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Further study on second order nonlocal problems monitored by an operator: an approach without compactness.

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Abstract. In this note we prove the existence of mild solutions for nonlocal problems governed by semilinear second order differential inclusions which involves a nonlinear term driven by an operator. A first result is obtained in suitable Banach spaces in the lack of compactness both on the fundamental operator, generated by the linear part, and on the nonlinear multivalued term. This purpose is achieved by combining a fixed point theorem, a selection theorem and a containment theorem. Further we provide another existence result in reflexive spaces by using the classical Hahn–Banach theorem and a new selection proposition, proved here, for a multimap guided by an operator. This setting allows us to remove some assumptions required in the previous existence theorem. As a consequence of this last result we obtain the controllability of a problem driven by a wave equation on which an appropriate perturbation acts.

Keywords: semilinear second order differential inclusion, perturbation effect, fundamental system, De Blasi measure of noncompactness, controllability problem, wave equation.

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1 Introduction

This research is a continuation of the recent papers [8] and [9]. In this paper we proceed the study, started in [8], concerning the existence of mild solutions for the following problem driven by a non-autonomous semilinear second order differential inclusion with nonlocal conditions

$$\begin{cases} x''(t) \in A(t)x(t) + F(t, N(t)x), \ t \in J = [0, a] \\ x(0) = g(x) \\ x'(0) = h(x) \end{cases} \qquad (\mathbf{P})_{\mathbf{N}}$$

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where $\{A(t)\}_{t\in J}$ is a family of linear operators generating a fundamental system and $F : J \times X \to \mathcal{P}(X)$, $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J;X);X)$, $g,h : \mathcal{C}(J;X) \to X$ are suitable maps, with X a appropriate Banach space and

$$\hat{\mathcal{C}}_w(\mathcal{C}(J;X);X) = \{f : \mathcal{C}(J;X) \to X : \ f \text{ is } (w\text{-}w)\text{sequentially continuous}\}. \tag{1.1}$$

We remember that in infinite dimensional spaces the nonlocal problems are investigated with different kinds of approach. By using topological methods the existence of mild solutions for these problems is studied with fixed point theorems applied to a suitable solution operator. This often requires strong compactness conditions, which are usually not satisfied in an infinite dimensional framework (see [4,5,8,10,21]).

The purpose of this work is twofold:

(1) to obtain existence results of mild solutions for the abstract problem $(\mathbf{P})_\mathbf{N}$ in the lack of compactness,

in order to establish

(2) controllability of the problem driven by the following non-autonomous wave equation

$$\frac{\partial^2 w}{\partial t^2}(t,\xi) = \frac{\partial^2 w}{\partial \xi^2}(t,\xi) + b(t)\frac{\partial w}{\partial \xi}(t,\xi) + f\Big(t,\xi,\frac{\mathfrak{Im}\, e(\hat{w})}{2e(\hat{w})+1}\int_0^t p(s)\,ds\Big) + u(t,\xi), \tag{\textbf{E}}$$

subjected to the "mixed conditions": the trajectory $w : [0,a] \times \mathbb{R} \to \mathbb{C}$ satisfies, with respect to the first variable, a "periodicity condition" and it has a fixed initial velocity. The control $u(t,\xi)$ belongs to a set of admissible controls. Moreover, $b \in \mathcal{C}^1(J)$, $p \in L^1(J)$, $f : J \times \mathbb{R} \times \mathbb{C} \to \mathbb{C}$, $e : \mathcal{C}(J, L^2(\mathbb{T},\mathbb{C})) \to \mathbb{C}$ are suitable maps, while $\hat{w} : J \to L^2(\mathbb{T},\mathbb{C})$, is defined as $\hat{w}(t) = w(t,\cdot)$ and $\mathfrak{Im}\, e(\hat{w})$ denotes the imaginary part of the complex number $e(\hat{w})$.

The study of aim (1) in *absence* of the operator N is already addressed in [9] by using a combination of two techniques: one technique is based on the concept of measure of noncompactness, while another makes use of the weak topology. This method is also used in [3] in order to prove the existence of mild solutions for problems monitored by semilinear first order differential inclusions.

On the other hand the controllability of the mentioned problem governed by (**E**) is brought back by using classical arguments (see, for example [42]) to purpose (1).

About the study referred in this note, let us recall that the theory of semilinear differential inclusions is well documented in literature. Various aspects of this field catch the attention of many researchers and are widely employed in the study of several dynamical problems arising from physics, economics, biology, social sciences. Several authors have studied abstract semilinear second order equations/inclusions in the autonomous case starting from the initial researches of Kato [25–27] (see, e.g. [15,28,32,36]). On the other hand, the theory dealing with non-autonomous second order abstract inclusions is only recently investigated, starting from Kozak's pioneering work [30]. On this subject we recall [8–10,13,21].

Moreover, with regard to nonlocal conditions we mention the reference [7] of Byszewski. In many cases it is advantageous to treat the nonlocal conditions since they are more appropriate then the classical initial conditions to describe natural phenomena (see [16,41] and the reference therein).

Finally, as is known, the controllability problems appear as a natural description of observed evolution phenomena of the real world. The attention of the researchers to such problems is increasing in literature. For example, for the notions and facts of controllability for first order differential equations/inclusions, the reader is referred to [1, 2, 14, 33], while we recall [15, 34, 35, 39] for nonlinear second order differential cases.

In this paper our main contributions are the followings:

(I) a new sufficient condition for the existence of mild solutions for the nonlocal problem $(\mathbf{P})_{\mathbf{N}}$ in weakly compactly generated Banach spaces (see Theorem 4.7, obtained as a consequence of our Propositions 4.3, 4.4, 4.5, 4.6);

(II) a new selection theorem for a multimap guided by an operator in reflexive Banach spaces (see Theorem 4.10);

(III) a version of the existence result for $(\mathbf{P})_{\mathbf{N}}$ in reflexive Banach spaces (see Theorem 4.13, proved by using our Propositions 4.4, 4.11, 4.12);

(IV) the existence of a mild solutions for an abstract problem satisfying a "periodicity condition" and having a fixed initial velocity (see Corollary 4.16);

(V) an application of Corollary 4.16 to the study of the controllability for the perturbed problem driven by (**E**) (see Theorem 5.1).

Regarding the proof of our first existence result for $(\mathbf{P})_{\mathbf{N}}$, in the setting of weakly compactly generated Banach spaces, we apply a fixed point theorem for multimaps, recently proved in [9]. This fixed point result allows us to work with weak topology and De Blasi measure of weak noncompactness. So we can avoid requests of compactness on the family generated by the linear part and on the multivalued term. The existence of mild solutions for the nonlocal problem $(\mathbf{P})_{\mathbf{N}}$ is also obtained as a consequence of a selection theorem and a containment theorem.

Then, to study the case of reflexive Banach spaces, in addition to use the fixed point theorem of [9], we need to achieve a new selection theorem for multimaps driven by a suitable operator. Combining this result with the classical Hahn–Banach Theorem and the weak upper semicontinuity property we are able to remove some assumptions required in the previous existence Theorem 4.7.

Finally we are in a position to study the purpose (2), thanks to the definition of a suitable operator N.

Let us note that, since we have not used the strong compactness property, our existence results extend in a broad sense those presented in [8]. On the other hand, although it is possible to reduce problem $(\mathbf{P})_{\mathbf{N}}$ to the one studied in [9] by considering an appropriate operator N (see Remark 4.15), the presence of the required boundedness property on N in our existence theorems makes us that problem $(\mathbf{P})_{\mathbf{N}}$ is not reduced to that examined in [9].

The paper is organized as follows. Section 2 is devoted to the collection of all notions, propositions and theorems known in literature and used in the sequel: so that the paper is self contained. The problem setting is presented in Section 3. Section 4 is divided into two subsections. The first one presents an existence theorem in weakly compactly generated Banach spaces, obtained by proving some preliminary propositions. The second one is aimed at examining the existence of mild solutions in the setting of reflexive Banach spaces. Finally, in Section 5 the controllability of the problem governed by (**E**) is given as a consequence of the last result presented in the previous section.

2 Preliminaries

In this section, we recall a few results, notations and definitions needed to establish our theorems. We introduce certain notations which are used throughout the article without any further mention. Let $(X, \|\cdot\|_X)$ be a Banach space, X^* be the dual space of X and τ_w be the weak topology on X. In this paper $\overline{B}_X(0,r)$ denotes the closed ball centered at the origin and of radius $r > 0$ of X. Moreover, we recall that a Banach space X is said to be *weakly compactly generated* (WCG, for short) if there exists a weakly compact subset H of X such that $X = \overline{\text{span}}\{H\}$ (see [20]). Let us note that every separable space is weakly compactly generated as well as the reflexive ones (see [20]).

Now, put $J = [0, a]$ an interval of the real line endowed with the usual Lebesgue measure μ, we denote by $\mathcal{M}(J)$ the family of all Lebesgue measurable sets, by $\mathcal{C}(J; X)$ the space of all continuous functions from J to X provided with the norm $\|\cdot\|_\infty$ of uniform convergence. We precise that to define the set $\hat{\mathcal{C}}_w(\mathcal{C}(J; X); X)$, presented in (1.1), we say that $f : \mathcal{C}(J; X) \to X$ is *(w-w)sequentially continuous* if for every sequence $(x_n)_n$, $x_n \in \mathcal{C}(J; X)$, $x_n \rightharpoonup x$, then $f(x_n) \rightharpoonup f(x)$.

We recall the following version of Theorem 4 of [29], which characterizes the weak convergence in the space $\mathcal{C}(J; X)$.

Proposition 2.1. *Let X be a normed space, $(g_n)_n$ be a sequence in $\mathcal{C}(J; X)$ and $g \in \mathcal{C}(J; X)$. Then $g_n \rightharpoonup g$ if and only if $(g_n - g)_n$ is uniformly bounded and $g_n(t) \rightharpoonup g(t)$, for every $t \in J$.*

Next, if (Ω, Σ) is a measurable space, a function $u : \Omega \to X$ is said to be $\Sigma \otimes \mathcal{B}(X)$-*measurable* if, for all $A \in \mathcal{B}(X)$, $u^{-1}(A) \in \Sigma$, where $\mathcal{B}(X)$ denotes the Borel σ-field of X (see [18, Definition 2.1.48]). In the case $(\Omega, \Sigma) = (J, \mathcal{M}(J))$, $u : J \to X$ is said to be *Bochner-measurable* (B-measurable, for short) if there is a sequence of simple functions which converges to u almost everywhere in J (see [18, Definition 3.10.1 (a)]) and $u : J \to X$ is said to be *weakly measurable* if for each $l \in X^*$, the real valued function $l(u)$ is measurable (see [18, Definition 3.10.1 (b)]).

Proposition 2.2 ([18, Corollary 3.10.5])**.** *If X is a separable Banach space and $u : J \to X$, then the following conditions are equivalent:*

(a) u is B-measurable;

(b) u is weakly measurable;

(c) u is $\mathcal{M}(J) \otimes \mathcal{B}(X)$-measurable.

Moreover, $L^1(J; X)$ is the space of all X-valued Bochner integrable functions on J with norm $\|u\|_{L^1(J;X)} = \int_0^a \|u(t)\|_X \, dt$ and $L^1_+(J) = \{f \in L^1(J; \mathbb{R}) : \ f(t) \geq 0, \ a.e. \ t \in J\}$. If $X = \mathbb{R}$ we put $\|\cdot\|_1 = \|\cdot\|_{L^1(J;\mathbb{R})}$. For L^1_+-functions the following result holds.

Proposition 2.3 ([12, Lemma 3.1])**.** *For every $k > 0$, $\nu \in L^1_+(J)$, there exists $n := n(k, \nu) \in \mathbb{N}$ such that*

$$\sup_{t \in J} \int_0^t k\nu(\xi) e^{-n(t-\xi)} \, d\xi < 1.$$

A set $A \subset L^1(J; X)$ has the property of *equi-absolute continuity of the integral* if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, for every $E \in \mathcal{M}(J)$, $\mu(E) < \delta_\varepsilon$, we have

$$\int_E \|f(t)\|_X \, dt < \varepsilon$$

whenever $f \in A$, while $A \subset L^1(J;X)$ is *integrably bounded* if there exists $\nu \in L^1_+(J)$ such that

$$\|f(t)\|_X \leq \nu(t), \quad a.e.\ t \in J, \text{ for every } f \in A.$$

Clearly every integrably bounded set has the property of equi-absolute continuity of the integral. We recall that the equi-absolute continuity of the integral is fundamental to characterize the relative weak compactness of a bounded set in $L^1(J;X)$.

Proposition 2.4 ([38, Corollary 9]). *Let A be a bounded subset of $L^1(J;X)$ such that it has the property of equi-absolute continuity of the integral and, for a.e. $t \in J$, the set $A(t) = \{f(t) : f \in A\}$ is relatively weakly compact. Then A is relatively weakly compact.*

Then, if H is a subset of the Banach space X, we denote by the symbol $\overline{H}^w$ the weak closure of H. As is well known, a bounded subset H of a reflexive space X is relatively weakly compact. Moreover, we recall that a subset H of a Banach space X is called *relatively weakly sequentially compact* if any sequence of points in H has a subsequence weakly convergent to a point in X (see [31]). Now we recall the classical Eberlein–Šmulian result.

Proposition 2.5 ([18, Theorem 3.5.3]). *A subset of a Banach space is relatively weakly compact if and only if it is relatively weakly sequentially compact. In particular, a subset of a Banach space is weakly compact if and only if it is weakly sequentially compact.*

In the sequel we use the following version of Theorem 3 obtained by H. Vogt in [40].

Proposition 2.6. *Let H be a relatively weakly compact subset of a Banach space X. Then H is weakly closed if and only if H is weakly sequentially closed.*

Further, if $\mathcal{P}(X)$ is the family of all nonempty subsets of X, we use the following notations:

$$\begin{aligned}
\mathcal{P}_b(X) &= \{H \in \mathcal{P}(X) : \ H \text{ bounded}\},\\
\mathcal{P}_f(X) &= \{H \in \mathcal{P}(X) : \ H \text{ closed}\},\\
\mathcal{P}_k(X) &= \{H \in \mathcal{P}(X) : \ H \text{ compact}\},\\
\mathcal{P}_{wk}(X) &= \{H \in \mathcal{P}(X) : \ H \text{ weakly compact}\}.
\end{aligned}$$

Now, let $(A_n)_n$ be a sequence, $A_n \in \mathcal{P}(X)$, we consider the *"Kuratowski limit superior"* (see [23, Definition 7.1.3])

$$w - \limsup_{n\to+\infty} A_n = \{x \in X : \ \exists (x_{n_k})_k,\ x_{n_k} \in A_{n_k},\ n_k < n_{k+1},\ x_{n_k} \rightharpoonup x\}$$

Proposition 2.7 ([23, Proposition 7.3.9]). *Let X be a Banach space, $1 \leq p < \infty$, $G : J \to \mathcal{P}_{wk}(X)$ and $(f_n)_n$, $f_n \in L^p(J;X)$, be a sequence such that*

i) there exists $f \in L^p(J;X)$ such that $f_n \rightharpoonup f$;

ii) $f_n(t) \in G(t)$ a.e. $t \in J$, $n \in \mathbb{N}$.

Then

$$f(t) \in \overline{\text{co}}\ w - \limsup_{n\to\infty}\{f_n(t)\}, \quad a.e.\ t \in J,$$

where $\overline{\text{co}}$ denotes the closure of the convex hull of a set.

Furthermore a multimap $F : \Omega \to \mathcal{P}(Y)$, where Y is a topological space, is said to be *measurable* if for every open set $V \subset Y$ one has $F^{-}(V) = \{x \in \Omega\ F(x) \cap V \neq \emptyset\} \in \Sigma$ (see [24, Definition 1.3.1]).

Proposition 2.8 ([18, Proposition 4.2.4]). *Let (Ω, Σ) be a measurable space and Y be a separable metric space. A multimap $F : \Omega \to \mathcal{P}(Y)$ is measurable if and only if for every $y \in Y$ the function $x \mapsto dist(y, F(x))$ is $\Sigma \otimes \mathcal{B}(Y)$-measurable.*

Proposition 2.9 ([18, Theorem 4.3.1]). *If (Ω, Σ) is a measurable space, Y is a Polish space and $F : \Omega \to \mathcal{P}_f(Y)$ is measurable, then F has a $\Sigma \otimes \mathcal{B}(Y)$-measurable selection.*

If T is a topological space, a multimap $F : T \to \mathcal{P}(Y)$ is said to be *upper semicontinuous* in T if, for every $\overline{x} \in T$, it is upper semicontinuous at $\overline{x}$, i.e. for every open $W \subset Y$ such that $F(\overline{x}) \subset W$, there exists a neighborhood $V(\overline{x})$ of $\overline{x}$ with the property $F(V(\overline{x})) \subset W$.

A multimap $F : T \to \mathcal{P}(Y)$ has *closed graph* if the set graph $F = \{(x, y) \in T \times Y :\ y \in F(x)\}$ is closed in $T \times Y$.

Moreover, F is said to be *compact* if $\overline{F(T)}$ is compact in Y, while F is said to be *locally compact* if every $x \in T$ there exists a neighborhood $V(x)$ such that the restriction $F_{|V(x)}$ is compact.

Proposition 2.10 ([24, Theorem 1.1.5]). *Let T, Y be topological spaces and $F : T \to \mathcal{P}_k(Y)$ be a closed and locally compact multimap. Then F is upper semicontinuos in T.*

If Y is a linear topological space, $F : X \to \mathcal{P}(Y)$ has *(s-w)sequentially closed graph* [*(w-w)sequentially closed graph*] if for every $(x_n)_n$, $x_n \in X$, $x_n \to x$ [$x_n \rightharpoonup x$] and for every $(y_n)_n$, $y_n \in F(x_n)$, $y_n \rightharpoonup y$, we have $y \in F(x)$. In the sequel, the "$(w$-$w)$sequential continuity" and the "$(w$-$w)$ sequentially closed graph property" are named "weak sequential continuity" and "weakly sequentially closed graph property" respectively.

A multimap $F : J \to \mathcal{P}(X)$ is said to have a *B-measurable selection* if there exists a B-measurable function $f : J \to X$ such that $f(t) \in F(t)$, a.e. $t \in J$.

Now, we recall the following results that ensures the existence of a B-selection for a multimap.

Theorem 2.11 ([24, Theorem 1.3.5]). *Let X, Y be Banach spaces and $F : J \times X \to \mathcal{P}_k(Y)$ be a multimap such that*

i) for every $x \in X$, $F(\cdot, x) : J \to \mathcal{P}_k(Y)$ has a B-measurable selection;

ii) for a.e. $t \in J$, $F(t, \cdot) : X \to \mathcal{P}_k(Y)$ is upper semicontinuous in X.

Then for every B-measurable function $q : J \to Y$, there exists a B-measurable selection $f : J \to X$ of the multimap $F(\cdot, q(\cdot))$.

Let us recall that for $v : J \to M$, where M is a metric space, the B-measurability is generalized in [22] by using again the simple functions. Thanks to this definition, the following result holds.

Theorem 2.12 ([9, Theorem 4.2] (Selection Theorem)). *Let M be a metric space, X be a Banach space and $F : J \times M \to \mathcal{P}(X)$ be a multimap such that*

f1) for a.e. $t \in J$, for every $x \in M$, the set $F(t, x)$ is convex ;

f2) for every $x \in M$, $F(\cdot, x) : J \to \mathcal{P}(X)$ has a B-measurable selection;

f3) for a.e. $t \in J$, $F(t,\cdot) : M \to \mathcal{P}(X)$ *has a (s-w)sequentially closed graph in* $M \times X$;

f4) there exists $\varphi : J \to [0,\infty)$, $\varphi \in L^1_+(J)$, *such that*

$$\sup_{z \in F(t,M)} \| z \| \leq \varphi(t), \text{ a.e. } t \in J;$$

f5) for almost all $t \in J$ *and every convergent sequence* $(x_n)_n$ *in* M, *the set* $\bigcup_n F(t,x_n)$ *is relatively weakly compact.*

Then, for every B-measurable $v : J \to M$, *there is a B-measurable* $y : J \to X$ *such that* $y(t) \in F(t,v(t))$ *for a.e.* $t \in J$.

Next, we recall that, if H is a subset of X, $F : H \to \mathcal{P}(X)$ is a multimap and $x_0 \in H$, a closed convex set $M_0 \subset H$ is said to be (x_0,F)*-fundamental*, if $x_0 \in M_0$ and $F(M_0) \subset M_0$ (see [3, p. 620]). In this setting we recall the following result which allows to characterize the *smallest* (x_0,F)*-fundamental* set

Theorem 2.13 ([3, Theorem 3.1])**.** *Let* X *be a locally convex Hausdorff space,* $H \subset X$ *and* $x_0 \in H$. *Let* $F : H \to \mathcal{P}(X)$ *be a multimap such that* $\overline{\text{co}}(F(H) \cup \{x_0\}) \subset H$. *Then*

1) $\mathcal{F} = \{H : \ H \text{ is a } (x_0,F)\text{-fundamental set}\} \neq \varnothing$;

2) put $M_0 = \bigcap_{H \in \mathcal{F}} H$, *we have* $M_0 \in \mathcal{F}$ *and* $M_0 = \overline{\text{co}}(F(M_0) \cup \{x_0\})$.

Now we present a fixed point result and a "Containment Theorem", which play a key role in our existence results.

Theorem 2.14 ([9, Corollary 4.4])**.** *Let* X *be a Banach space,* $H \subset X$, $x_0 \in H$ *and* $F : H \to \mathcal{P}(X)$ *be a multimap such that*

i) $F(x)$ *convex, for every* $x \in H$;

ii) $\overline{\text{co}}(F(H) \cup \{x_0\}) \subset H$;

iii) M_0 *is weakly compact;*

iv) $F_{|M_0}$ *has weakly sequentially closed graph,*

where M_0 *is the smallest* (x_0,F)*-fundamental set.*

Then there exists at least one point $\overline{x} \in M_0$ *such that* $\overline{x} \in F(\overline{x})$.

Theorem 2.15 ([3, Theorem 4.4] (Containment Theorem))**.** *Let* X *be a Banach space and* $G_n, G : J \to \mathcal{P}(X)$ *be such that*

α) a.e. $t \in J$, *for every* $(u_n)_n$, $u_n \in G_n(t)$, *there exists a subsequence* $(u_{n_k})_k$ *of* $(u_n)_n$ *and* $u \in G(t)$ *such that* $u_{n_k} \rightharpoonup u$;

αα) there exists a sequence $(y_n)_n$, $y_n : J \to X$, *having the property of equi-absolute continuity of the integral, such that* $y_n(t) \in G_n(t)$, *a.e.* $t \in J$, *for all* $n \in \mathbb{N}$.

Then there exists a subsequence $(y_{n_k})_k$ *of* $(y_n)_n$ *such that* $y_{n_k} \rightharpoonup y$ *in* $L^1(J;X)$ *and, moreover,* $y(t) \in \overline{\text{co}}G(t)$, *a.e.* $t \in J$.

Now, a function $\omega : \mathcal{P}_b(X) \to \mathbb{R}^+_0$ is said to be a *measure of weak noncompactness* (MwNC, for short) if the following properties are satisfied (see [11, Definition 4.1]):

ω_1) ω is a Sadowskii functional, i.e. $\omega(\overline{\text{co}}(H)) = \omega(H)$, for every $H \in \mathcal{P}_b(X)$;

ω_2) ω is regular, i.e. $\omega(H) = 0$ if and only if $\overline{H}^w$ is weakly compact.

Further, a measure of weak noncompactness $\omega : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ is said to be:
monotone if $H_1, H_2 \in \mathcal{P}_b(X) :\ H_1 \subset H_2$ implies $\omega(H_1) \le \omega(H_2)$;
nonsingular if $\omega(\{x\} \cup H) = \omega(H)$, $x \in X$, $H \in \mathcal{P}_b(X)$;
x_0-stable if, fixed $x_0 \in X$, $\omega(\{x_0\} \cup H) = \omega(H)$, $H \in \mathcal{P}_b(X)$;
invariant under closure if $\omega(\overline{H}) = \omega(H)$, $H \in \mathcal{P}_b(X)$;
invariant with respect to the union with compact set if $\omega(H \cup C) = \omega(H)$, for every relatively compact set $C \subset X$ and $H \in \mathcal{P}_b(X)$.

In particular in [17] De Blasi introduces the MwNC function $\beta : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ so defined

$$\beta(H) = \inf\{\varepsilon \in [0, \infty[: \text{there exists } C \subset X \text{ weakly compact} :\ H \subseteq C + B_X(0, \varepsilon)\},$$

(named in literature *De Blasi*–MwNC) and he proves that β has all the properties mentioned before and it is also *algebrically subadditive*, i.e. $\beta\left(\sum_{k=1}^n H_k\right) \le \sum_{k=1}^n \beta(H_k)$, where $H_k \in \mathcal{P}_b(X)$, $k = 1, \dots, n$.

Moreover, for every bounded linear operator $L : X \to X$ the following property holds (see [19, Lemma 1])

$$\beta(L(H)) \le \|L\|\, \beta(H),\ H \in \mathcal{P}_b(X), \tag{2.1}$$

where $\|L\|$ denotes the norm of the operator L.

We recall the following interesting result for the De Blasi–MwNC $\beta : \mathcal{P}_b(X) \to \mathbb{R}_0^+$.

Proposition 2.16 ([3, Theorem 2.7])**.** *Let (Ω, Σ, μ) be a finite positive measure space and X be a weakly compactly generated Banach space. Then for every countable bounded set $C \subset L^1(J; X)$ having the property of equi-absolute continuity of the integral, the function $\beta(C(\cdot))$ is $\mathcal{M}(J) \otimes \mathcal{M}(\mathbb{R})$ measurable and*

$$\beta\left(\left\{\int_\Omega x(s)\, ds :\ x \in C\right\}\right) \le \int_\Omega \beta(C(s))\, ds.$$

In the sequel, fixed $\alpha \in \mathbb{R}$, we use the following Sadowskii functional $\beta_\alpha : \mathcal{P}_b(\mathcal{C}(J; X)) \to \mathbb{R}_0^+$, so defined (see [3, Definition 3.9])

$$\beta_\alpha(M) = \sup_{\substack{C \subset M \\ \text{countable}}} \sup_{t \in J} \beta(C(t)) e^{-\alpha t}, \quad M \in \mathcal{P}_b(\mathcal{C}(J; X)), \tag{2.2}$$

where β is the De Blasi MwNC and, for every $t \in J$, $C(t) = \{x(t) :\ x \in C\}$. We recall that the Sadowskii functional β_α is *x_0-stable* and *monotone* (see [3, Proposition 3.10]) and β_α has the two following properties (see [9, Remark 2.11])

(I) β_α is algebraically subadditive;

(II) $M \subset \mathcal{C}(J; X)$ is relatively weakly compact $\Rightarrow \beta_\alpha(M) = 0$.

3 Problem setting

First of all, on the linear part of the second order differential inclusion, presented in the nonlocal problem $(\mathbf{P})_\mathbf{N}$, we assume the following property:

(A) $\{A(t)\}_{t\in J}$ is a family of linear operators $A(t) : D(A) \to X$, where $D(A)$, independent on $t \in J$, is a subset dense in X such that, for each $x \in D(A)$, the function $t \mapsto A(t)x$ is continuous on J and generating a fundamental system $\{S(t,s)\}_{t,s\in J}$.

The notion of fundamental system is introduced by Kozak in [30] and it is recently used in [8–10, 21]. In some works, for every $t \in J$, the linear operator $A(t) : D(A) \to X$ is also closed (see [21, 30]) and bounded (see [8–10]), but we leave out these properties on $A(t)$ since they are not necessary in order to prove the existence of mild solutions (see [37]).

Definition 3.1. A family $\{S(t,s)\}_{t,s\in J}$ of bounded linear operators $S(t,s) : X \to X$ is called a *fundamental system* generated by the family $\{A(t)\}_{t\in J}$ if

S1. for each $x \in X$, the function $S(\cdot,\cdot)x : J \times J \to X$ is of class $\mathcal{C}^1$ and

  a. for every $t \in J$, $S(t,t)x = 0,\ x \in X$;

  b. for every $t, s \in J$ and every $x \in X$, $\frac{\partial}{\partial t} S(t,s)\big|_{t=s} x = x$ and $\frac{\partial}{\partial s} S(t,s)\big|_{t=s} x = -x$;

S2. for all $t, s \in J$, $x \in D(A)$, then $S(t,s)x \in D(A)$, the map $S(\cdot,\cdot)x : J \times J \to X$ is of class $\mathcal{C}^2$ and

  a′. $\frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x$;

  b′. $\frac{\partial^2}{\partial s^2} S(t,s)x = S(t,s)A(s)x$;

  c′. $\frac{\partial^2}{\partial t \partial s} S(t,s)\big|_{t=s} x = 0$;

S3. for all $t, s \in J$, $x \in D(A)$, then $\frac{\partial}{\partial s} S(t,s)x \in D(A)$. Moreover, there exist $\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x$ such that

  a″. $\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x = A(t)\frac{\partial}{\partial s} S(t,s)x$;

  b″. $\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x = \frac{\partial}{\partial t} S(t,s)A(s)x$;

  and, for all $x \in D(A)$, the function $(t,s) \mapsto A(t)\frac{\partial}{\partial s} S(t,s)x$ is continuous in $J \times J$.

As in [21], a map $S : J \times J \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the space of all bounded linear operators in X with the norm $\|\cdot\|_{\mathcal{L}(X)}$, is said to be a *fundamental operator* if the family $\{S(t,s)\}_{t,s\in J}$ is a fundamental system.

Moreover, for every $(t,s) \in J \times J$, we consider the linear operator, named *"cosine operator"*, $C(t,s) = -\frac{\partial}{\partial s} S(t,s) : X \to X$.

In [10] it is pointed out that the Banach–Steinhaus Theorem allows to establish the existence of two constant $K, K^* > 0$ such that

p1. $\|C(t,s)\|_{\mathcal{L}(X)} \le K$, $(t,s) \in J \times J$;

p2. $\|S(t,s)\|_{\mathcal{L}(X)} \le K|t-s|$, $(t,s) \in J \times J$;

p3. $\|S(t,s)\|_{\mathcal{L}(X)} \le Ka$, $(t,s) \in J \times J$;

p4. $\|S(t_2,s) - S(t_1,s)\|_{\mathcal{L}(X)} \le K^*|t_2 - t_1|,\ t_1, t_2, s \in J$.

Further, as in [10], we denote with $G_S : L^1(J;X) \to \mathcal{C}(J;X)$ the *fundamental Cauchy operator* defined as

$$G_S f(t) = \int_0^t S(t,s)f(s)\,ds, \quad t \in J,\ f \in L^1(J;X). \tag{3.1}$$

Let us note that, fixed $t \in J$, the map $p_t : [0,t] \times X \to X$ such that $p_t(\xi,x) = S(t,\xi)x$, $(\xi,x) \in [0,t] \times X$, satisfies all the assumptions of Theorem 2.11 (see S1.). Hence, for every $f \in L^1(J;X)$, by p3. it easy to deduce that $p_t(\cdot, f(\cdot))$ is B-integrable in $[0,t]$. Then, by using p4. we also have $G_S f \in \mathcal{C}(J;X)$. So G_S is well posed. Moreover, the fundamental Cauchy operator has the properties declared in the following

Proposition 3.2 ([9, Proposition 4.1]). *If $\{S(t,s)\}_{(t,s)\in J\times J}$ is a fundamental system, then the fundamental Cauchy operator $G_S : L^1(J;X) \to \mathcal{C}(J;X)$ is linear, bounded and weakly continuous (so it is also weakly sequentially continuous).*

We investigate the existence of mild solutions for the nonlocal problem $(\mathbf{P})_\mathbf{N}$ (see [8, Definition 2.2])

Definition 3.3. A continuous function $u : J \to X$ is a *mild solution* for $(\mathbf{P})_\mathbf{N}$ if

$$u(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi)f(\xi)\,d\xi, \quad t \in J,$$

where $f \in S^1_{F(\cdot,N(\cdot)u)} = \{f \in L^1(J;X) :\ f(t) \in F(t,N(t)u),\ a.e.\ t \in J\}$.

4 Existence results for the nonlocal problem $(\mathbf{P})_\mathbf{N}$

In this section, put X a Banach space, we consider the following properties on the multimap $F : J \times X \multimap \mathcal{P}(X)$ and on the map $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J;X);X)$

F1 for every $(t,x) \in J \times X$, the set $F(t,x)$ is convex;

F2 for every $x \in X$, $F(\cdot,x) : J \multimap \mathcal{P}(X)$ admits a B-measurable selection;

F3 for a.e. $t \in J$, $F(t,\cdot) : X \multimap \mathcal{P}(X)$ has a weakly sequentially closed graph;

F4 there exists $(\varphi_n)_n$, $\varphi_n \in L^1_+(J)$ such that

$$\limsup_{n\to\infty} \frac{Ka\int_0^a \varphi_n(t)\,dt}{n} < 1 \tag{4.1}$$

and

$$\|F(t,\overline{B}_X(0,n))\| \le \varphi_n(t), \quad a.e.\ t \in J,\ n \in \mathbb{N}; \tag{4.2}$$

where K is the constant presented in p1. of Section 3;

FN there exists $A \subset J$, $\mu(A) = 0$: for all $n \in \mathbb{N}$ there exists $\nu_n \in L^1_+(J)$ such that, for every $t \in J \setminus A$

$$\beta(C_1) \le \nu_n(t)\beta(C_0(t)), \tag{4.3}$$

for all countable C_0, C_1 with

$$C_0 \subseteq \overline{B}_{\mathcal{C}(J;X)}(0,n), \quad C_1 \subseteq F(t, C_0(t) \cup N(t)C_0),$$

where β is the De Blasi measure of weak noncompactness;

N1 for every $u \in \mathcal{C}(J;X)$, $N(\cdot)u$ is B-measurable;

N2 there exists $\bar{c} \in \mathbb{N}$ such that $\|N(t)u\|_X \leq \bar{c}$, for all $t \in J$, $u \in \mathcal{C}(J;X)$.

Moreover, we consider the following properties on the functions $g, h : \mathcal{C}(J;X) \to X$

gh1 g, h are weakly sequentially continuous;

gh2 for every countable $H \subset \mathcal{C}(J;X)$, the sets $g(H)$ and $h(H)$ are relatively weakly compact;

gh3 for every bounded and closed subset M of $\mathcal{C}(J;X)$, the sets

$$C(\cdot,0)g(M) \quad \text{and} \quad S(\cdot,0)h(M)$$

are relatively weakly compact in $\mathcal{C}(J;X)$.

Remark 4.1. First of all we note that, under assumptions F1, F3 and FN,

$$F(t, N(t)x) \text{ is closed in } X, \text{ for a.e. } t \in J \text{ and for every } x \in \mathcal{C}(J;X).$$

Denoted by H^* a null measure set such that F3 and FN hold in $J \setminus H^*$, we fix $t \in J \setminus H^*$ and $x \in \mathcal{C}(J;X)$. Now we prove that the set $F(t, N(t)x)$ is relatively w-compact. To this aim we consider $C_0 = \{x\}$ and $C_1 = \{y_n : n \in \mathbb{N}\}$, where $y_n \in F(t, N(t)x)$, $n \in \mathbb{N}$. Note that $C_0 \subset \overline{B}_{\mathcal{C}(J;X)}(0, p)$, for a suitable $p \in \mathbb{N}$, and $C_1 \subset F(t, C_0(t) \cup N(t)C_0)$, so by (4.3) we have $\beta(C_1) \leq \nu_p(t)\beta(C_0(t)) = 0$. By the regularity of β the set C_1 is relatively w-compact and so there exist $(y_{n_k})_k \subset (y_n)_n$ and $y \in X$ such that $y_{n_k} \rightharpoonup y$. Then by the arbitrariness of the sequence $(y_n)_n$, by using the Eberlein–Šmulian Theorem we have that the set $F(t, N(t)x)$ is relatively weakly compact too. By virtue of F1 and F3 we also know that the set $F(t, N(t)x)$ is convex and weakly sequentially closed. So, by using Proposition 2.6 we have that the set $F(t, N(t)x)$ is closed in X.

Remark 4.2. We note that, in the setting of reflexive Banach spaces and under assumptions F1, F3 and F4, by using again Proposition 2.6 we have

$$F(t, x) \text{ is closed, for a.e. } t \in J \text{ and for every } x \in X, \tag{4.4}$$

(see the beginning of the proof of Theorem 5.3 of [9].)

4.1 Existence of mild solutions in WCG Banach spaces

In this subsection, by combining the Containment Theorem 2.15 and a selection result, which is a consequence of Theorem 2.12, we obtain the existence of mild solutions to the nonlocal problem $(\mathbf{P})_\mathbf{N}$, assuming that X is a WCG Banach space. Note that our technique allows us to avoid hypotheses of compactness both on the family generated by the linear part and on the nonlinear multivalued term. We achive our goal by applying the fixed point Theorem 2.14 to the following multioperator $T : \mathcal{C}(J;X) \to \mathcal{P}(\mathcal{C}(J;X))$ defined as (see (3.1))

$$Tu = \{y \in \mathcal{C}(J;X) : y(t) = C(t,0)g(u) + S(t,0)h(u) + G_S f(t),\ t \in J,\ f \in S^1_{F(\cdot,N(\cdot)u)}\}, \tag{4.5}$$

where

$$S^1_{F(\cdot,N(\cdot)u)} = \{f \in L^1(J;X) : f(t) \in F(t, N(t)u) \ a.e.\ t \in J\}. \tag{4.6}$$

To make the propositions that we will present below of greater applicability, allow us to request, at first, that the following property holds

(T) $\quad S^1_{F(\cdot,N(\cdot)u)} \neq \emptyset,\ u \in \mathcal{C}(J;X),$

so we have the multioperator T is well posed.

Obviously the fixed points of the integral multioperator T are mild solutions for the problem $(\mathbf{P})_{\mathbf{N}}$.

At first, thanks to the Containment Theorem 2.15, we establish the following property on T.

Proposition 4.3. *Let X be a Banach space, under assumptions* (A), (T), F1, F3, F4, FN, N2 *and* gh1, *the multioperator T has a weakly sequentially closed graph.*

Proof. Let $(q_n)_n$ and $(x_n)_n$ be two sequences of $\mathcal{C}(J;X)$ such that

$$x_n \in Tq_n, \quad n \in \mathbb{N} \tag{4.7}$$

and

$$q_n \rightharpoonup q, \quad x_n \rightharpoonup x, \tag{4.8}$$

where $q, x \in \mathcal{C}(J;X)$. We have to show that $x \in Tq$.

First of all, by Proposition 2.1 the weak convergence of $(q_n)_n$ implies that

$$q_n(t) \rightharpoonup q(t), \quad t \in J \tag{4.9}$$

and the existence of $\overline{n} \in \mathbb{N}$ such that

$$\|q_n\|_{\mathcal{C}(J;X)} \leq \overline{n}, \quad n \in \mathbb{N}. \tag{4.10}$$

Now we prove that Containment Theorem 2.15 can be applied to the multimaps $G_n : J \to \mathcal{P}(X)$, $n \in \mathbb{N}$ and $G : J \to \mathcal{P}(X)$ respectively so defined

$$G_n(t) = F(t, N(t)q_n), \quad t \in J, \tag{4.11}$$

$$G(t) = F(t, N(t)q), \quad t \in J. \tag{4.12}$$

First we establish α) of Theorem 2.15. To this aim we consider the null measure set H^* for which F3 and FN hold. Fixed $t \in J \setminus H^*$, we consider a sequence $(u_n)_n$ such that

$$u_n \in G_n(t), \quad n \in \mathbb{N}. \tag{4.13}$$

Now we fix the countable subset of $\overline{B}_{\mathcal{C}(J;X)}(0, \overline{n})$, where $\overline{n}$ is presented in (4.10), so defined

$$C_0 = \{q_n : n \in \mathbb{N}\} \tag{4.14}$$

and the countable set of X

$$C_1 = \{u_n : \ n \in \mathbb{N}\}$$

satisfying (see (4.13), (4.11) and (4.14))

$$C_1 \subset F(t, \{N(t)q_n\}_n) \subset F(t, C_0(t) \cup N(t)C_0).$$

So, by FN there exists $\nu_{\overline{n}} \in L^1_+(J)$ such that

$$\beta(C_1) \leq \nu_{\overline{n}}(t)\beta(C_0(t)). \tag{4.15}$$

Now, since the set $C_0(t)$ is relatively weakly sequentially compact (see (4.14) and (4.9)), by the regularity of β we have $\beta(C_0(t)) = 0$. By the virtue of (4.15) and of the Eberlein–Šmulian Theorem, we deduce that C_1 is relatively weakly sequentially compact too, i.e. there exist $(u_{n_k})_k \subset (u_n)_n$ and $u \in X$ such that

$$u_{n_k} \rightharpoonup u. \tag{4.16}$$

Therefore, taking into account the weak sequential continuity of $N(t)$ and F3, from (4.8), (4.11), (4.12), (4.13) and (4.16) we have $u \in G(t)$. So $\alpha)$ of Theorem 2.15 holds.

Next, we prove $\alpha\alpha)$ of Theorem 2.15. By (4.7), (4.5) and (T), for every $n \in \mathbb{N}$, there exists $f_n \in S^1_{F(\cdot,N(\cdot)q_n)}$ such that

$$x_n(t) = C(t,0)g(q_n) + S(t,0)h(q_n) + G_S f_n(t), \quad t \in J.$$

First we observe that the sequence $(f_n)_n$, $f_n : J \to X$, is such that

$$f_n(t) \in G_n(t) = F(t, N(t)q_n), \quad \text{a.e. } t \in J,\ n \in \mathbb{N}. \tag{4.17}$$

Moreover, thanks to (4.17) and to hypotheses N2 and F4, the sequence $(f_n)_n$ is integrably bounded. So $(f_n)_n$ has the property of equi-absolute continuity of the integral, i.e. $\alpha\alpha)$ holds.

Now we are in a position to apply the mentioned Containment Theorem, so there exists a subsequence $(f_{n_k})_k \subset (f_n)_n$ such that

$$f_{n_k} \rightharpoonup f \quad \text{in } L^1(J;X),$$

where (see (4.12), F1 and, taking into account F3 and FN, Remark 4.1)

$$f(t) \in \overline{\text{co}}G(t) = \overline{\text{co}}F(t, N(t)q) = F(t, N(t)q), \quad a.e.\ t \in J.$$

Hence we can conclude that

$$f \in S^1_{F(\cdot,N(\cdot)q)}. \tag{4.18}$$

Moreover, the weak continuity of the fundamental Cauchy operator G_S (see Proposition 3.2) implies that $G_S f_{n_k} \rightharpoonup G_S f$. Then, by using again Proposition 2.1, hypothesis gh1, continuity and linearity of $S(t,0)$ and $C(t,0)$, $t \in J$, we have

$$x_{n_k}(t) \rightharpoonup C(t,0)g(q) + S(t,0)h(q) + G_S f(t) =: \tilde{x}(t), \quad t \in J. \tag{4.19}$$

On the other hand by (4.8) we deduce $x_{n_k}(t) \rightharpoonup x(t)$, for all $t \in J$ and then the uniqueness of the weak limit implies

$$x(t) = \tilde{x}(t),\ t \in J. \tag{4.20}$$

Finally, from (4.20), (4.19), (4.18) and (4.5) we have that $x \in Tq$. Therefore we can conclude that T has a weakly sequentially closed graph. □

Proposition 4.4. *Let X be a Banach space, under assumptions* (A), (T), F4, N2 *and* gh2, *there exists $\overline{r} \in \mathbb{N}$, $\overline{r} > \overline{c}$, such that the operator T maps the closed ball $K_{\overline{r}} = \overline{B}_{C(J;X)}(0, \overline{r})$ into itself, where* 0 *denotes the null function of $C(J;X)$.*

Proof. First of all, from N2, we know that there exists a constant $\overline{c} \in \mathbb{N}$ such that $\|N(t)x\|_X \leq \overline{c}$, for every $t \in J$, $x \in C(J;X)$.

We show that there exists $\overline{r} \in \mathbb{N}$, $\overline{r} > \overline{c}$, such that

$$T(K_{\overline{r}}) \subset K_{\overline{r}}. \tag{4.21}$$

Let assume by contradiction that, for every $n \in \mathbb{N}$ such that $n > \bar{c}$ we have

$$T(K_n) \not\subseteq K_n.$$

Then, there exist $q_n \in \mathcal{C}(J; X)$ with $\|q_n\|_{\mathcal{C}(J;X)} \leq n$ and $x_{q_n} \in Tq_n$ such that $\|x_{q_n}\|_{\mathcal{C}(J;X)} > n$.

Being $\|x_{q_n}\|_{\mathcal{C}(J;X)} > n$, there exists $t_n \in J = [0, a]$: $\|x_{q_n}(t_n)\|_X > n$. By gh2 we have that $g(\{q_n, \ n \in \mathbb{N} : \ n > \bar{c}\})$ and $h(\{q_n, \ n \in \mathbb{N} : \ n > \bar{c}\})$ are relatively weakly compact. Hence there exists a subsequence $(q_{n_k})_k$ such that $(g(q_{n_k}))_k$ and $(h(q_{n_k}))_k$ are weakly convergent, so there exists $Q > 0$ such that (see [6], Proposition 3.5 (iii)) $\|g(q_{n_k})\|_X \leq Q$, $\|h(q_{n_k})\|_X \leq Q$, for every $n_k \in \mathbb{N}$, $n_k > \bar{c}$. Now, being $f_{q_{n_k}} \in S^1_{F(\cdot, N(\cdot)q_{n_k})}$, taking into account p1. and p3. we can write

$$\begin{aligned} n_k < \|x_{q_{n_k}}(t_{n_k})\|_X &\leq \|C(t_{n_k}, 0)\|_{\mathcal{L}(X)} \|g(q_{n_k})\|_X + \|S(t_{n_k}, 0)\|_{\mathcal{L}(X)} \|h(q_{n_k})\|_X \\ &+ \int_0^{t_{n_k}} \|S(t_{n_k}, \xi)\|_{\mathcal{L}(X)} \|f_{q_{n_k}}(\xi)\|_X \, d\xi \leq KQ + KaQ + Ka \int_0^a \|f_{q_{n_k}}(\xi)\|_X \, d\xi, \end{aligned} \tag{4.22}$$

Next, from N2 we have $\|N(t)q_{n_k}\|_X \leq \bar{c} < n_k$, $t \in J$. So $f_{q_{n_k}}(t) \in F(t, N(t)q_{n_k}) \subset F(t, \overline{B}_X(0, n_k))$, a.e. $t \in J$. Now by (4.2) of F4 there exists $\varphi_{n_k} \in L^1_+(J)$ such that

$$\|f_{q_{n_k}}(t)\|_X \leq \varphi_{n_k}(t), \quad a.e.\ t \in J,$$

then by (4.22) the following inequality holds

$$n_k < \|x_{q_{n_k}}(t_{n_k})\|_X \leq KQ + KaQ + Ka \int_0^a \varphi_{n_k}(\xi) \, d\xi. \tag{4.23}$$

Therefore, since (4.23) is true for every natural number $n_k > \bar{c}$, we have

$$1 \leq \frac{KQ + KaQ}{n_k} + \frac{Ka \int_0^a \varphi_{n_k}(\xi) \, d\xi}{n_k}, \quad n_k \in \mathbb{N}, \ n_k > \bar{c}.$$

Hence, passing to the superior limit, by (4.1) we deduce the following contradiction

$$1 \leq \limsup_{k \to \infty} \left(\frac{KQ + KaQ}{n_k} + \frac{Ka \int_0^a \varphi_{n_k}(\xi) \, d\xi}{n_k} \right) \leq \limsup_{n \to \infty} \frac{Ka \int_0^a \varphi_n(\xi) \, d\xi}{n} < 1.$$

Therefore we can conclude that (4.21) is true, i.e. there exists $\bar{r} \in \mathbb{N}$ with $\bar{r} > \bar{c}$ such that

$$K_{\bar{r}} = \overline{B}_{\mathcal{C}(J;X)}(0, \bar{r}) \tag{4.24}$$

is invariant under the action of the operator T. □

If the Banach space X is also WCG, we have the following result for the multimap $T_{\bar{r}} = T_{|K_{\bar{r}}} : K_{\bar{r}} \to \mathcal{P}(\mathcal{C}(J; X))$, which is the restriction of the multimap T on the set $K_{\bar{r}}$ defined in (4.24).

Proposition 4.5. *If X is a weakly compactly generated Banach space, under assumptions* (A), (T), F4, FN, N2, gh2 *and* gh3 *there exists M_0 the smallest $(0, T_{\bar{r}})$-fundamental set which is weakly compact, with $\bar{r} > \bar{c}$ such that $T(K_{\bar{r}}) \subset K_{\bar{r}}$.*

Proof. First of all, we consider $x_0 = 0 \in \mathcal{C}(J;X)$ and the set $K_{\overline{r}}$ in the locally convex Hausdorff space $\mathcal{C}(J;X)$ equipped with the weak topology. Since $T_{\overline{r}}(K_{\overline{r}}) \subset K_{\overline{r}}$, clearly we have $\overline{\text{co}}(T_{\overline{r}}(K_{\overline{r}}) \cup \{0\}) \subset K_{\overline{r}}$. Hence, being true the assumptions of Theorem 2.13, there exists the smallest $(0, T_{\overline{r}})$-fundamental set $M_0 \subset \mathcal{C}(J;X)$ such that

$$M_0 \subset K_{\overline{r}} = \overline{B}_{\mathcal{C}(J;X)}(0, \overline{r}) \tag{4.25}$$

and

$$M_0 = \overline{\text{co}}(T_{\overline{r}}(M_0) \cup \{0\}). \tag{4.26}$$

Now, we prove that M_0 is weakly compact.

We consider the Sadovskij functional β_α, where $\alpha \in \mathbb{R}^+$, defined in (2.2). Being β_α 0-stable we can write (see (4.26))

$$\beta_\alpha(T_{\overline{r}}(M_0)) = \beta_\alpha(M_0). \tag{4.27}$$

Hence, since β_α satisfies (I) and (II), by (4.27), (4.5) and gh3 we have (see (2.2) and (3.1))

$$\begin{aligned}
\beta_\alpha(M_0) &= \beta_\alpha\left(\{C(\cdot,0)g(u) + S(\cdot,0)h(u) + G_S f :\ f \in S^1_{F(\cdot,N(\cdot)u)},\ u \in M_0\}\right) \\
&\le \beta_\alpha(C(\cdot,0)g(M_0)) + \beta_\alpha(S(\cdot,0)h(M_0)) + \beta_\alpha(\{G_S f :\ f \in S^1_{F(\cdot,N(\cdot)u)},\ u \in M_0\}) \\
&= \beta_\alpha(\{G_S f :\ f \in S^1_{F(\cdot,N(\cdot)u)},\ u \in M_0\}) \\
&= \sup_{\substack{C \subset S^1_{F(\cdot,N(\cdot)M_0)} \\ C \text{ countable}}} \ \sup_{t \in J} \beta\left(\left\{\int_0^t S(t,\xi) f(\xi)\, d\xi :\ f \in C\right\}\right) e^{-\alpha t}.
\end{aligned} \tag{4.28}$$

Now, fixed $t \in J$ and a countable set $C \subset S^1_{F(\cdot,N(\cdot)M_0)}$ we define

$$C^*_t = \{S(t,\cdot) f(\cdot) :\ f \in C\}.$$

Recalling that $\overline{r} > \overline{c}$, by using p3. and F4, we can say that the set C^*_t is integrably bounded and so it is bounded in $L^1(J;X)$ and it has the property of equi-absolute continuity of the integral. Then, by recalling that X is a WCG Banach space, we are in the position to apply Proposition 2.16 to the countable set C^*_t, so we have

$$\begin{aligned}
\beta\left(\left\{\int_0^t S(t,\xi) f(\xi)\, d\xi :\ f \in C\right\}\right) &\le \int_0^t \beta(C^*_t(\xi))\, d\xi \\
&= \int_0^t \beta(\{S(t,\xi) f(\xi) :\ f \in C\})\, d\xi.
\end{aligned} \tag{4.29}$$

Further let us note that for every $f \in C$ we can consider, by the Axiom of Choice, a continuous map $q_f \in M_0$ such that $f(\xi) \in F(\xi, N(\xi)q_f)$, a.e. $\xi \in J$. So the set $C_0^C = \{q_f \in M_0 :\ f \in C\}$ is countable too. Now, taking into account the numerability of C, there exists a null measure set $I \subset J$ containing the set A defined in FN, such that

$$f(\xi) \in F(\xi, N(\xi)q_f),\ \xi \in J \setminus I,\ f \in C,$$

where $q_f \in C_0^C$.

Hence, fixed $\xi \in J \setminus I$, we observe that $C(\xi) \subset F(\xi, C_0^C(\xi) \cup N(\xi)C_0^C)$. Now, since the countable set $C_0^C \subset M_0 \subset K_{\overline{r}}$, by FN there exists $\nu_{\overline{r}} \in L^1_+(J)$ such that

$$\beta(C(\xi)) \le \nu_{\overline{r}}(\xi)\beta(C_0^C(\xi)).$$

The above considerations allow us to claim that for every countable set $C \subset S^1_{F(\cdot,N(\cdot)M_0)}$ there exists a countable subset $C_0^C \subset M_0$ such that

$$\int_0^t \beta(C(\xi))\,d\xi \le \int_0^t \nu_{\bar{r}}(\xi)\beta(C_0^C(\xi))\,d\xi \le \int_0^t \nu_{\bar{r}}(\xi) \sup_{\substack{C_0\subset M_0\\ C_0 \text{ countable}}} \beta(C_0(\xi))\,d\xi. \tag{4.30}$$

Now, by using (4.28), (4.29), (2.1), p3. and (4.30) we can write (see (2.2))

$$\begin{aligned}
\beta_\alpha(M_0) &\le \sup_{\substack{C\subset S^1_{F(\cdot,N(\cdot)M_0)}\\ C \text{ countable}}} \sup_{t\in J} \left(\int_0^t \|S(t,\xi)\|_{\mathcal{L}(X)} \beta(C(\xi))\,d\xi \right) e^{-\alpha t} \\
&\le \sup_{\substack{C\subset S^1_{F(\cdot,N(\cdot)M_0)}\\ C \text{ countable}}} \sup_{t\in J} \left(Ka \int_0^t \beta(C(\xi))\,d\xi \right) e^{-\alpha t} \\
&\le \sup_{\substack{C\subset S^1_{F(\cdot,N(\cdot)M_0)}\\ C \text{ countable}}} \sup_{t\in J} \left(Ka \int_0^t \nu_{\bar{r}}(\xi) \sup_{\substack{C_0\subset M_0\\ C_0 \text{ countable}}} \beta(C_0(\xi))\,d\xi \right) e^{-\alpha t} \\
&\le \sup_{t\in J} \left(Ka \int_0^t e^{-\alpha(t-\xi)} \nu_{\bar{r}}(\xi) \sup_{\substack{C_0\subset M_0\\ C_0 \text{ countable}}} \sup_{\xi\in J} e^{-\alpha\xi} \beta(C_0(\xi))\,d\xi \right) \\
&= \beta_\alpha(M_0) \sup_{t\in J} \int_0^t Ka e^{-\alpha(t-\xi)} \nu_{\bar{r}}(\xi)\,d\xi.
\end{aligned} \tag{4.31}$$

By virtue of Proposition 2.3 we can say that there exists $m \in \mathbb{N}$ such that

$$\sup_{t\in J} \int_0^t Ka e^{-m(t-\xi)} \nu_{\bar{r}}(\xi)\,d\xi < 1. \tag{4.32}$$

Now, if we assume that $\beta_m(M_0) > 0$, where m is the constant characterized in (4.32), from (4.31) we have the contradiction

$$\beta_m(M_0) \le \beta_m(M_0) \sup_{t\in J} \int_0^t Ka e^{-m(t-\xi)} \nu_{\bar{p}}(\xi)\,d\xi < \beta_m(M_0).$$

So we can claim

$$\beta_m(M_0) = 0. \tag{4.33}$$

By definition of $\beta_m(M_0)$, we have that, for every $t \in J$, the set $M_0(t)$ is relatively weakly sequentially compact. Indeed, fixed $t \in J$ and a sequence $(q_n(t))_n$, $q_n(t) \in M_0(t)$, $n \in \mathbb{N}$, we consider the countable set

$$\tilde{C}(t) = \{q_n(t) : \; n \in \mathbb{N}\}.$$

By (4.33) we have $\beta(\tilde{C}(t)) = 0$ and, since β is regular the set $\tilde{C}(t)$ is relatively weakly compact. So, by the Eberlein–Šmulian Theorem $\tilde{C}(t)$ is relatively weakly sequentially compact too. Hence there exists a subsequence $(q_{n_k}(t))_k$ of $(q_n(t))_n$ such that $q_{n_k}(t) \rightharpoonup q(t) \in X$. Therefore, by the arbitrariness of the sequence $(q_n(t))_n$ we can conclude that $M_0(t)$ is relatively weakly sequentially compact, and, by using again Proposition 2.5, $M_0(t)$ is relatively weakly compact too.

Now, we use Proposition 2.4 to prove that the set $S^1_{F(\cdot,N(\cdot)M_0)}$ is relatively weakly compact in $L^1(J;X)$.

First of all, since $N(t)M_0 \subset \overline{B}_X(0,\overline{r})$ for every $t \in J$ (see N2 and recalling that $\overline{r} > \overline{c}$), we prove that $S^1_{F(\cdot,N(\cdot)M_0)}$ is integrably bounded. By (4.2) of F4 we have that there exists $\varphi_{\overline{r}} \in L^1_+(J)$ such that

$$\|f(t)\|_X \leq \varphi_{\overline{r}}(t), \quad \text{a.e. } t \in J,\ f \in S^1_{F(\cdot,N(\cdot)M_0)}.$$

Therefore we can deduce that $S^1_{F(\cdot,N(\cdot)M_0)}$ is bounded in $L^1(J;X)$ and it has the property of equi-absolute continuity of the integral.

Finally we show that $S^1_{F(t,N(t)M_0)}$ is relatively weakly compact in X, for a.e. $t \in J$.

Let us fix $t \in J \setminus H^*$, where H^* is the null measure set containing the set A presented in FN and such that $\|F(t, N(t)M_0\|_X \leq \varphi_{\overline{r}}(t)$, for every $t \in J \setminus H^*$. First of all, we note that $S^1_{F(t,N(t)M_0)}$ is norm bounded in X by the constant $\varphi_{\overline{r}}(t)$.

Next, we consider a sequence $(y_n)_n \subset S^1_{F(t,N(t)M_0)}$. Obviously there exists a sequence $(f_n)_n$ such that $f_n \in S^1_{F(\cdot,N(\cdot)M_0)}$ and $f_n(t) = y_n$, $n \in \mathbb{N}$. So we have

$$y_n \in F(t, N(t)M_0), \quad n \in \mathbb{N}. \tag{4.34}$$

Let us note that, for every $n \in \mathbb{N}$, from (4.34), there exists $q_n \in M_0$ such that

$$y_n \in F(t, N(t)q_n). \tag{4.35}$$

Now, by considering the two countable sets $C_0 = \{q_n : n \in \mathbb{N}\} \subset K_{\overline{r}}$ (see (4.25)) and $C_1 = \{y_n : n \in \mathbb{N}\}$ we have (see (4.35))

$$C_1 \subset F(t, N(t)C_0) \subset F(t, C_0(t) \cup N(t)C_0).$$

So, by FN and recalling that $M_0(t)$ is relatively weakly compact, we write

$$0 \leq \beta(C_1) \leq \nu_{\overline{r}}(t)\beta(C_0(t)) \leq \nu_{\overline{r}}(t)\beta(M_0(t)) = 0,$$

so $\beta(C_1) = 0$, i.e. C_1 is relatively weakly compact. Hence there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ weakly convergent.

By the arbitrariness of $(y_n)_n$ in $S^1_{F(t,N(t)M_0)}$, we can conclude that $S^1_{F(t,N(t)M_0)}$ is relatively weakly sequentially compact. By using again the Eberlein–Šmulian Theorem the set $S^1_{F(t,N(t)M_0)}$ is relatively weakly compact.

Therefore, we are in the position to apply Proposition 2.4, hence $S^1_{F(\cdot,N(\cdot)M_0)}$ is relatively weakly compact in $L^1(J;X)$.

In order to prove the weak compactness of M_0, by (4.26) it is sufficient to show that $T(M_0)$ is relatively weakly compact.

To this aim we fix a sequence $(x_n)_n$, $x_n \in T(M_0)$. Then there exists $(q_n)_n$, $q_n \in M_0$, such that $x_n \in Tq_n$, $n \in \mathbb{N}$. Hence

$$x_n(t) = C(t,0)g(q_n) + S(t,0)h(q_n) + \int_0^t S(t,\xi)f_n(\xi)\,d\xi, \quad t \in J,$$

where $f_n \in S^1_{F(\cdot,N(\cdot)q_n)} \subset S^1_{F(\cdot,N(\cdot)M_0)}$.

By the relative weak sequential compactness of $S^1_{F(\cdot,N(\cdot)M_0)}$ we can find a subsequence $(f_{n_k})_k$ of $(f_n)_n$, such that $f_{n_k} \rightharpoonup f$ in $L^1(J;X)$ and by using the mentioned Proposition 3.2, we have

$$G_S f_{n_k} \rightharpoonup G_S f. \tag{4.36}$$

Moreover, if we consider the countable set $\{q_{n_k} : n_k \in \mathbb{N}\}$, thanks to gh2, there exist a subsequence of $(q_{n_k})_k$, w.l.o.g we name also $(q_{n_k})_k$, and $x, y \in X$ such that

$$g(q_{n_k}) \rightharpoonup x \quad \text{and} \quad h(q_{n_k}) \rightharpoonup y. \tag{4.37}$$

Now, let us consider the subsequence $(x_{n_k})_k$ of $(x_n)_n$. First of all, for every $t \in J$, since $S(t,0)$ and $C(t,0)$ are linear and bounded, from (4.36) and (4.37) we deduce

$$x_{n_k}(t) \rightharpoonup C(t,0)x + S(t,0)y + G_S f(t) := \overline{x}(t),$$

where $\overline{x} : J \to X$ is a continuous function.

Then, since $x_{n_k} \in T(M_0)$, $k \in \mathbb{N}$, by (4.25) we can write

$$\|x_{n_k} - \overline{x}\|_{C(J;X)} \leq \overline{r} + \|\overline{x}\|_{C(J;X)},$$

i.e. the sequence $(x_{n_k} - \overline{x})_k$ is uniformly bounded. So, thanks to Proposition 2.1 $x_{n_k} \rightharpoonup \overline{x}$. Then we deduce that $T(M_0)$ is relatively weakly sequentially compact and so $T(M_0)$ is relatively weakly compact.

Finally, recalling (4.26) we can conclude that M_0 is weakly compact. □

Now we state some conditions on F and N in order to have property (T) true in a Banach space X, i.e. the following selection proposition.

Proposition 4.6. *Let X be a Banach space, $F : J \times X \to \mathcal{P}(X)$ be a multimap having the properties* F1, F2, F3, F4 *and $N : J \to \hat{C}_w(C(J;X);X)$ be a map satisfying* N1, N2. *If* FN *holds, then for every $u \in C(J;X)$ the set $S^1_{F(\cdot,N(\cdot)u)}$ is nonempty.*

Proof. First of all, fixed $u \in C(J;X)$, we define the following function $q_u : J \to X$

$$q_u(t) = N(t)u, \quad t \in J \tag{4.38}$$

and by N1 we say that q_u is B-measurable. Moreover, in correspondence of the costant $\overline{c} \in \mathbb{N}$ of N2, there exists $r_u \in \mathbb{N}$, $r_u > \overline{c}$ such that $\|u(t)\|_X \leq r_u$, $t \in J$. Now, we consider $F_{|J \times M_u} : J \times M_u \to \mathcal{P}(X)$, where

$$M_u = \overline{B}_X(0, r_u). \tag{4.39}$$

and we consider on M_u the metric d induced by that on the Banach space X.

We will show that this multimap satisfies all the hypotheses of Theorem 2.12.

First of all *f1)*, *f2)* and *f3)* of Theorem 2.12 are true for the restriction $F_{|J \times M_u}$ (see F1, F2 and F3 respectively).

Now, for the fixed r_u in (4.39), from F4 there exists $\varphi_{r_u} \in L^1_+(J)$ such that (see (4.2))

$$\sup_{z \in F(t,M_u)} \|z\| \leq \varphi_{r_u}(t), \quad a.e.\ t \in J,$$

i.e. *f4)* of Theorem 2.12 is true for $F_{|J \times M_u}$.

Finally, fixed a sequence $(u_n)_n$, $u_n \in M_u$, such that $u_n \to v$ in M_u, we consider, for every $n \in \mathbb{N}$, $g_n : J \to X$ so defined

$$g_n(t) = u_n, \quad t \in J. \tag{4.40}$$

Clearly $g_n \in \overline{B}_{C(J;X)}(0, r_u)$ (see (4.39)). Then, fixed $t \in J \setminus A$, where A is defined in FN, we show that $\bigcup_n F(t, u_n)$ is relatively weakly compact. Indeed, considering a sequence $(x_p)_p$, $x_p \in$

$\bigcup_n F(t,u_n)$, we fix the following countable sets $C_0 = \{g_n : n \in \mathbb{N}\}$ and $C_1 = \{x_p : p \in \mathbb{N}\}$. Now $C_0 \subset \overline{B}_{\mathcal{C}(J;X)}(0,r_u)$ and C_1 has the property (see (4.40))

$$C_1 \subset \bigcup_n F(t,u_n) = \bigcup_n F(t,g_n(t)) \subset F(t,C_0(t) \cup N(t)C_0).$$

By hypothesis FN there exists $\nu_{r_u} \in L^1_+(J)$ such that

$$\beta(C_1) \le \nu_{r_u}(t)\beta(C_0(t)).$$

Recalling the convergence $u_n \to v$ we have $\beta(C_0(t)) = 0$ and so $\beta(C_1) = 0$, i.e. C_1 is relatively weakly compact. Then there exists a subsequence of $(x_p)_p$ weakly convergent in X. By the arbitrariness of $(x_p)_p$, the set $\bigcup_n F(t,u_n)$ is relatively sequentially weakly compact and so it is also relatively weakly compact. Therefore property *f5)* of Theorem 2.12 holds for $F_{|J\times M_u}$.

Hence, in correspondence of $q_u : J \to X$ defined in (4.38), there exists a B-selection $g_{q_u} : J \to X$, for the multimap $F_{|J\times M_u}(\cdot,q_u(\cdot))$. Now, recalling that $r_u > \bar{c}$ for every $t \in J$, $N(t)u \in M_u$ (see (4.39) and N2), and by (4.38) we have $F_{|J\times M_u}(t,q_u(t)) = F(t,N(t)u)$. So

$$g_{q_u}(t) \in F(t,N(t)u), \quad \text{a.e. } t \in J. \tag{4.41}$$

Moreover, by using F4 there exists $\varphi_{r_u} \in L^1_+(J)$ such that (see (4.39))

$$\|g_{q_u}(t)\|_X \le \varphi_{r_u}(t), \quad \text{a.e. } t \in J.$$

Hence $g_{q_u} \in L^1(J;X)$ and, by (4.41), we have $g_{q_u} \in S^1_{F(\cdot,N(\cdot)u)}$, i.e. the set $S^1_{F(\cdot,N(\cdot)u)}$ is nonempty. □

Finally we are in a position to establish the following existence result of mild solutions for $\mathbf{(P)_N}$ in weakly compactly generated Banach spaces.

Theorem 4.7. *Let X be a WCG Banach space and $\{A(t)\}_{t\in J}$ a family of operators which satisfies the property* (A).

Let $F : J \times X \to \mathcal{P}(X)$ be a multimap and $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J;X);X)$ be a map satisfying F1, F2, F3, F4 *and* N1, N2 *respectively, and* FN. *Let $g,h : \mathcal{C}(J;X) \to X$ be two functions satisfying* gh1, gh2 *and* gh3.

Then there exists at least one mild solution for the nonlocal problem $\mathbf{(P)_N}$.

Proof. First of all by using Proposition 4.4 and Proposition 4.6 we can say that there exists $\bar{r} > \bar{c}$, $T(K_{\bar{r}}) \subset K_{\bar{r}}$, such that the map $T_{\bar{r}} = T_{|K_{\bar{r}}} : K_{\bar{r}} \to \mathcal{P}(\mathcal{C}(J;X))$ defined as in (4.5) is well posed, where $K_{\bar{r}}$ is presented in (4.24).

In order to obtain the thesis we want to apply the fixed point Theorem 2.14 to $T_{\bar{r}}$. At first, by F1 we deduce that $T_{\bar{r}}$ takes convex values, i.e. *i)* of Theorem 2.14 is satisfied.

Moreover, since $T_{\bar{r}}(K_{\bar{r}}) \subset K_{\bar{r}}$, we have $\overline{\text{co}}(T_{\bar{r}}(K_{\bar{r}} \cup \{0\})) \subset K_{\bar{r}}$, i.e. hypothesis *ii)* of Theorem 2.14 holds.

Next, Proposition 4.5 ensures the existence and the weakly compactness of the smallest $(0,T_{\bar{r}})$-fundamental set M_0, i.e. *iii)* of Theorem 2.14 is true.

Finally, thanks to Proposition 4.3, the restriction of $T_{\bar{r}}$ to the set weakly compact set M_0 has weakly sequentially closed graph, i.e. *iv)* of Theorem 2.14 holds.

In conclusion we can apply Theorem 2.14 to $T_{\bar{r}}$. Hence the multioperator T has a fixed point in M_0, i.e. there exists $x \in M_0$ such that

$$x(t) = C(t,0)g(x) + S(t,0)h(x) + \int_0^t S(t,\xi)f(\xi)\,d\xi, \quad t \in J$$

where $f \in S^1_{F(\cdot,N(\cdot)x)}$. Of course, x is a mild solution for $\mathbf{(P)_N}$. □

Clearly, an immediate consequence of Theorem 4.7 is the following existence result for Cauchy problems.

Corollary 4.8. *Let X be a WCG Banach space and $x_0, x_1 \in X$. Under the assumptions* (*A*)*,* F1*,* F2*,* F3*,* F4*,* FN *and* N1*,* N2*, there exists at least one mild solution for the Cauchy problem*

$$\begin{cases} x''(t) \in A(t)x(t) + F(t, N(t)x), \; t \in J \\ x(0) = x_0 \\ x'(0) = x_1. \end{cases}$$

Remark 4.9. Let us note that Theorem 4.7 extends in broad sense Proposition 4.3 of [8]. In particular we remove the hypothesis of compactness on the operators $S(t,s)$, $(t,s) \in J \times J$, and we use the weak topology instead of the strong one on the maps involved in the nonlocal problem.

4.2 Existence of mild solutions in reflexive spaces

In this subsection we discuss the existence of mild solutions to the problem $(\mathbf{P})_\mathbf{N}$ in the particular case of reflexive Banach spaces. In this case we are able to omit assumptions FN and gh3 of Theorem 4.7. Let us note that the lack of these hypotheses implies that this result is new with respect to Theorem 4.7 since the reflexivity does not imply that gh3 holds. For this reason it is necessary to modify in some points the proof of the previous existence result and also to prove a variant of Theorem 2.12. Let us note that the reflexivity of the space allows us to remove hypothesis *f5)* of Theorem 2.12 in order to establish the existence of a selection for multimaps perturbed by an operator.

This new proposition plays a key role in order to show the good position of the solution multioperator in this new setting. Moreover, in order to prove the existence of at least a mild solution for the nonlocal problem $(\mathbf{P})_\mathbf{N}$, we use again the fixed point Theorem 2.14, but instead of the Containment Theorem, we work with the property of weak upper semicontinuity and with the classical Hahn–Banach Theorem.

Theorem 4.10. *Let $J = [0, a]$, M be a metric space, X a reflexive Banach space, $F : J \times M \to \mathcal{P}(X)$ a multimap having the properties f1), f2), f3), f4) of Theorem 2.12 and $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J; X); M)$ be a map satifying* N1*.*

Then, for every $u \in \mathcal{C}(J; X)$, there exists $y \in L^1(J; X)$ with $y(t) \in F(t, N(t)u)$ for a.e. $t \in J$.

Proof. First of all, by *f2)* we easily can deduce

f2)$_w$ for every simple function $s : J \to M$, the multimap $F(\cdot, s(\cdot))$ has a B-measurable selection.

Now, fixed $u \in \mathcal{C}(J; X)$, we define $q^u : J \to M$ as

$$q^u(t) = N(t)u, \quad t \in J \tag{4.42}$$

Clearly, by N1 the map q^u is B-measurable and so there exists a sequence of simple functions $(s_n^u)_n$, $s_n^u : J \to M$, such that

$$s_n^u(t) \to q^u(t), \quad \text{a.e. } t \in J. \tag{4.43}$$

Hence, fixed $n \in \mathbb{N}$, in correspondence of the simple function s_n^u, by *f2)$_w$* there exists a B-measurable function $y_n^u : J \to X$ such that

$$y_n^u(t) \in F(t, s_n^u(t)) \quad \text{a.e. } t \in J. \tag{4.44}$$

Now, let us consider $A = \{y_n^u,\ n \in \mathbb{N}\} \subset L^1(J;X)$. By *f4)* and (4.44) we have that

$$\|y_n^u(t)\|_X \leq \varphi(t), \quad \text{a.e. } t \in J, n \in \mathbb{N}, \tag{4.45}$$

i.e. $A(t)$ is bounded in X, a.e. $t \in J$. So, since X is a reflexive Banach space, $A(t)$ is relatively weakly compact, for a.e. $t \in J$. Moreover, (4.45) implies that A is bounded in $L^1(J;X)$ and it has the property of equi-absolute continuity of the integral. Therefore, since the set A satisfies all the hypotheses of Proposition 2.4, we can conclude that A is relatively weakly compact. Hence there exist $(y_{n_k}^u)_k \subset (y_n^u)_n$ and $y \in L^1(J;X)$ such that $y_{n_k}^u \rightharpoonup y$.

Now, we can apply Proposition 2.7 to the multimap $G : J \to \mathcal{P}_{wk}(X)$, defined by $G(s) = \overline{A(s)}^w$, for every $s \in J$, and to the sequence $(y_{n_k}^u)_k$ of $L^1(J;X)$, so we deduce

$$y(t) \in \overline{\mathrm{co}}\ w - \limsup_{k\to\infty} \{y_{n_k}^u(t)\}, \quad \text{a.e. } t \in J. \tag{4.46}$$

Next we fix H the null measure subset of J such that (4.43), (4.44), (4.46), *f1)* and *f3)* hold for every $t \in J \setminus H$. By (4.44) we can claim

$$\overline{\mathrm{co}}\ w - \limsup_{k\to\infty} \{y_{n_k}(t)\} \subset \overline{\mathrm{co}}\ w - \limsup_{k\to\infty} F(t, s_{n_k}^u(t)), \quad t \in J \setminus H. \tag{4.47}$$

Moreover, we are able to prove

$$w - \limsup_{k\to\infty} F(t, s_{n_k}^u(t)) \subset F(t, q^u(t)), \quad t \in J \setminus H. \tag{4.48}$$

To this aim, let us fix $t \in J \setminus H$ and $z \in w - \limsup_{k\to\infty} F(t, s_{n_k}^u(t))$. Then there exists $z_{n_{k_p}} \in F(t, s_{n_{k_p}}^u(t))$ such that $z_{n_{k_p}} \rightharpoonup z$, where $(n_{k_p})_p$ is an increasing sequence. Now, by (4.43) we know that

$$s_{n_{k_p}}^u(t) \to q^u(t),$$

therefore, since $t \notin H$, hypothesis *f3)* implies $z \in F(t, q^u(t))$. For the arbitrariness of z we can conclude that (4.48) is true.

In virtue of *f3)* the convex set $F(t, q^u(t))$ is closed in X so, by (4.48) and (4.42) we can write

$$\overline{\mathrm{co}}\ w - \limsup_{k\to\infty} F(t, s_{n_k}^u(t)) \subset F(t, N(t)u). \tag{4.49}$$

Finally, thanks to (4.46), (4.47), (4.49), we can conclude that the map $y \in L^1(J;X)$ satisfies $y(t) \in F(t, N(t)u)$ a.e. $t \in J$, so the thesis holds. □

Now we show that in reflexive Banach spaces we can omit some assumptions on the multimap F and on the map N required in Proposition 4.3 and Proposition 4.5.

Proposition 4.11. *Let X be a reflexive Banach space. Under assumptions* (A), (T), F1, F3, F4, N2 *and* gh1, *the multioperator T has a weakly sequentially closed graph.*

Proof. As in Proposition 4.3 we fix two sequences $(q_n)_n \subset \mathcal{C}(J;X)$ and $(x_n)_n \subset \mathcal{C}(J;X)$, weakly convergent to $q, x \in \mathcal{C}(J;X)$ respectively, with $x_n \in Tq_n$, $n \in \mathbb{N}$.

By (T) we can say that, for every $n \in \mathbb{N}$, there exists (see (4.6))

$$f_n \in S^1_{F(\cdot, N(\cdot)q_n)} \tag{4.50}$$

such that (see (4.5))

$$x_n(t) = C(t,0)g(q_n) + S(t,0)h(q_n) + \int_0^t S(t,\xi)f_n(\xi)\,d\xi, \quad t \in J.$$

Now we want to prove that the set $A = \{f_n : n \in \mathbb{N}\}$ satisfies all the hypotheses of Proposition 2.4. Obviously, by (4.50), A is a subset of $L^1(J;X)$ and we have

$$f_n(t) \in F(t, N(t)q_n) \subset F(t, \overline{B}_X(0,\bar{c})), \quad a.e.\ t \in J,\ n \in \mathbb{N}, \tag{4.51}$$

where $\bar{c}$ is the constant presented in N2.

Now, put H the null measure set for which F4 and (4.51) hold, there exists $\varphi_{\bar{c}} \in L^1_+(J)$ such that (see (4.2))

$$\|f_n(t)\|_X \leq \varphi_{\bar{c}}(t), \quad t \in J \setminus H,\ n \in \mathbb{N}.$$

So A is bounded in $L^1(J;X)$ and A has also the property of equi-absolute continuity of the integral. Moreover, for every $t \in J \setminus H$, $A(t)$ being bounded on the reflexive space X, $A(t)$ is relatively weakly compact.

Hence, applying Proposition 2.4, the set A is relatively weakly compact in $L^1(J;X)$. So there exist a subsequence $(f_{n_k})_k$ of $(f_n)_n$ and $f \in L^1(J;X)$ such that

$$f_{n_k} \rightharpoonup f. \tag{4.52}$$

Now, from (4.52) and by using the classical Mazur Theorem there exists a sequence $(\tilde{f}_{n_k})_k$ made up of convex combinations of f_{n_k}'s, such that $\tilde{f}_{n_k} \to f$ in $L^1(J;X)$. So, up to a subsequence, we have

$$\tilde{f}_{n_{k_p}}(t) \to f(t), \quad a.e.\ t \in J. \tag{4.53}$$

Let H^* be the null measure set for which (4.53), (4.4), (4.51), F3 and F4 hold. In order to show that $f \in S^1_{F(\cdot,N(\cdot)q)}$ we prove that

$$f(t) \in F(t, N(t)q), \quad t \in J \setminus H^*. \tag{4.54}$$

If we assume that (4.54) is false, there exists $\bar{t} \in J \setminus H^*$ such that $f(\bar{t}) \notin F(\bar{t}, N(\bar{t})q)$.

First, we want to establish that the multimap $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},\cdot)$ is weakly upper semicontinuous and, to this aim, we show that all the hypotheses of Proposition 2.10 are satisfied.

For every $x \in \overline{B}_X(0,\bar{c})$ from F4 we can write $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},x) \subset \overline{B}_X(0,\varphi_{\bar{c}}(\bar{t}))$. Therefore, since X is reflexive, from (4.4) (see F1, F3 and F4) we can say that the bounded set $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},x)$ is weakly compact. Moreover, the weak compactness of $\overline{B}_X(0,\varphi_{\bar{c}}(\bar{t}))$ obviously implies that the multimap $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},\cdot)$ is weakly compact. Further, recalling hypothesis F3, from Proposition 2.6 we deduce that $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},\cdot)$ is a weakly closed multimap.

Hence, since all the hypotheses of Proposition 2.10 are satisfied, the multimap $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},\cdot)$ is weakly upper semicontinuous.

Now, let us consider the convex, closed set $F_{|\overline{B}_X(0,\bar{c})}(\bar{t}, N(\bar{t})q)$ and the compact set $\{f(\bar{t})\}$. Since we have assumed that $f(\bar{t}) \notin F_{|\overline{B}_X(0,\bar{c})}(\bar{t}, N(\bar{t})q)$, by the classical Hahn–Banach Theorem there exists a weakly open convex set $V \supset F_{|\overline{B}_X(0,\bar{c})}(\bar{t}, N(\bar{t})q)$ satisfying

$$f(\bar{t}) \notin \overline{V} = \overline{V}^w \tag{4.55}$$

Next, taking into account the weak upper semicontinuity of $F_{|\overline{B}_X(0,\bar{c})}(\bar{t},\cdot)$, there exists a weak neighborhood $W_{N(\bar{t})q}$ of the point $N(\bar{t})q$ such that

$$F_{|\overline{B}_X(0,\bar{c})}(\bar{t},x) \subset V, \quad x \in W_{N(\bar{t})q} \cap \overline{B}_X(0,\bar{c}). \tag{4.56}$$

Moreover, by the weak convergence of $(q_n)_n$ to q and the weak sequential continuity of $N(\bar{t})$, the subsequence $(N(\bar{t})q_{n_{k_p}})_p$, indexed as in (4.53), weakly converges to $N(\bar{t})q$. So, there exists $\overline{N} \in \mathbb{N}$ such that, for every $n_{k_p} \geq \overline{N}$, $N(\bar{t})q_{n_{k_p}} \in W_{N(\bar{t})q}$. Since from N2 $N(\bar{t})q_{n_{k_p}} \in \overline{B}_X(0,\bar{c})$, $n_{k_p} \geq \overline{N}$, we deduce that (see (4.56))

$$f_{n_{k_p}}(\bar{t}) \in F_{|\overline{B}_X(0,\bar{c})}(\bar{t}, N(\bar{t})q_{n_{k_p}})) \subset V,\ n_{k_p} \geq \overline{N}.$$

Now, thanks to the convexity of V, we can claim that the convex combinations $\tilde{f}_{n_{k_p}}$, satisfying (4.53), have the following property

$$\tilde{f}_{n_{k_p}}(\bar{t}) \in V, \quad n_{k_p} \geq \overline{N}$$

and so

$$f(\bar{t}) \in \overline{V} = \overline{V}^{w},$$

which contradicts (4.55). Therefore (4.54) is true. By recalling that $f \in L^1(J;X)$ we obtain $f \in S^1_{F(\cdot,N(\cdot)q)}$. Now, by using gh1 and the same technique of the final part of Proposition 4.3 we obtain $x \in Tq$.

Therefore we can conclude that T has a weakly sequentially closed graph. □

Proposition 4.12. *Let X be a reflexive Banach space and assumptions* (A), (T), F4, N2 *and* gh2 *hold. If there exists $\bar{r} > \bar{c}$ such that $T(K_{\bar{r}}) \subset K_{\bar{r}}$, where $K_{\bar{r}} = \overline{B}_{C(J;X)}(0,\bar{r})$, then there exists M_0 the smallest $(0, T_{\bar{r}})$-fundamental set which is weakly compact, where $T_{\bar{r}}$ is the restriction of T to the set $K_{\bar{r}}$.*

Proof. First of all, by using Theorem 2.13 and reasoning as at the beginning of Proposition 4.5, there exists

$$M_0 \subset \overline{B}_{C(J;X)}(0,\bar{r}) = K_{\bar{r}}$$

such that

$$M_0 = \overline{\mathrm{co}}(T_{\bar{r}}(M_0) \cup \{0\}). \tag{4.57}$$

Now, we prove that M_0 is weakly compact. To this end we establish that the set $T_{\bar{r}}(M_0)$ is relatively weakly compact.

Let $(q_n)_n$ be a sequence in M_0 and $(x_n)_n$ be a sequence in $C(J;X)$ such that $x_n \in T_{\bar{r}}q_n$, $n \in \mathbb{N}$. Now, by (T), there exists a sequence $(f_n)_n$, $f_n \in S^1_{F(\cdot,N(\cdot)q_n)}$, such that (see (4.5))

$$x_n(t) = C(t,0)g(q_n) + S(t,0)h(q_n) + \int_0^t S(t,\xi)f_n(\xi)\,d\xi, \quad t \in J.$$

Next, put $A = \{f_n :\ n \in \mathbb{N}\}$, reasoning as in Proposition 4.11, we can apply Proposition 2.4 (see N2 and F4). Therefore we have that the subset A of $L^1(J;X)$ is relatively weakly compact. So there exist $(f_{n_k})_k$ subsequence of $(f_n)_n$ and $f \in L^1(J;X)$ such that $f_{n_k} \rightharpoonup f$.

Now, by the weak sequential continuity of G_S (see Proposition 3.2), we can write

$$G_S f_{n_k} \rightharpoonup G_S f.$$

Next, thanks to hypothesis gh2, reasoning as in the final part of the proof of Proposition 4.5, the subsequence $(x_{n_k})_k$ of $(x_n)_n$ weakly converges to a continuous function. Therefore $T(M_0)$ is relatively weakly sequentially compact and so $T(M_0)$ is also relatively weakly compact. Recalling (4.57) we deduce that the subset M_0 of $C(J;X)$ is weakly compact. □

Theorem 4.13. *Let X be a reflexive Banach space, $J = [0, a]$ and $\{A(t)\}_{t\in J}$ a family which satisfies the property* (A).

Let $F : J \times X \to \mathcal{P}(X)$ be a multimap and $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J; X); X)$ be a map satisfying F1, F2, F3, F4 *and* N1, N2 *respectively. Let $g, h : \mathcal{C}(J; X) \to X$ be two functions having the properties* gh1 *and* gh2.

Then there exists at least one mild solution for the nonlocal problem $(\mathbf{P})_{\mathbf{N}}$.

Proof. First of all, in the setting of theorem we have that property (T) is true. Indeed, fixed $u \in \mathcal{C}(J; X)$, we define a function $q_u : J \to X$ as in (4.38). Moreover, in correspondence of $\bar{c} \in \mathbb{N}$, presented in N2, there exists $r_u \in \mathbb{N}$, $r_u > \bar{c}$, such that $\|u(t)\|_X \leq r_u$, for every $t \in J$. Now we consider $F_{|J\times M_u} : J \times M_u \to \mathcal{P}(X)$, where $M_u = \overline{B}_X(0, r_u)$, and we note that hypotheses F1, F2, F3, F4 imply *f1), f2), f3), f4)* of Theorem 4.10 respectively. Moreover, the map N satisfies N1 of Theorem 4.10. So, by considering on M_u the metric d induced by that on X, we can conclude that there exists $g_u \in S^1_{F(\cdot,N(\cdot)u)}$ (see (4.6)). Therefore, since (T) holds, the map $T_{\overline{r}} = T_{|K_{\overline{r}}} : K_{\overline{r}} \to \mathcal{P}(\mathcal{C}(J; X))$, where $K_{\overline{r}}$ is defined in (4.24), is well posed.

Thanks to Propositions 4.4, 4.11, 4.12 and analogous arguments used in the proof of Theorem 4.7 we are in a position to apply Theorem 2.14.

So there exists at least one one mild solution for $(\mathbf{P})_{\mathbf{N}}$. □

Remark 4.14. Let us note that in reflexive Banach spaces we can omit hypothesis FN in Corollary 4.8.

Remark 4.15. We observe that $(\mathbf{P})_{\mathbf{N}}$ can be rewritten as the problem studied in [9]

$$\begin{cases} x''(t) \in A(t)x(t) + F(t, x(t)),\ t \in J \\ x(0) = g(x) \\ x'(0) = h(x) \end{cases} \tag{P}$$

by considering the map $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J; X); X)$ so defined

$$N(t)u = u(t), \quad t \in J,\ u \in \mathcal{C}(J; X).$$

Let us note that N is well posed by using Proposition 2.1.

Unfortunately Theorems 4.7 and 4.13 does not allow us to prove the existence of mild solutions for (**P**) because the map N has not the property N2.

Finally we deduce as a consequence of Theorem 4.13 the existence of mild solutions satisfying a "periodicity condition" and having a fixed initial velocity.

Corollary 4.16. *Let X be a reflexive Banach space and $\overline{x} \in X$. Under the assumptions* (A), F1, F2, F3, F4 *and* N1, N2 *there exists at least one mild solution for the problem*

$$\begin{cases} x''(t) \in A(t)x(t) + F(t, N(t)x),\ t \in J = [0, a] \\ x(0) = x(a) \\ x'(0) = \overline{x}. \end{cases} \tag{$\mathbf{P}$)$'_{\mathbf{N}}$}$$

Proof. By considering the maps $g, h : \mathcal{C}(J; X) \to X$ so defined

$$g(x) = x(a) \qquad \text{and} \qquad h(x) = \overline{x}, \qquad x \in \mathcal{C}(J; X)$$

it is easy to see by using Proposition 2.1 that gh1 is true. Moreover, the reflexivity of X allow us to say that gh2 holds too. Since all the hypotheses of Theorem 4.13 are satisfied, we conclude that there exists at least one mild solution of the problem $(\mathbf{P})'_{\mathbf{N}}$. □

5 Controllability for a problem driven by (E)

Now, we are in a position to study the controllability for the problem mentioned in the introduction, subjected to mixed conditions and governed by the wave equation (E) under the action of a suitable operator.

First of all we use, as in [21], the identification between functions defined on the quotient group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with values in $\mathbb{C}$ and 2π-periodic functions from $\mathbb{R}$ to $\mathbb{C}$. Moreover, we consider the space $L^2(\mathbb{T},\mathbb{C})$, i.e. the space of all functions defined in $\mathbb{R}$ and assuming values in $\mathbb{C}$, 2-integrable in $[0,2\pi]$ and 2π-periodic, with the usual norm $\|\cdot\|_{L^2(\mathbb{T},\mathbb{C})}$.

In particular, we want to study the following problem

$$\begin{cases} \frac{\partial^2 w}{\partial t^2}(t,\xi) = \frac{\partial^2 w}{\partial \xi^2}(t,\xi) + b(t)\frac{\partial w}{\partial \xi}(t,\xi) + f(t,\xi,\frac{\Im\mathrm{m}\ e(\hat{w})}{2e(\hat{w})+1}\int_0^t p(s)\,ds) + u(t,\xi) \\ w(t,0) = w(t,2\pi),\ t \in J \\ \frac{\partial w}{\partial \xi}(t,0) = \frac{\partial w}{\partial \xi}(t,2\pi),\ t \in J \\ w(0,\xi) = w(a,\xi),\ \text{a.e. } \xi \in \mathbb{R} \\ \frac{\partial w}{\partial t}(0,\xi) = x_0,\ \text{a.e. } \xi \in \mathbb{R} \\ u(t,\xi) \in U(t),\ \text{a.e. } t \in J,\ \xi \in \mathbb{R} \end{cases} \tag{C}$$

where $x_0 \in \mathbb{C}$, $J = [0,a]$, $a > 0$, $b \in \mathcal{C}^1(J)$, $p \in L^1(J)$, $f : J \times \mathbb{R} \times \mathbb{C} \to \mathbb{C}$, $e : \mathcal{C}(J, L^2(\mathbb{T},\mathbb{C})) \to \mathbb{C}$ are suitable maps, $\hat{w} : J \to L^2(\mathbb{T},\mathbb{C})$, where $\hat{w}(t) = w(t,\cdot)$, and $U : J \to \mathcal{P}(\mathbb{C})$.

In order to rewrite problem (C) into the abstract form $(\mathbf{P})'_{\mathbf{N}}$, it is necessary to define the Banach space X, the family $\{A(t) :\ t \in J\}$ and the nonlinear term F.

First of all we assume the Banach space $X = L^2(\mathbb{T},\mathbb{C})$. Moreover, we denote by $H^1(\mathbb{T},\mathbb{C})$ and by $H^2(\mathbb{T},\mathbb{C})$ respectively the following Sobolev spaces

$$H^1(\mathbb{T},\mathbb{C}) = \left\{ x \in L^2(\mathbb{T},\mathbb{C}) :\ \text{there exists } \frac{dx}{d\xi} \in L^2(\mathbb{T},\mathbb{C}) \right\}$$

$$H^2(\mathbb{T},\mathbb{C}) = \left\{ x \in L^2(\mathbb{T},\mathbb{C}) :\ \text{there exist } \frac{dx}{d\xi}, \frac{d^2x}{d\xi^2} \in L^2(\mathbb{T},\mathbb{C}) \right\}.$$

Further we consider the operator $A_0 : H^2(\mathbb{T},\mathbb{C}) \to L^2(\mathbb{T},\mathbb{C})$ so defined

$$A_0 x = \frac{d^2 x}{d\xi^2}, \quad x \in H^2(\mathbb{T},\mathbb{C}) \tag{5.1}$$

and we recall that the A_0 is the infinitesimal generator of a strongly continuous cosine family $\{C_0(t)\}_{t\in\mathbb{R}}$, where $C_0(t) : L^2(\mathbb{T},\mathbb{C}) \to L^2(\mathbb{T},\mathbb{C})$, for every $t \in \mathbb{R}$ (see [21]).

Then we fixed the function $P : J \to \mathcal{L}(H^1(\mathbb{T},\mathbb{C}), L^2(\mathbb{T},\mathbb{C}))$ defined as

$$P(t)x = b(t)\frac{dx}{d\xi}, \quad t \in J,\ x \in H^1(\mathbb{T},\mathbb{C}). \tag{5.2}$$

Now we can introduce the family $\{A(t) :\ t \in J\}$ where, for every $t \in J$, $A(t) : D(A) = H^2(\mathbb{T},\mathbb{C}) \to L^2(\mathbb{T},\mathbb{C})$ is the following operator

$$A(t) := A_0 + P(t). \tag{5.3}$$

Let us note that the family $\{A(t) :\ t \in J\}$ generates a fundamental system $\{S(t,s)\}_{(t,s)\in J\times J}$ (see [21, Lemma 4.1]) and, for every $x \in D(A)$, the map $t \mapsto A(t)x$ is continuous.

Moreover, let us consider $e : \mathcal{C}(J, L^2(\mathbb{T},\mathbb{C})) \to \mathbb{C}$ an operator such that e)$_1$ e linear, bounded such that $e(h) \neq -\frac{1}{2}$, for all $h \in \mathcal{C}(J, L^2(\mathbb{T}, \mathbb{C}))$,

the map $f : J \times \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ having the following properties

f)$_1$ $f(t, \xi + 2\pi, z) = f(t, \xi, z)$, for all $(t, \xi, z) \in J \times \mathbb{R} \times \mathbb{C}$;

f)$_2$ $f(t, \cdot, y(\cdot)) \in L^2(\mathbb{T}, \mathbb{C})$, for every $t \in J$, $y \in L^2(\mathbb{T}, \mathbb{C})$;

f)$_3$ for every $y \in L^2(\mathbb{T}, \mathbb{C})$, the map $t \mapsto f(t, \cdot, y(\cdot))$ is weakly measurable;

f)$_4$ there exists $\alpha \in L^1_+(J)$ such that

$$Ka \int_0^a \alpha(\theta)\, d\theta < 1$$

where K is presented in Section 3 and

$$\|f(t, \xi, z) - f(t, \xi, w)\|_{\mathbb{C}} \leq \alpha(t)\|z - w\|_{\mathbb{C}}, \quad z, w \in \mathbb{C},\ \xi \in \mathbb{R},\ \text{a.e. } t \in J; \tag{5.4}$$

f)$_5$ for a.e. $t \in J$ and every $(y_n)_n$, $y_n \in L^2(\mathbb{T}, \mathbb{C})$ such that $y_n \rightharpoonup y$, $y \in L^2(\mathbb{T}, \mathbb{C})$, the sequence $(f(t, \cdot, y_n(\cdot)))_n$ uniformly converges to $f(t, \cdot, y(\cdot))$ in $\mathbb{R}$;

f)$_6$ for every $t \in J$, $f(t, \cdot, 0) \in L^2(\mathbb{T}, \mathbb{C})$ and the map $t \mapsto \|f(t, \cdot, 0)\|_{L^2(\mathbb{T}, \mathbb{C})}$ is in $L^1(J)$;

and the multimap $U : J \to \mathcal{P}(\mathbb{C})$ having the following properties:

U)$_1$ for every $t \in J$, $U(t)$ is closed and convex;

U)$_2$ for every $y \in L^2(\mathbb{T}, \mathbb{C})$, the map $t \mapsto \inf_{z \in U(t)} \left(\int_0^{2\pi} \|y(\xi) - z\|^2_{\mathbb{C}}\, d\xi \right)^{\frac{1}{2}}$ is $\mathcal{M}(J) \otimes \mathcal{B}(\mathbb{R})$-measurable.

Let us note that, since our goal is to prove the existence of a mild solution, the existence of derivatives is not necessary. So it is sufficient to consider that $w(t, \cdot) \in L^2(\mathbb{T}, \mathbb{C})$, instead of $w(t, \cdot) \in H^2(\mathbb{T}, \mathbb{C})$. In that follows, we revise functions $w, u : J \times \mathbb{R} \to \mathbb{C}$ such that $w(t, \cdot), u(t, \cdot) \in L^2(\mathbb{T}, \mathbb{C})$, for every $t \in J$, as two maps $x, v : J \to L^2(\mathbb{T}, \mathbb{C})$ respectively so defined

$$x(t)(\xi) = w(t, \xi), \quad t \in J,\ \xi \in \mathbb{R}$$

$$v(t)(\xi) = u(t, \xi), \quad t \in J,\ \xi \in \mathbb{R}.$$

Now by using f)$_2$ we can define the function $\tilde{f} : J \times L^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$ such that

$$\tilde{f}(t, y)(\xi) = f(t, \xi, y(\xi)), \quad t \in J,\ \xi \in \mathbb{R},\ y \in L^2(\mathbb{T}, \mathbb{C}). \tag{5.5}$$

Next we consider the multimap $\tilde{U} : J \to \mathcal{P}(L^2(\mathbb{T}, \mathbb{C}))$

$$\tilde{U}(t) = \{v \in L^2(\mathbb{T}, \mathbb{C}) :\ \exists z \in U(t) \text{ such that } v(\xi) = z, \text{ a.e. } \xi \in \mathbb{R}\}, \quad t \in J, \tag{5.6}$$

which is obviously well defined. Thanks to hypothesis U)$_1$ we deduce that $\tilde{U}(t)$ is closed and convex, for every $t \in J$. Then, taking into account U)$_2$ and the separability of $L^2(\mathbb{T}, \mathbb{C})$, Proposition 2.8 implies the measurability of $\tilde{U}$.

Moreover, we define the multimap $F : J \times L^2(\mathbb{T}, \mathbb{C}) \to \mathcal{P}(L^2(\mathbb{T}, \mathbb{C}))$ in the following way

$$F(t, y) = \{\tilde{f}(t, y) + v :\ v \in \tilde{U}(t)\}, \quad t \in J,\ y \in L^2(\mathbb{T}, \mathbb{C}). \tag{5.7}$$

Finally, by using the linear and bounded operator $e : \mathcal{C}(J, L^2(\mathbb{T}, \mathbb{C})) \to \mathbb{C}$ and the L^1-map $p : J \to \mathbb{R}$, we construct $N : J \to \hat{\mathcal{C}}_w(\mathcal{C}(J; L^2(\mathbb{T}, \mathbb{C})); L^2(\mathbb{T}, \mathbb{C}))$ such that, for every $t \in J$ and every $h \in \mathcal{C}(J; L^2(\mathbb{T}, \mathbb{C}))$, the map $N(t)h$ is so defined (see e)$_1$)

$$[N(t)h](\xi) = \frac{\mathfrak{Im}\, e(h)}{2e(h)+1} \int_0^t p(s)\, ds, \quad \xi \in \mathbb{R}. \tag{5.8}$$

Clearly, being the map $N(t)h$ constant on $\mathbb{R}$, we have that $N(t)h \in L^2(\mathbb{T}, \mathbb{C})$ and, for every $t \in J$, $N(t)$ is weakly sequentially continuous. Therefore N is correctly defined too.

So by recalling (5.1), (5.2), (5.3), (5.5), (5.6), (5.7) and (5.8), problem **(C)** can be rewritten in the abstract form

$$\begin{cases} x''(t) \in A_0 x(t) + P(t)x(t) + F(t, N(t)x) = A(t)x(t) + F(t, N(t)x),\ t \in J \\ x(0) = x(a) \\ x'(0) = \hat{x_0} \end{cases}$$

where $\hat{x_0} : \mathbb{R} \to \mathbb{C}$ is the function of $L^2(\mathbb{T}, \mathbb{C})$ such that

$$\hat{x_0}(\xi) = x_0, \quad \xi \in \mathbb{R}. \tag{5.9}$$

At this point let us show that we can apply Corollary 4.16.

First of all, we note that the Banach space $L^2(\mathbb{T}, \mathbb{C})$ is obviously reflexive. Moreover, hypothesis (A) is clearly true thanks to the construction of the family $\{A(t) : t \in J\}$.

Now, let us show that hypotheses N1 and N2 are satisfied.

First of all, fixed $h \in \mathcal{C}(J; L^2(\mathbb{T}, \mathbb{C}))$ we note that $N(\cdot)h$ is a continuous map on J. Indeed, fixed $\bar{t} \in J$, we write (see (5.8))

$$\begin{aligned} \|N(t)h - N(\bar{t})h\|_{L^2(\mathbb{T},\mathbb{C})} &= \left\{ \int_0^{2\pi} \|[N(t)h](\xi) - [N(\bar{t})h](\xi)\|^2_{\mathbb{C}}\, d\xi \right\}^{\frac{1}{2}} \\ &= \left| \int_{\bar{t}}^t p(s)\, ds \right| \left\| \frac{\mathfrak{Im}\, e(h)}{2e(h)+1} \right\|_{\mathbb{C}} \sqrt{2\pi} \end{aligned}$$

and, by the absolute continuity of the integral we have the continuity of $N(\cdot)h$ in $\bar{t}$. Now, being $N(\cdot)h \in \mathcal{C}(J; L^2(\mathbb{T}, \mathbb{C}))$, obviously N satisfies hypothesis N1.

Moreover, for every $t \in J$ and every $h \in \mathcal{C}(J; L^2(\mathbb{T}, \mathbb{C}))$, since $\|\mathfrak{Im}\, e(h)\|_{\mathbb{C}} \le \|2e(h)+1\|_{\mathbb{C}}$, we have

$$\|N(t)h\|_{L^2(\mathbb{T},\mathbb{C})} \le \sqrt{2\pi}\|p\|_1 := \bar{c},$$

i.e. hypothesis N2 holds.

Now we show that the multimap F satisfies hypotheses F1–F4.

First of all, since $\tilde{U}$ has convex values, we can say that F takes convex values too, i.e. F1 holds.

Next we prove that, fixed $y \in L^2(\mathbb{T}, \mathbb{C})$, the multimap $F(\cdot, y)$ has a B-selection.

Since $L^2(\mathbb{T}, \mathbb{C})$ is a separable Banach space and taking into account f)$_3$, Proposition 2.2 allows us to say that $\tilde{f}(\cdot, y)$ is B-measurable.

Moreover, by recalling that $\tilde{U}$ is measurable and takes closed values, by using Proposition 2.9 there exists a $\mathcal{M}(J) \otimes \mathcal{B}(L^2(\mathbb{T}, \mathbb{C}))$-measurable $\tilde{u} : J \to L^2(\mathbb{T}, \mathbb{C})$ such that $\tilde{u}(t) \in \tilde{U}(t)$, for every $t \in J$. Then, by using the separability of the space $L^2(\mathbb{T}, \mathbb{C})$, $\tilde{u}$ is also B-measurable (see again Proposition 2.2).

So, $q_y : J \to L^2(\mathbb{T},\mathbb{C})$ so defined $q_y(t) := \tilde{f}(t,y) + \tilde{u}(t)$, $t \in J$, is B-measurable and such that $q_y(t) \in F(t,y)$, for every $t \in J$ (see (5.7)). Therefore, hypothesis F2 holds.

Now, we show that also hypothesis F3 is true.

Put H the null measure set for which f)$_5$ holds, we fix $t \in J \setminus H$ and $(y_n)_n$, $(q_n)_n$ two sequences in $L^2(\mathbb{T},\mathbb{C})$ such that $y_n \rightharpoonup y$, $q_n \rightharpoonup q$, where $y, q \in L^2(\mathbb{T},\mathbb{C})$, and $q_n \in F(t,y_n)$, for every $n \in \mathbb{N}$. Now, from (5.7), there exists $v_n \in \tilde{U}(t)$ such that

$$q_n = \tilde{f}(t,y_n) + v_n, \quad n \in \mathbb{N}. \tag{5.10}$$

Recalling f)$_5$ we have that $(f(t,\cdot,y_n(\cdot)))_n$ uniformly converges to $f(t,\cdot,y(\cdot))$ hence, by (5.5), we can write for every $\varepsilon > 0$ there exists $\overline{n}_t = \overline{n}_t(\frac{\varepsilon}{\sqrt{2\pi}}) \in \mathbb{N}$ such that, for every $n \geq \overline{n}_t$, $\|\tilde{f}(t,y_n)(\xi) - \tilde{f}(t,y)(\xi)\|_{\mathbb{C}} < \frac{\varepsilon}{\sqrt{2\pi}}$, $\xi \in \mathbb{T}$, and so we have $\|\tilde{f}(t,y_n) - \tilde{f}(t,y)\|_{L^2(\mathbb{T},\mathbb{C})} < \varepsilon$. Then we can say

$$\tilde{f}(t,y_n) \to \tilde{f}(t,y) \quad \text{in } L^2(\mathbb{T},\mathbb{C}). \tag{5.11}$$

Now, by (5.10) we have

$$v_n = q_n - \tilde{f}(t,y_n), \quad n \in \mathbb{N},$$

so (5.11) and the weak convergence of $(q_n)_n$ imply

$$v_n \rightharpoonup q - \tilde{f}(t,y) =: v,$$

where $v \in L^2(\mathbb{T},\mathbb{C})$.

Further, since $(v_n)_n$ is a sequence in the weakly closed set $\tilde{U}(t)$, the weak limit $v \in \tilde{U}(t)$ hence, by (5.7), we deduce $q = \tilde{f}(t,y) + v \in F(t,y)$. Therefore F3 holds.

Finally we prove that F4 is also true.

First of all, let $\tilde{H}$ the null measure set for which (5.4) of f)$_4$ holds. For every $n \in \mathbb{N}$, let us fix $y \in \overline{B}_{L^2(\mathbb{T},\mathbb{C})}(0,n)$ and $t \in J \setminus \tilde{H}$. Now, fixed $q \in F(t,y)$, by (5.7) there exists $v \in \tilde{U}(t)$ such that $q = \tilde{f}(t,y) + v$ and, named $z \in \mathbb{C}$ such that $v(\xi) = z$ a.e. $\xi \in \mathbb{T}$ (see (5.6)), we have (see (5.5) and f)$_4$)

$$\begin{aligned}
\|q\|_{L^2(\mathbb{T},\mathbb{C})} &\leq \|\tilde{f}(t,y)\|_{L^2(\mathbb{T},\mathbb{C})} + \|v\|_{L^2(\mathbb{T},\mathbb{C})} \\
&\leq \left\{ \int_0^{2\pi} [\alpha(t)\|y(\xi)\|_{\mathbb{C}} + \|f(t,\xi,0)\|_{\mathbb{C}}]^2 \, d\xi \right\}^{\frac{1}{2}} + \left\{ \int_0^{2\pi} \|z\|^2_{\mathbb{C}} \, d\xi \right\}^{\frac{1}{2}} \\
&\leq \left\{ \int_0^{2\pi} \alpha^2(t)\|y(\xi)\|^2_{\mathbb{C}} \, d\xi \right\}^{\frac{1}{2}} + \left\{ \int_0^{2\pi} \|f(t,\xi,0)\|^2_{\mathbb{C}} \, d\xi \right\}^{\frac{1}{2}} \\
&\quad + \left\{ \int_0^{2\pi} 2\alpha(t)\|f(t,\xi,0)\|_{\mathbb{C}}\|y(\xi)\|_{\mathbb{C}} \, d\xi \right\}^{\frac{1}{2}} + \sqrt{2\pi}\|z\|_{\mathbb{C}} \\
&\leq \alpha(t)\|y\|_{L^2(\mathbb{T},\mathbb{C})} + \|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})} \\
&\quad + \sqrt{2\alpha(t)}\sqrt{\|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})}}\sqrt{\|y\|_{L^2(\mathbb{T},\mathbb{C})}} + \sqrt{2\pi}\|z\|_{\mathbb{C}}.
\end{aligned}$$

So, recalling that $y \in \overline{B}_{L^2(\mathbb{T},\mathbb{C})}(0,n)$, we have

$$\|q\|_{L^2(\mathbb{T},\mathbb{C})} \leq \alpha(t)n + \|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})} + \sqrt{2n\alpha(t)}\sqrt{\|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})}} + \sqrt{2\pi}\|z\|_{\mathbb{C}}.$$

Therefore, put $\varphi_n : J \to \mathbb{R}^+_0$ so defined

$$\varphi_n(t) := \alpha(t)n + \|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})} + \sqrt{2n}\sqrt{\alpha(t)}\sqrt{\|f(t,\cdot,0)\|_{L^2(\mathbb{T},\mathbb{C})}} + \sqrt{2\pi}\|z\|_{\mathbb{C}},$$

by the arbitrariness of $q \in F(t,y)$ and $y \in \overline{B}_{L^2(\mathbb{T},\mathbb{C})}(0,n)$ we deduce

$$\|F(t,\overline{B}_{L^2(\mathbb{T},\mathbb{C})}(0,n))\| \leq \varphi_n(t) \quad \text{a.e. } t \in J. \tag{5.12}$$

As a consequence of f)$_4$, f)$_6$ and by using Hölder inequality it is easy to see that $\varphi_n \in L^1_+(J)$.

Moreover, by f)$_4$ we also have

$$\limsup_{n\to\infty} \frac{Ka\int_0^a \varphi_n(t)\,dt}{n} = Ka\int_0^a \alpha(t)\,dt < 1. \tag{5.13}$$

So (5.12) and (5.13) establish F4.

By virtue of arguments above presented, we are in the position to apply Corollary 4.16. Then there exists a continuous function $\hat{x} : J \to L^2(\mathbb{T},\mathbb{C})$ such that

$$\hat{x}(t) = C(t,0)\hat{x}(a) + S(t,0)\hat{x}_0 + \int_0^t S(t,s)\hat{q}(s)\,ds, \quad t \in J \tag{5.14}$$

$$\hat{x}(0) = \hat{x}(a) \qquad \hat{x}'(0) = \hat{x}_0, \tag{5.15}$$

where $\hat{q} \in S^1_{F(\cdot,N(\cdot)\hat{x})} = \{q \in L^1(J;L^2(\mathbb{T},\mathbb{C})) : \ q(t) \in F(t,N(t)\hat{x}) \text{ a.e. } t \in J\}$.

Now we consider $v_{\hat{x}} : J \to L^2(\mathbb{T},\mathbb{C})$ as

$$v_{\hat{x}}(t) = \hat{q}(t) - \tilde{f}(t,N(t)\hat{x}), \ t \in J. \tag{5.16}$$

In order to prove that $v_{\hat{x}}$ is B-measurable, we begin by showing that $\tilde{f}(\cdot,N(\cdot)\hat{x})$ is B-measurable. To this aim we consider the multimap $G : J \times L^2(\mathbb{T},\mathbb{C}) \to \mathcal{P}_k(L^2(\mathbb{T},\mathbb{C}))$ so defined (see (5.5))

$$G(t,y) = \{\tilde{f}(t,y)\}, \quad t \in J, \ y \in L^2(\mathbb{T},\mathbb{C})$$

and we establish that G satisfies all the hypotheses of Theorem 2.11.

First of all, fixed $y \in L^2(\mathbb{T},\mathbb{C})$, thanks to hypothesis f)$_3$ and to the separability of $L^2(\mathbb{T},\mathbb{C})$, $G(\cdot,y) = \{\tilde{f}(\cdot,y)\}$ has obviously a B-measurable selection, so hypothesis *i)* of Theorem 2.11 holds. Next, let us fix $t \in J \setminus H$, where H is the null measure set for which (5.4) is true, and $\overline{y} \in L^2(\mathbb{T},\mathbb{C})$. Then by (5.5) we have

$$\|\tilde{f}(t,\overline{y}) - \tilde{f}(t,y)\|_{L^2(\mathbb{T},\mathbb{C})} = \alpha(t)\|\overline{y} - y\|_{L^2(\mathbb{T},\mathbb{C})}, \quad y \in L^2(\mathbb{T},\mathbb{C}).$$

So, passing to the limit for $y \to \overline{y}$ we obtain that $\tilde{f}(t,\cdot)$ is continuous in $\overline{y}$. Obviously, for a.e. $t \in J$, $G(t,\cdot)$ is upper semicontinuous in $L^2(\mathbb{T},\mathbb{C})$, i.e. hypothesis *ii)* of Theorem 2.11.

Finally, since we have already proved N1, we know that $N(\cdot)\hat{x}$ is B-measurable. By using Theorem 2.11, $\tilde{f}(\cdot,N(\cdot)\hat{x})$ is B-measurable too. Therefore, being $\hat{q}$ B-measurable, also $v_{\hat{x}}$ is B-measurable.

At this point, put $w : J \times \mathbb{R} \to \mathbb{C}$ and $u : J \times \mathbb{R} \to \mathbb{C}$ respectively so defined

$$w(t,\xi) = \hat{x}(t)(\xi), \ u(t,\xi) = v_{\hat{x}}(t)(\xi), \quad t \in J, \ \xi \in \mathbb{R}, \tag{5.17}$$

we show that $\{w,u\}$ is an admissible mild-pair for **(C)**. By (5.14) and (5.15) we immediately have that, for every $\xi \in \mathbb{R}$, $w(\cdot,\xi)$ is continuous on J, $w(\xi,0) = w(\xi,a)$ and, for every $t \in J$, $w(t,\cdot)$ is 2-integrable on $[0,2\pi]$ and 2π-periodic. Let us note that, for every $\xi \in \mathbb{R}$ for which $w(\cdot,\xi)$ is derivable at 0, we have $\frac{\partial w}{\partial t}(0,\xi) = x_0$ (see (5.15) and (5.9)).

Then, fixed $t \in J \setminus H$, where H is the null measure set such that $\hat{q}(\cdot) \in F(\cdot,N(\cdot)\hat{x})$ in $J \setminus H$, by (5.7) there exists $v_t \in \tilde{U}(t)$ such that

$$\hat{q}(t) = \tilde{f}(t,N(t)\hat{x}) + v_t.$$

On the other hand, from (5.16) we have

$$\hat{q}(t) = \tilde{f}(t, N(t)\hat{x}) + v_{\hat{x}}(t).$$

Therefore, $v_{\hat{x}}(t) = v_t$. Hence $v_{\hat{x}}(t) \in \tilde{U}(t)$, a.e. $t \in J$. Since by (5.6) we have $v_{\hat{x}}(t)(\xi) \in U(t)$, a.e. $t \in J$, $\xi \in \mathbb{R}$, by (5.17) we can write $u(t,\xi) \in U(t)$, a.e. $t \in J$, $\xi \in \mathbb{R}$. Then, by the B-measurability of $v_{\hat{x}}$, for a.e. $\xi \in \mathbb{R}$, the map $u(\cdot,\xi)$ is B-measurable too. Clearly for every $t \in J$, $u(t,\cdot)$ is 2-integrable on $[0,2\pi]$ and 2π-periodic.

Hence, we can conclude that $\{w,u\}$ is an admissible mild-pair for **(C)**.

Finally we are able to enunciate the following result.

Theorem 5.1. *In the framework above described, there exist $w,u : J \times \mathbb{R} \to \mathbb{C}$ satisfying the following properties*

(w1) for every $t \in J$, $w(t,\cdot)$ is 2-integrable on $[0,2\pi]$ and 2π-periodic;

(w2) for every $\xi \in \mathbb{R}$, $w(\cdot,\xi)$ is continuous on J;

(w3) $w(0,\xi) = w(a,\xi)$, for every $\xi \in \mathbb{R}$;

(w4) for every $\xi \in \mathbb{R}$ such that $w(\cdot,\xi)$ is derivable at 0, we have $\frac{\partial w}{\partial t}(0,\xi) = x_0$;

(u1) for every $t \in J$, $u(t,\cdot)$ is 2-integrable on $[0,2\pi]$ and 2π-periodic;

(u2) for every $\xi \in \mathbb{R}$, $u(\cdot,\xi)$ is B-measurable and such that $u(t,\xi) \in U(t)$, for a.e. $t \in J$, $\xi \in \mathbb{R}$,

*i.e. $\{w,u\}$ is an admissible pair for **(C)** such that*

$$w(t,\xi) = [C(t,0)w(a,\cdot)](\xi) + [S(t,0)\hat{x}_0](\xi) + \int_0^t [S(t,s)q(s,\cdot)](\xi)\,ds, \quad t \in J,\ \xi \in \mathbb{R}$$

where $\hat{x}_0(\xi) = x_0$ for every $\xi \in \mathbb{R}$ and $q : J \times \mathbb{R} \to \mathbb{C}$ is so defined

$$q(t,\xi) = f\left(t, \xi, \frac{\mathfrak{Im}\, e(\hat{w})}{2e(\hat{w})+1}\int_0^t p(s)\,ds\right) + u(t,\xi), \quad t \in J,\ \xi \in \mathbb{R},$$

being $\hat{w} : J \to L^2(\mathbb{T},\mathbb{C})$, a map such that $\hat{w}(t) = w(t,\cdot)$, for every $t \in J$.

6 Conclusions and future studies

In this paper, the existence of mild solutions to a nonlocal problem governed by a semilinear second order differential inclusion in Banach spaces is investigated. The novelty with respect to the known results of [9] is the presence of an operator which intervenes on the dynamics described through a second order differential inclusion. Our first result is obtained with a fixed point approach, by applying ideas about measures of weak noncompactness, a selection theorem and a containment theorem. Further, in order to analyze the case of reflexive spaces a new selection theorem is proved and a combination of this result with the classical Hahn–Banach Theorem and the weak upper semicontinuity property is used. The applied method enables us obtaining the existence results without any compactness requirement both on the family generated by the linear part and on the nonlinear multivalued term. Finally our theoretical theorems are applied to study the controllability of a problem driven by a wave equation.

A possible future direction of research related to this topic could be to broaden the class of models to which it can be applied. For example could be interesting to remove the boundedness type property on the perturbation operator N, perhaps using a different fixed point theorem. As we noted in Remark 4.15, this assumption does not allow to see a non-perturbed problem as a particular case of a perturbed one. Moreover, in a contest of lack of compactness, this boundedness property on N does not make it possible to investigate problems involving operators N having a stabilization effect on the solution, like those studied in [8] under strong compactness assumptions.

Author contributions

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