 0$, Theorem A is clearly sharp. For, e.g.,

$$\lambda_1 = \lambda_2 = p_0 = 0.5, \quad \alpha = 3, \tag{4.9}$$

we conclude that, by Theorem A, (4.6) is oscillatory if

$$q_0 > 0.0464 \tag{4.10}$$

and, by (4.8), (4.6) has a nonoscillatory solution if

$$q_0 \le 0.0094,$$

meaning that the behavior of (4.6) subject to (4.9) is unknown for $q_0 \in (0.0094, 0.0464]$.

By Theorem 3.9 (C_3), (4.6) is oscillatory if the system

$$\begin{cases} \dfrac{\beta_0^*}{(1-p_0^2)^\alpha} = \dfrac{b^\alpha(1-b)\lambda_2^b}{\left(1 - p_0\lambda_1^{-(1-g)/\alpha}\right)^\alpha} \\[2ex] \dfrac{\beta_0^*}{(1-p_0^2)^\alpha} = \dfrac{g(1-g)^\alpha\lambda_2^b}{\left(1 - p_0\lambda_1^{-(1-g)/\alpha}\right)^\alpha} \end{cases} \tag{4.11}$$

does not have a solution $\{b, g\}$ on $(0, 1)$, what happens if, by Corollary 3.11,

$$\frac{\beta_0^*}{\left(1-p_0^2\right)^\alpha} > \max\left\{\frac{b^\alpha(1-b)\lambda_2^b}{\left(1 - p_0\lambda_1^{-(1-g)/\alpha}\right)^\alpha} : 0 < g < 1, \text{ where } b = \frac{\ln \frac{\beta_0^*(1-p_0\lambda_1^{-(1-g)/\alpha})^\alpha}{(1-p_0^2)^\alpha g(1-g)^\alpha}}{\ln \lambda_2}\right\}. \tag{4.12}$$

To show the improvement over Theorem A, assume (4.9) and

$$q_0 > 0.0158.$$

Although (4.10) fails to apply, it can be verified using numerical software that (4.12) is satisfied and the system (4.11) does not possess a positive solution, i.e., (4.6) is oscillatory. An alternative approach to attain the same conclusion is to use Theorem 3.7 by initiating an iterative process (e.g., 2 iterations are needed for $q_0 = 0.04$, 11 iterations for $q_0 = 0.017$, 63 iterations for $q_0 = 0.0158$). How to fill the gap $q_0 \in (0.0094, 0.0158]$ remains open at the moment.

Now, assume $\lambda_1 > 1$. By Theorem A, (4.6) is oscillatory if

$$\beta_0^* > \frac{\max\{b^\alpha(1-b)\lambda_2^b : 0 < b < 1\}}{(1-p_0)^\alpha}. \tag{4.13}$$

Here, we would like to point out an oversight we made in [10, Example 1], where we stated that (4.7) (instead of (4.13)) is sufficient for oscillation of (4.6). To look at the improvement, we find that by Corollary 3.12, (4.6) is oscillatory if

$$\beta_0^* > \left(1 - \left(p_0 \lambda_1^{-1/\alpha}\right)^2\right)^\alpha \max\left\{ \frac{b^\alpha (1-b)\lambda_2^b}{\left(1 - p_0 \lambda_1^{-b/\alpha}\right)^\alpha} : 0 < b < 1 \right\}. \tag{4.14}$$

It is obvious to see that, in contrast with (4.14), the criterion (4.13) does not take the influence of λ_1 into account. Clearly, for $p_0 \neq 0$,

$$\max\left\{ \frac{b^\alpha (1-b)\lambda_2^b}{\left(1 - p_0 \lambda_1^{-b/\alpha}\right)^\alpha} : 0 < b < 1 \right\} < \frac{\max\{b^\alpha(1-b)\lambda_2^b : 0 < b < 1\}}{(1-p_0)^\alpha}$$

and

$$\left(1 - \left(p_0 \lambda_1^{-1/\alpha}\right)^2\right)^\alpha < 1,$$

and hence the progress is observable.

Remark 4.3. For $k = 0$, the results established in this paper complement those from [21], where (1.1) subject to

$$\pi(t_0) = \infty$$

was studied. We stress that obtaining a corresponding variant of Lemma 3.3 would immediately improve oscillation criteria from [21]. Another interesting task left for further research is to consider the same problem with $p_0 \geq 1$ or $p_0 < 0$.

5 Summary

The aim of the present paper was to continue studying the oscillation problem of (1.1) under conditions (H_1)–(H_5) and to provide new results which improve Theorem A when $p_0 \neq 0$ and $\lambda_* < \infty$. Our results improve all existing works (i.e., the cited related papers and references therein) on this subject so far.

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