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Ulam–Hyers stability and exponentially dichotomic evolution equations in Banach spaces

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For finite-dimensional linear differential systems with bounded coefficients we prove that their exponential dichotomy on $\mathbb{R}$ is equivalent to their Ulam–Hyers stability on $\mathbb{R}$ with uniqueness. We also consider abstract non-autonomous evolution equations which are exponentially bounded and exponentially dichotomic and prove that Ulam–Hyers stability with uniqueness is maintained when perturbing them with a nonlinear term having a sufficiently small Lipschitz constant.

Keywords: Ulam–Hyers stability, evolution family, nonautonomous, mild solution, exponential dichotomy, small Lipschitz constant.

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1 Introduction

Ulam–Hyers stability of different types of equations is intensively studied in the literature, especially in the last years. The idea of this notion was given by Ulam in 1940. Note that there exists generalizations of the initial notion (see [15]). As far as we know, the first studies on the Ulam–Hyers stability of differential equations were presented by Obłoza [12, 13] in 1993 and 1997, and by Alsina–Ger [1] in 1998.

The special case of finite dimensional linear differential systems with constant and, respectively, continuous periodic coefficients, was considered by Jung [10] in 2006, Bușe–Salieri–Tabassum [5] in 2014, Barbu–Bușe–Tabassum [4] in 2015, and, respectively, by Buică–Tőtős [3] in 2022. These papers emphasized the relation of Ulam–Hyers stability on unbounded intervals of finite dimensional linear differential systems, with their exponential dichotomy.

Ulam–Hyers stability of some nonlinear differential equations were also studied, especially on a compact interval of time. Anyway, it seems that Ulam–Hyers stability on a compact interval is a property of any linear differential system and of the most of the nonlinear ones. I. A. Rus proved this using the Gronwall Lemma technique and other techniques in [14]. In [2] we showed that exponentially stable abstract linear evolution equations are Ulam–Hyers stable on the interval $[0, \infty)$. We also proved that this property is maintained when perturbing this type of equations with a nonlinear term having a sufficiently small Lipschitz constant.

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In this work we show that exponentially dichotomic on $\mathbb{R}$ abstract linear evolution equations are Ulam–Hyers stable on $\mathbb{R}$ with uniqueness. We study the special case of finite dimensional linear differential systems with bounded coefficients and prove that their exponential dichotomy is equivalent to their Ulam–Hyers stability with uniqueness (Theorem 3.5 in Section 3). We also prove that Ulam–Hyers stability with uniqueness is maintained when perturbing this type of linear abstract evolution equations with a nonlinear term having a sufficiently small Lipschitz constant (Theorem 4.2, Theorem 4.4 and Theorem 4.6 in Section 4).

2 Exponential dichotomy of an evolution family. Definition and equivalent condition

Let $(X, |\cdot|)$ be a real or complex Banach space. The zero vector in X will be denoted by 0. $\mathcal{L}(X)$ will stand for the space of bounded linear operators from X into itself. The corresponding norm in $\mathcal{L}(X)$ will also be denoted by $|\cdot|$. The identity operator on X is $I \in \mathcal{L}(X)$. For notations, notions and results presented in this section we used [6, 11].

Definition 2.1 ([6, Definition 3.1])**.** A family of operators $\{U(\theta,\tau)\}_{\theta\geq\tau} \subset \mathcal{L}(X)$, with $\theta, \tau \in \mathbb{R}$, is called an evolution family if

(i) $U(\theta,s)U(s,\tau) = U(\theta,\tau)$ and $U(\theta,\theta) = I$ for all $\theta \geq s \geq \tau$; and

(ii) for each $x \in X$, the function $(\theta,\tau) \mapsto U(\theta,\tau)x$ is continuous for $\theta \geq \tau$.

An evolution family $\{U(\theta,\tau)\}_{\theta\geq\tau}$ is said to be exponentially bounded if, in addition,

(iii) there exist real constants $C \geq 1$ and $\gamma > 0$ such that

$$|U(\theta,\tau)| \leq Ce^{\gamma(\theta-\tau)}, \quad \theta \geq \tau.$$

We now give the definition of exponential dichotomy for an evolution family. Let $P : \mathbb{R} \to \mathcal{L}(X)$ be a projection-valued function (i.e. $P(\theta)P(\theta) = P(\theta)$ for each $\theta \in \mathbb{R}$). The function whose values are the complementary projections is denoted by $Q(\theta) = I - P(\theta)$ for each $\theta \in \mathbb{R}$. If, for all $\theta \geq \tau$, we have

$$P(\theta)U(\theta,\tau) = U(\theta,\tau)P(\tau),$$

then we denote by

$$U_P(\theta,\tau) := P(\theta)U(\theta,\tau)P(\tau), \quad U_Q(\theta,\tau) := Q(\theta)U(\theta,\tau)Q(\tau),$$

the restrictions of the operator $U(\theta,\tau)$ on $\operatorname{Im} P(\tau)$ and $\operatorname{Im} Q(\tau)$, respectively. We stress that $U_P(\theta,\tau)$ is an operator from $\operatorname{Im} P(\tau)$ to $\operatorname{Im}(\theta)$ while $U_Q(\theta,\tau)$ is an operator from $\operatorname{Im} Q(\tau)$ to $\operatorname{Im} Q(\theta)$.

Definition 2.2 ([6, Definition 3.6])**.** An evolution family $\{U(\theta,\tau)\}_{\theta\geq\tau}$ is said to have an exponential dichotomy (with constants $M > 0$ and $\omega > 0$ if there exists a projection-valued function $P : \mathbb{R} \to \mathcal{L}(X)$ such that, for each $x \in X$, the function $\theta \mapsto P(\theta)x$ is continuous and bounded, and, for all $\theta \geq \tau$, the following conditions hold.

(i) $P(\theta)U(\theta,\tau) = U(\theta,\tau)P(\tau)$.

(ii) $U_Q(\theta,\tau)$ is invertible as an operator from $\operatorname{Im} Q(\tau)$ to $\operatorname{Im} Q(\theta)$.

(iii) $|U_P(\theta,\tau)| \le Me^{-\omega(\theta-\tau)}$.

(iv) $|[U_Q(\theta,\tau)]^{-1}| \le Me^{-\omega(\theta-\tau)}$.

Denote by $C_b(\mathbb{R},X) = \{g : \mathbb{R} \to X \text{ continuous and bounded}\}$. It is known that $C_b(\mathbb{R},X)$ with the norm $\|u\| = \max_{t\in\mathbb{R}} |u(t)|$ is a Banach space.

Condition (M). For every $g \in C_b(\mathbb{R},X)$, there exists a unique function $u \in C_b(\mathbb{R},X)$ such that

$$u(\theta) = U(\theta,\tau)u(\tau) + \int_\tau^\theta U(\theta,s)g(s)ds, \quad \theta \ge \tau. \tag{2.1}$$

Theorem 2.3 (Theorem 4.28 in [6]). *An exponentially bounded evolution family has an exponential dichotomy if and only if Condition (M) is satisfied. Moreover, if this is the case, for each $g \in C_b(\mathbb{R},X)$ the solution $u^* \in C_b(\mathbb{R},X)$ of the integral equation (2.1) is given by*

$$u^*(\theta) = \int_{-\infty}^\theta U_P(\theta,\tau)g(\tau)d\tau - \int_\theta^\infty [U_Q(\tau,\theta)]^{-1}g(\tau)d\tau, \quad \theta \in \mathbb{R}. \tag{2.2}$$

Proposition 2.4. *In the hypotheses of Theorem 2.3, the function given by (2.2) satisfies*

$$\|u^*\| \le \frac{2M}{\omega}\|g\|. \tag{2.3}$$

When either $P(t) = I$ for all $t \in \mathbb{R}$, or $Q(t) = I$ for all $t \in \mathbb{R}$, the estimation can be improved as

$$\|u^*\| \le \frac{M}{\omega}\|g\|. \tag{2.4}$$

Proof. For any $t \in \mathbb{R}$ we have

$$\begin{aligned}|u^*(t)| &\le \left|\int_{-\infty}^t U_P(t,s)g(s)ds\right| + \left|\int_t^\infty [U_Q(s,t)]^{-1}g(s)ds\right| \\ &\le \left|\int_{-\infty}^t |U_P(t,s)|\cdot|g(s)|ds\right| + \left|\int_t^\infty |[U_Q(s,t)]^{-1}|\cdot|g(s)|ds\right| \\ &\le M\|g\|\left[\int_{-\infty}^t e^{-\omega(t-s)}ds + \int_t^\infty e^{-\omega(s-t)}ds\right] = \frac{2M}{\omega}\|g\|.\end{aligned}$$

In each of the particular cases $P = I$ or $Q = I$, only one of the two integrals appear in the expression (2.2) of u^*. Thus, also in the last line of the display above appears only one of the two integrals, each of them being equal to $1/\omega$. □

3 Exponential dichotomy and Ulam–Hyers stability of finite dimensional linear differential systems

Let $A \in C(\mathbb{R},\mathcal{L}(\mathbb{C}^n))$. We consider the differential system in $X = \mathbb{C}^n$

$$x' = A(t)x. \tag{3.1}$$

We present now the notion of Ulam–Hyers stability on the time interval $\mathbb{R}$ of the finite dimensional linear differential system (3.1).

Definition 3.1. We say that the equation (3.1) is Ulam–Hyers stable when there exists a constant $m > 0$ such that, for any $\varepsilon > 0$ and any $\varphi \in C^1(\mathbb{R}, \mathbb{C}^n)$ with

$$|\varphi'(t) - A(t)\varphi(t)| \leq \varepsilon, \quad t \in \mathbb{R},$$

there exists $\psi \in C^1(\mathbb{R}, X^n)$ a solution of (3.1), such that $(\varphi - \psi) \in C_b(\mathbb{R}, \mathbb{C}^n)$ and

$$\|\varphi - \psi\| \leq m\varepsilon.$$

We say that the equation (3.1) is Ulam–Hyers stable with uniqueness when, for a given φ as above, there exists a unique ψ.

Remark 3.2. Assume, in addition, that there exists $T > 0$ such that $A(T + t) = A(t)$ for all $t \in \mathbb{R}$. It is known that, in this particular case, if equation (3.1) is Ulam–Hyers stable then it is Ulam–Hyers stable with uniqueness. One can see, for example [3].

An important result proved in [3] is the following.

Lemma 3.3 ([3])**.** *The equation $x' = A(t)x$ is Ulam–Hyers stable if and only if for any $g \in C_b(\mathbb{R}, \mathbb{C}^n)$ there is a solution in $C_b(\mathbb{R}, \mathbb{C}^n) \cap C^1(\mathbb{R}, \mathbb{C}^n)$ of $x' = A(t)x + g$.*

Let $Y(t) \in \mathcal{L}(\mathbb{C}^n)$ be the fundamental matrix solution of (3.1) such that $Y(0)$ is the identity matrix, and define

$$U(\theta, \tau) = Y(\theta)Y^{-1}(\tau), \quad \theta, \tau \in \mathbb{R}.$$

It is known (or it can be easily checked) that $\{U(\theta, \tau)\}_{\theta \geq \tau}$ is an evolution family and we have $[U(\theta, \tau)]^{-1} = U(\tau, \theta)$ for all $\theta, \tau \in \mathbb{R}$.

We say that *the equation $x' = A(t)x$ has an exponential dichotomy* whenever $\{U(\theta, \tau)\}_{\theta \geq \tau}$ defined above has an exponential dichotomy (as in Definition 2.2).

In addition, we have the following.

Lemma 3.4 ([7])**.** *If A is a bounded function then $\{U(\theta, \tau)\}_{\theta \geq \tau}$ is exponentially bounded.*

Proof. Fix $\tau \in \mathbb{R}$. Then $U(\cdot, \tau)$ is a matrix solution of the initial value problem $x' = A(t)x$, $x(\tau) = I_n$ (the identity matrix). Then

$$U(\theta, \tau) = I_n + \int_\tau^\theta A(s)U(s, \tau)ds, \quad \theta \geq \tau.$$

Applying the Gronwall inequality we immediately obtain $|U(\theta, \tau)| \leq e^{\gamma(\theta - \tau)}$, $\theta \geq \tau$, where $\gamma > 0$ is such that $|A(t)| \leq \gamma$ for all $t \in \mathbb{R}$. □

As a consequence of Lemma 3.3, Lemma 3.4 and Theorem 2.3 we obtain the following characterizations, which is the main result of this section.

Theorem 3.5. *Let $A \in C(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ be a bounded function. The following conditions are equivalent.*

(i) The equation (3.1) is Ulam–Hyers stable with uniqueness.

(ii) Condition (M) is satisfied for the equation (3.1).

(iii) The equation (3.1) has an exponential dichotomy.

Using Remark 3.2, Theorem 3.5, and a result from [7] we obtain the following corollary. In the statement appears the fundamental matrix solution $Y(t)$ defined before.

Corollary 3.6. *Let $A \in C(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ be a T-periodic function. The following conditions are equivalent.*

(i) The equation (3.1) is Ulam–Hyers stable with uniqueness.

(ii) Condition (M) is satisfied for the equation (3.1).

(iii) The equation (3.1) has an exponential dichotomy.

(iv) No eigenvalue of $Y(T)$ lies on the unit circle.

In the case when $A \in \mathcal{L}(\mathbb{C}^n)$ (is constant) Corollary 3.6 holds true with condition (iv) replaced by "*No eigenvalue of A has zero real part.*". These two corollaries are known, but they were justified using other tools. One can see [3,4].

4 Main abstract result and applications

The main result of this section concludes the Ulam–Hyers stability of mild solutions of some nonlinear abstract nonautonomous evolution equations. We start by proving a lemma which is essential in the proof of the main result. We present with details two applications of the main abstract result for finite dimensional nonautonomous differential systems and for an abstract autonomous evolution equation whose linear part is the generator of a C_0-semigroup.

Lemma 4.1. *Let $\{U(\theta,\tau)\}_{\theta\geq\tau}$ be an exponentially bounded evolution family on X. In addition, assume that it has an exponential dichotomy and let the constants $M > 0$ and $\omega > 0$ be like in Definition 2.2.*

Let $L > 0$, $g \in C_b(\mathbb{R}, X)$ and $F \in C(\mathbb{R} \times X, X)$ with $F(s, 0) = 0$ for any $s \in \mathbb{R}$. Assume that

(i) $|F(s,u_1) - F(s,u_2)| \leq L|u_1 - u_2|$, $s \in \mathbb{R}$, $u_1, u_2 \in X$,

(ii) $2L < \omega/M$.

Then there exists a unique solution $u^* \in C_b(\mathbb{R}, X)$ of the following integral equation.

$$u(t) = \int_{-\infty}^{t} U_P(t,s)[F(s,u(s)) + g(s)]ds - \int_{t}^{\infty} [U_Q(s,t)]^{-1}[F(s,u(s)) + g(s)]ds. \tag{4.1}$$

Moreover, we have

$$\|u^*\| \leq \frac{M}{\omega/2 - LM}\|g\|. \tag{4.2}$$

When either $P(t) = I$ for all $t \in \mathbb{R}$, or $Q(t) = I$ for all $t \in \mathbb{R}$, condition (ii) *can be replaced by* (ii)′ *$L < \omega/M$ and the estimation (4.2) can be improved as*

$$\|u^*\| \leq \frac{M}{\omega - LM}\|g\|. \tag{4.3}$$

Proof. Consider the operator

$$B : C_b(\mathbb{R}, X) \to C(\mathbb{R}, X)$$

defined for any $u \in C_b(\mathbb{R}, X)$ and for any $t \in \mathbb{R}$ by

$$B(u)(t) = \int_{-\infty}^{t} U_P(t,s)[F(s,u(s)) + g(s)]ds - \int_{t}^{\infty} [U_Q(s,t)]^{-1}[F(s,u(s)) + g(s)]ds.$$

We claim that B is a contraction with the Lipschitz constant $2LM/\omega$. For any $u_1, u_2 \in C_b(\mathbb{R}, X)$ and $t \in \mathbb{R}$ we have

$$\begin{aligned}
|B(u_1)(t) - B(u_2)(t)| &\leq \left| \int_{-\infty}^{t} U_P(t,s)[F(s,u_1(s)) - F(s,u_2(s))]ds \right| \\
&\quad + \left| \int_{t}^{\infty} [U_Q(s,t)]^{-1}[F(s,u_1(s)) - F(s,u_2(s))]ds \right| \\
&\leq L \left| \int_{-\infty}^{t} |U_P(t,s)| \cdot |u_1(s) - u_2(s)| ds \right| \\
&\quad + L \left| \int_{t}^{\infty} |[U_Q(s,t)]^{-1}| \cdot |u_1(s) - u_2(s)| ds \right| \\
&\leq LM \|u_1 - u_2\| \left[\int_{-\infty}^{t} e^{-\omega(t-s)} ds + \int_{t}^{\infty} e^{-\omega(s-t)} ds \right] \\
&\leq \frac{2LM}{\omega} \|u_1 - u_2\|.
\end{aligned}$$

Then

$$\|B(u_1) - B(u_2)\| \leq \frac{2LM}{\omega} \|u_1 - u_2\|, \quad u_1, u_2 \in C_b(\mathbb{R}, X). \tag{4.4}$$

Thus, the claim is proved.

By Theorem 2.3 we have $B(0) \in C_b(\mathbb{R}, X)$ since its expression is given by (2.2). Then using (2.3) from Proposition 2.4 we have

$$\|B(0)\| \leq \frac{2M}{\omega} \|g\|. \tag{4.5}$$

Relation (4.4) implies that

$$\|B(u)\| \leq \frac{2LM}{\omega} \|u\| + \|B(0)\|, \quad u \in C_b(\mathbb{R}, X). \tag{4.6}$$

Then

$$Bu \in C_b(\mathbb{R}, X), \quad u \in C_b(\mathbb{R}, X),$$

meaning that $C_b(\mathbb{R}, X)$ is invariant for B. The Contraction Mapping Principle assures the existence of a unique fixed point, denoted u^*, of B in $C_b(\mathbb{R}, X)$. Moreover, from (4.6) we deduce that

$$\|u^*\| \leq \frac{2LM}{\omega} \|u^*\| + \|B(0)\|,$$

which, together with (4.5) implies (4.2).

For the last part one needs to use (2.4) instead of (2.3). □

Theorem 4.2. *Let $\{U(\theta,\tau)\}_{\theta\geq\tau}$ be an exponentially bounded evolution family on X. In addition, assume that it has an exponential dichotomy and let the constants $M > 0$ and $\omega > 0$ be like in Definition 2.2.*

Let $f \in C(\mathbb{R} \times X, X)$, $L > 0$ be such that

(i) $|f(s,u_1) - f(s,u_2)| \leq L|u_1 - u_2|$, $s \in \mathbb{R}$, $u_1, u_2 \in X$,

(ii) $2L < \omega/M$.

Let $g \in C_b(\mathbb{R}, X)$. If $\varphi \in C(\mathbb{R}, X)$ is a solution of

$$y(\theta) = U(\theta,\tau)y(\tau) + \int_\tau^\theta U(\theta,s)[f(s,y(s) + g(s)]ds, \quad \theta \geq \tau, \tag{4.7}$$

then there exists a unique solution $\psi \in C(\mathbb{R}, X)$ of

$$x(\theta) = U(\theta,\tau)x(\tau) + \int_\tau^\theta U(\theta,s)f(s,x(s)ds, \quad \theta \geq \tau, \tag{4.8}$$

such that $(\varphi - \psi) \in C_b(\mathbb{R}, X)$ and

$$\|\varphi - \psi\| \leq \frac{M}{\omega/2 - LM}\|g\|. \tag{4.9}$$

When either $P(t) = I$ for all $t \in \mathbb{R}$, or $Q(t) = I$ for all $t \in \mathbb{R}$, condition (ii) *can be replaced by* (ii)′ *$L < \omega/M$ and the estimation (4.9) can be improved as*

$$\|\varphi - \psi\| \leq \frac{M}{\omega - LM}\|g\|. \tag{4.10}$$

Proof. Consider the function $F : \mathbb{R} \times X \to X$ defined by

$$F(s,u) = f(s,\varphi(s)) - f(s,\varphi(s) - u), \quad (s,u) \in \mathbb{R} \times X.$$

It is not difficult to see that F satisfies the hypotheses of Lemma 4.1. In fact, all the hypotheses of this theorem are fulfilled. Then let $u^* \in C_b(\mathbb{R}, X)$ be the unique bounded solution of equation (4.1). Consider the function $g^*(s) = F(s, u^*(s)) + g(s)$, $s \in \mathbb{R}$ which satisfies $g^* \in C_b(\mathbb{R}, X)$. Then, from (4.1) we have that

$$u^*(\theta) = \int_{-\infty}^\theta U_P(\theta,\tau)g^*(\tau)d\tau - \int_\theta^\infty [U_Q(\tau,\theta)]^{-1}g^*(\tau)d\tau, \quad \theta \in \mathbb{R}. \tag{4.11}$$

By Theorem 2.3, the above relation implies that u^* is the unique bounded solution of

$$u(\theta) = U(\theta,\tau)u(\tau) + \int_\tau^\theta U(\theta,s)g^*(s)ds, \quad \theta \geq \tau. \tag{4.12}$$

Now define

$$\psi = \varphi - u^*$$

and note that $\psi \in C(\mathbb{R}, X)$ is a solution of (4.7) which, in addition, by Lemma 4.1, satisfies (4.9). The uniqueness of ψ with mentioned properties follows by the uniqueness of u^* as in Theorem 2.3. □

4.1 Application. Finite dimensional differential systems

Let $A \in C(\mathbb{R}, \mathcal{L}(\mathbb{C}^n))$ and $f \in C(\mathbb{R} \times \mathbb{C}^n, \mathbb{C}^n)$. We consider the nonlinear differential system in $X = \mathbb{C}^n$

$$x' = A(t)x + f(t,x). \tag{4.13}$$

Recall that we refer to the linear system $x' = A(t)x$ in Section 3.

Definition 4.3. We say that the equation (4.13) is Ulam–Hyers stable when there exists a constant $m > 0$ such that, for any $\varepsilon > 0$ and any $\varphi \in C^1(\mathbb{R}, \mathbb{C}^n)$ with

$$|\varphi'(t) - A(t)\varphi(t) - f(t, \varphi(t))| \leq \varepsilon, \quad t \in \mathbb{R},$$

there exists $\psi \in C^1(\mathbb{R}, \mathbb{C}^n)$ a solution of (4.13), such that $(\varphi - \psi) \in C_b(\mathbb{R}, \mathbb{C}^n)$ and

$$\|\varphi - \psi\| \leq m\varepsilon.$$

We say that the equation (4.13) is Ulam–Hyers stable with uniqueness when, for a given φ as above, there exists a unique ψ.

As a consequence of Theorem 4.2, using also Lemma 3.4, we obtain the following result.

Theorem 4.4. *Assume that A is a bounded function and that the system $x' = A(t)x$ has an exponential dichotomy. Let $M > 0$ and $\omega > 0$ be like in Definition 2.2. Assume that there exists $L > 0$ with $2L < \omega/M$ and such that*

$$|f(s,y) - f(s,x)| \leq L|x - y|, \quad \textit{for all } s \in \mathbb{R}, \quad x, y \in \mathbb{C}^n.$$

Then system (4.13) is Ulam–Hyers stable with uniqueness and with constant

$$m = M/(\omega/2 - LM).$$

4.2 Application. Semigroups

For the definition of a C_0-semigroup and other useful results we used [8,9].

Definition 4.5. If the evolution family $\{U(\theta, \tau)\}_{\theta \geq \tau}$ on the Banach space X satisfies in addition

$$U(\theta, \tau)x = U(\theta - \tau, 0)x, \quad \theta \geq \tau, \quad x \in X,$$

then it is called a C_0-semigroup.

Assume from now that $\{U(\theta, \tau)\}_{\theta \geq \tau}$ is a C_0-semigroup. An important remark is that there exists a dense set $D \subset X$ and a linear operator $A : D \to X$ such that if $x \in D$,

$$\lim_{\theta \downarrow 0} \frac{U(\theta, 0)x - x}{\theta} = Ax.$$

The mapping A is in general unbounded and is called the infinitesimal generator of the semigroup. Sometimes the following notation is used

$$e^{tA} := U(t, 0), \quad t \geq 0$$

and it is said that $\{e^{tA}\}_{t\geq 0}$ is a one-parameter C_0-semigroup.

Let $f \in C(\mathbb{R} \times X, X)$ and consider the abstract evolution equation

$$x' + Ax = f(t,x), \tag{4.14}$$

and the abstract evolution inequation

$$|x' + Ax - f(t,x)| \leq \varepsilon. \tag{4.15}$$

We say that $\psi \in C(\mathbb{R}, X)$ is a *mild solution* of equation (4.14) if ψ is a solution of the integral equation (4.8).

We say that $\varphi \in C(\mathbb{R}, X)$ is a *mild solution* of inequation (4.15) if there exists $g \in C(\mathbb{R}, X)$ with $|g(s)| \leq \varepsilon$, $s \in \mathbb{R}$ such that φ is a solution of the integral equation (4.7).

Let $m > 0$. We say that the evolution equation (4.14) is *Ulam–Hyers stable with constant m* if for any $\varepsilon > 0$ and for any mild solution $\varphi \in C(\mathbb{R}, X)$ of inequation (4.15) there exists a mild solution $\psi \in C(\mathbb{R}, X)$ of (4.14) such that $(\varphi - \psi) \in C_b(\mathbb{R}, X)$ and

$$\|\varphi - \psi\| \leq m\varepsilon.$$

We say that the equation (4.14) is Ulam–Hyers stable with uniqueness when, for a given φ as above, there exists a unique ψ.

As a consequence of Theorem 4.2 we obtain the following result.

Theorem 4.6. *Let $A : D \subset X \to X$ be the infinitesimal generator of an exponentially bounded and exponentially dichotomic C_0-semigroup $\{U(\theta,\tau)\}_{\theta\geq\tau}$. Let M and ω be like in Definition 2.2. Assume that there exists $L > 0$ with $2L < \omega/M$ such that*

$$|f(s,y) - f(s,x)| \leq L|x-y|, \quad \text{for all } s \in \mathbb{R}, \quad x,y \in X.$$

Then the abstract evolution equation (4.14) is Ulam–Hyers stable with uniqueness and with constant $m = M/(\omega/2 - LM)$.

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