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The logistic equation in the context of Stieltjes differential and integral equations

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Abstract. In this paper, we introduce logistic equations with Stieltjes derivatives and provide explicit solution formulas. As an application, we present a population model which involves intraspecific competition, periods of hibernation, as well as seasonal reproductive cycles. We also deal with various forms of Stieltjes integral equations, and find the corresponding logistic equations. We show that our work extends earlier results for dynamic equations on time scales, which served as an inspiration for this paper.

Keywords: logistic equation, Stieltjes differential equation, Stieltjes integral equation, dynamic equation, population dynamics.

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1 Introduction

The logistic equation is ubiquitous in population dynamics. The simplest version of this equation, which was proposed by Pierre-François Verhulst in 1838 (see [2]), has the form

$$x'(t) = rx(t)\left(1 - \frac{x(t)}{K}\right),$$

where $x(t)$ represents the size of a population at time t, r is the population growth rate, and K is the carrying capacity of the environment, i.e., the maximum population size that can be sustained by the environment. More realistic models assume that r and K are no longer constants and are, in fact, functions of time, which leads to the equation

$$x'(t) = r(t)x(t)\left(1 - \frac{x(t)}{K(t)}\right). \tag{1.1}$$

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Observe that the logistic equation above is nonlinear; however, dividing Eq. (1.1) by $-x(t)^2$ and substituting $y(t) = 1/x(t)$, we obtain the nonhomogeneous linear equation

$$y'(t) = -r(t)y(t) + \frac{r(t)}{K(t)},$$

whose solution can be obtained using the variation of constants formula. Conversely, starting with the general first-order nonhomogenous linear equation

$$y'(t) = p(t)y(t) + f(t)$$

and performing the change of variables $x(t) = 1/y(t)$, we get the logistic equation

$$x'(t) = -p(t)x(t) - f(t)x(t)^2.$$

Thus, the logistic equation can be regarded as an equation for $x = 1/y$, where y is a (nonzero) solution of a first-order nonhomogenous linear equation. This idea has been employed in [3], which deals with dynamic equations on time scales. Beginning with the first-order nonhomogeneous linear Δ-dynamic equation

$$u^{\Delta}(t) = p(t)u(t) + f(t), \tag{1.2}$$

the authors found that $y = 1/u$ satisfies

$$y^{\Delta}(t) = -(p(t) + f(t)y(t))y(\sigma(t)), \tag{1.3}$$

where σ is the forward jump operator. Similarly, starting with the adjoint equation of Eq. (1.2), namely,

$$v^{\Delta}(t) = -p(t)v(\sigma(t)) + f(t)$$

they found that $x = 1/v$ satisfies

$$x^{\Delta}(t) = (p(t) - f(t)x(\sigma(t))x(t). \tag{1.4}$$

Hence, it is reasonable to refer to Eq. (1.3) and Eq. (1.4) as to logistic dynamic equations.

In the present paper, we deal with two classes of equations that are more general than dynamic equations, and whose solutions need not be continuous. First, we focus on Stieltjes differential equations, which were introduced and studied e.g. in [7–12, 14]. The concept of the Stieltjes derivative of a function with respect to a left-continuous nondecreasing function g is recalled in Section 2, where we also recall some basic facts on linear Stieltjes differential equations. In Section 3, we show that if u is a (nonzero) solution of the Stieltjes differential equation

$$u'_g(t) = p(t)u(t) + f(t), \tag{1.5}$$

then $y = 1/u$ is a solution of

$$y'_g(t)(1 + (p(t) + f(t)y(t))\Delta^+ g(t)) + p(t)y(t) + f(t)y(t)^2 = 0 \tag{1.6}$$

(where $\Delta^+ g(t) = g(t+) - g(t)$), or equivalently

$$y'_g(t) = -(p(t) + f(t)y(t))y(t+).$$

Similarly, if v is a solution of the adjoint linear equation to Eq. (1.5), i.e.,

$$v'_g(t) = -\frac{p(t)}{1+p(t)\Delta^+g(t)}v(t) + \frac{f(t)}{1+p(t)\Delta^+g(t)}, \tag{1.7}$$

then $x = 1/v$ satisfies

$$x'_g(t)(1+\Delta^+g(t)f(t)x(t)) - p(t)x(t) + f(t)x(t)^2 = 0, \tag{1.8}$$

or equivalently

$$x'_g(t) = (p(t) - f(t)x(t+))x(t).$$

In view of these facts, we refer to Eq. (1.6) and Eq. (1.8) as to logistic equations with Stieltjes derivatives. We provide explicit solution formulas for both equations.

In Section 4, we show that the logistic Δ-dynamic equations (1.3) and (1.4) represent special cases of the Stieltjes differential equations (1.6) and (1.8) corresponding to a suitable function g.

Theoretical results on logistic differential equations with Stieltjes derivatives are illustrated on an example in Section 5. It describes a simple model of grizzly bears, whose population dynamics involves competition between individuals, periods of hibernation, as well as a seasonal reproductive cycle.

The second part of the paper deals with Stieltjes integral equations. In Section 6, we recall some basic properties of Stieltjes integrals, and present a generalization of the quotient rule; as far as we are aware, this is the first appearance of the quotient rule for Stieltjes integrals in the literature.

In Section 7, we consider three types of linear nonhomogeneous Stieltjes integral equations, namely

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + f(s))\,\mathrm{d}g(s),$$

as well as the pair of dual equations

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s-) + f(s))\,\mathrm{d}g(s),$$
$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(s+) + f(s))\,\mathrm{d}g(s),$$

which were recently studied in [13, 20]. For each of the three equations, we find the corresponding logistic equation satisfied by the function $y = 1/x$. In comparison with earlier sections, we only assume that g has bounded variation, and do not require left-continuity. Because of this, the corresponding theory covers not only Δ-dynamic equations on time scales (where the corresponding g is left-continuous), but also ∇-dynamic equations (where g is right-continuous). These facts are utilized in Section 8, where we explore the relations between Stieltjes integral versions of the logistic equation and both types of dynamic equations.

2 Preliminaries on Stieltjes derivatives

Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing and left-continuous function. We shall denote by μ_g the Lebesgue–Stieltjes measure associated to g given by

$$\mu_g([c,d)) = g(d) - g(c), \quad c, d \in \mathbb{R},\ c < d,$$

see [6, 17, 18]. We will use the term "g-measurable" for a set or function to refer to μ_g-measurability in the corresponding sense, and we denote by $\mathcal{L}^1_g(X, \mathbb{R})$ the set of Lebesgue–Stieltjes μ_g-integrable functions on a g-measurable set X with values in $\mathbb{R}$, whose integral we denote by

$$\int_X f(s)\,\mathrm{d}\mu_g(s), \quad f \in \mathcal{L}^1_g(X, \mathbb{R}).$$

Similarly, we will talk about properties holding g-almost everywhere in a set X (shortened to g-a.e. in X), or holding for g-almost all (or simply, g-a.a.) $x \in X$, as a simplified way to express that they hold μ_g-almost everywhere in X or for μ_g-almost all $x \in X$, respectively.

Define

$$C_g = \{t \in \mathbb{R} : g \text{ is constant on } (t - \varepsilon, t + \varepsilon) \text{ for some } \varepsilon > 0\},$$
$$D_g = \{t \in \mathbb{R} : \Delta^+ g(t) > 0\}.$$

Observe that, as pointed out in [9], the set C_g has null g-measure and it is open in the usual topology, so it can be uniquely expressed as the countable union of open disjoint intervals, say

$$C_g = \bigcup_{n \in \mathbb{N}} (a_n, b_n),$$

for some $a_n, b_n \in [-\infty, +\infty]$, $n \in \mathbb{N}$. With this notation, we also define

$$N_g^- = \{a_n \in \mathbb{R} : n \in \mathbb{N}\} \setminus D_g, \quad N_g^+ = \{b_n \in \mathbb{R} : n \in \mathbb{N}\} \setminus D_g, \quad N_g = N_g^- \cup N_g^+.$$

We are now in position to introduce the definition of the Stieltjes derivative of a real-valued function as in [9, 11].

Definition 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R} \setminus C_g$. We define the *Stieltjes derivative*, or *g-derivative*, of f at t as follows, provided the corresponding limit exists:

$$f'_g(t) = \begin{cases} \displaystyle\lim_{s \to t} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \notin D_g \cup N_g, \\ \displaystyle\lim_{s \to t^-} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \in N_g^-, \\ \displaystyle\lim_{s \to t^+} \frac{f(s) - f(t)}{g(s) - g(t)}, & t \in D_g \cup N_g^+, \end{cases}$$

In that case, we say that f is *g-differentiable at t*.

Remark 2.2. It is important to note that, as explained in [11, Remark 2.2], for $t \in N_g$ we have

$$f'_g(t) = \lim_{s \to t} \frac{f(s) - f(t)}{g(s) - g(t)},$$

as the domain of the quotient function gives the corresponding one-sided limit. Furthermore, since g is a regulated function, it follows that the g-derivative of f at a point $t \in D_g$ exists if and only if $f(t+)$ exists and, in that case,

$$f'_g(t) = \frac{\Delta^+ f(t)}{\Delta^+ g(t)}.$$

The following result, [11, Proposition 2.5], contains some basic properties of Stieltjes derivatives such as linearity and the product and quotient rules.

Proposition 2.3. *Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be two g-differentiable functions at $t \in \mathbb{R} \setminus C_g$. Then:*

- *The function $\lambda_1 f_1 + \lambda_2 f_2$ is g-differentiable at t for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and*

$$(\lambda_1 f_1 + \lambda_2 f_2)'_g(t) = \lambda_1 (f_1)'_g(t) + \lambda_2 (f_2)'_g(t).$$

- *The product $f_1 f_2$ is g-differentiable at t and*

$$(f_1 f_2)'_g(t) = (f_1)'_g(t) f_2(t) + (f_2)'_g(t) f_1(t) + (f_1)'_g(t)\,(f_2)'_g(t) \Delta^+ g(t).$$

- *If $f_2(t)\,(f_2(t) + (f_2)'_g(t)\,\Delta^+ g(t)) \neq 0$, the quotient f_1/f_2 is g-differentiable at t and*

$$\left(\frac{f_1}{f_2}\right)'_g(t) = \frac{(f_1)'_g(t) f_2(t) - (f_2)'_g(t) f_1(t)}{f_2(t)\,(f_2(t) + (f_2)'_g(t)\,\Delta^+ g(t))}. \tag{2.1}$$

Next, we present the concept of g-absolute continuity introduced in [9], as well as some of its properties. For simplicity, we introduce such concept as part of the following result from [9, Proposition 5.4].

Theorem 2.4. *Let $F : [a,b] \to \mathbb{R}$. The following conditions are equivalent:*

1. *The function F is* g-absolutely continuous on $[a,b]$ *according to the following definition: for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every open pairwise disjoint family of subintervals $\{(a_n, b_n)\}_{n=1}^m$ satisfying*

$$\sum_{n=1}^m (g(b_n) - g(a_n)) < \delta,$$

we have

$$\sum_{n=1}^m |F(b_n) - F(a_n)| < \varepsilon.$$

2. *The function F satisfies the following conditions:*
   - (i) *there exists $F'_g(t)$ for g-a.a. $t \in [a,b)$;*
   - (ii) *$F'_g \in \mathcal{L}^1_g([a,b), \mathbb{R})$;*
   - (iii) *for each $t \in [a,b]$,*

$$F(t) = F(a) + \int_{[a,t)} F'_g(s)\, \mathrm{d}\mu_g(s). \tag{2.2}$$

Remark 2.5. Observe that the equality in Eq. (2.2) is, indeed, true for $t = a$ as we are considering the integral over the empty set, which makes the integral null.

We denote by $\mathcal{AC}_g([a,b], \mathbb{R})$ the set of g-absolutely continuous functions in $[a,b]$ with values on $\mathbb{R}$. With this notation, we present [9, Proposition 2.4], a result that, in a way, is the converse of Theorem 2.4.

Theorem 2.6. *Let $f \in \mathcal{L}^1_g([a,b), \mathbb{R})$. Then, the function $F : [a,b] \to \mathbb{R}$ defined as*

$$F(t) = \int_{[a,t)} f(s)\, \mathrm{d}\mu_g(s),$$

is an element of $\mathcal{AC}_g([a,b], \mathbb{R})$ and $F'_g(t) = f(t)$ for g-a.a. $t \in [a,b)$.

We include the following lemma that, to the best of our knowledge, is not available in the literature. (Only the fact that if f has bounded variation and is bounded away from zero, then $1/f$ has bounded variation, is known; see for example [1, Exercise 1.1].) This result shows that, under certain conditions, the multiplicative inverse of a g-absolutely continuous function is also g-absolutely continuous.

Lemma 2.7. *Let $f : [a,b] \to \mathbb{R}$ be a regulated function such that*

$$f(t) \neq 0,\ t \in [a,b]; \quad f(t+) \neq 0,\ t \in [a,b); \quad f(t-) \neq 0,\ t \in (a,b].$$

Then, there exists $M > 0$ such that $|f(t)| \geq M$ for all $t \in [a,b]$. Furthermore:

(i) *If f has bounded variation on $[a,b]$, then so does $1/f$.*

(ii) *If f is g-absolutely continuous on $[a,b]$, then so is $1/f$.*

Proof. First, note that for each $t \in (a,b)$, $f(t-), f(t+) \neq 0$ so we can find $\delta_t > 0$ such that

$$|f(s)| > \frac{|f(t-)|}{2},\ s \in (t-\delta_t, t) \quad \text{and} \quad |f(s)| > \frac{|f(t+)|}{2},\ s \in (t, t+\delta_t).$$

Consequently, $|f(s)| \geq M_t := \min\{|f(t-)|/2, |f(t+)|/2, |f(t)|\} > 0$ for all $s \in (t-\delta_t, t+\delta_t)$. A similar reasoning shows that there exist $\delta_a, \delta_b > 0$ such that

$$|f(s)| \geq M_a := \min\left\{\frac{|f(a+)|}{2}, |f(a)|\right\}, \quad s \in [a, a+\delta_a),$$
$$|f(s)| \geq M_b := \min\left\{\frac{|f(b-)|}{2}, |f(b)|\right\}, \quad s \in (b-\delta_b, b].$$

Note that the family $\{(t-\delta_t, t+\delta_t)\}_{t\in[a,b]}$ is an open cover of $[a,b]$, which is compact, so there must be a finite subcover, i.e., there exist $t_1, t_2, \ldots, t_N \in [a,b]$ such that $\{(t_k - \delta_{t_k}, t_k + \delta_{t_k})\}_{k=1}^N$ covers $[a,b]$. Now, it is enough to take $M = \min\{M_{t_1}, M_{t_2}, \ldots, M_{t_N}\}$ to obtain the first part of the result.

Now, in order to prove (i)–(ii), note that given $c, d \in [a,b]$, $c < d$, we have

$$\left|\frac{1}{f(d)} - \frac{1}{f(c)}\right| = \left|\frac{f(c)-f(d)}{f(d)f(c)}\right| \leq \frac{|f(c)-f(d)|}{M^2}. \tag{2.3}$$

Assume that f has bounded variation on $[a,b]$. Let $a = t_0 < t_1 < \cdots < t_m = b$ be a partition of $[a,b]$. Then, (2.3) yields

$$\sum_{i=1}^m \left|\frac{1}{f(t_i)} - \frac{1}{f(t_{i-1})}\right| \leq \frac{1}{M^2}\sum_{i=1}^m |f(t_{i-1}) - f(t_i)| \leq \frac{1}{M^2}\operatorname{var}(f,[a,b]),$$

which shows that $1/f$ has bounded variation on $[a,b]$.

Finally, assume that f is g-absolutely continuous on $[a,b]$ and let $\varepsilon > 0$. In that case, there exists $\delta > 0$ such that if $\{(a_n, b_n)\}_{n=1}^m$ is a family of open pairwise disjoint subintervals of $[a,b]$ satisfying that $\sum_{n=1}^m (g(b_n) - g(a_n)) < \delta$, then

$$\sum_{n=1}^m |f(b_n) - f(a_n)| < M^2\varepsilon.$$

Consequently, if $\{(a_n, b_n)\}_{n=1}^m$ is a family of open pairwise disjoint subintervals satisfying $\sum_{n=1}^m (g(b_n) - g(a_n)) < \delta$, using (2.3) we have

$$\sum_{n=1}^{m} \left| \frac{1}{f(b_n)} - \frac{1}{f(a_n)} \right| \leq \frac{1}{M^2} \sum_{n=1}^{m} |f(b_n) - f(a_n)| < \varepsilon,$$

which proves that $1/f \in \mathcal{AC}_g([a, b], \mathbb{R})$. □

As shown in [8, Proposition 5.5], every g-absolutely continuous function is g-continuous according to the following definition from [8].

Definition 2.8. A function $f : [a, b] \to \mathbb{R}$ is *g-continuous* at a point $t \in [a, b]$, or *continuous with respect to g* at t, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(t) - f(s)| < \varepsilon, \quad \text{for all } s \in [a, b],\ |g(t) - g(s)| < \delta.$$

If f is g-continuous at every point $t \in A \subset [a, b]$, we say that f is g-continuous on A.

The following result, [8, Proposition 3.2], describes some properties of g-continuous functions, and thus, of g-absolutely continuous functions.

Proposition 2.9. *If $f : [a, b] \to \mathbb{R}$ is g-continuous on $[a, b]$, then:*

- *f is continuous from the left at every $t \in (a, b]$;*
- *if g is continuous at $t \in [a, b)$, then so is f;*
- *if g is constant on some $[\alpha, \beta] \subset [a, b]$, then so is f.*

In particular, g-continuous functions on $[a, b]$ are continuous on $[a, b]$ when g is continuous on $[a, b)$.

Finally, we provide some context and information on differential problems with Stieltjes derivatives of the form

$$u'_g(t) = F(t, u(t)), \qquad u(t_0) = u_0, \tag{2.4}$$

with $t_0, T, u_0 \in \mathbb{R}$, $T > 0$, and $F : [t_0, t_0 + T] \times \mathbb{R} \to \mathbb{R}$. Let us start by clarifying the concept of solution for this type of equations.

Definition 2.10. Given $\tau \in (0, T]$, a *solution* of Eq. (2.4) on $[t_0, t_0 + \tau]$ is a function $u \in \mathcal{AC}_g([t_0, t_0 + \tau], \mathbb{R})$ such that $u(t_0) = u_0$ and

$$u'_g(t) = F(t, u(t)), \quad g\text{-a.a. } t \in [t_0, t_0 + \tau).$$

As usual, one of the most important equations in the context of Stieltjes derivatives is the linear differential equation, which has been deeply studied in [7,8,11]. In the following result, which can be found in [11, Theorem 3.2] or, more generally, in [7, Theorem 4.3], we introduce the g-exponential map, the unique solution of the homogeneous linear problem.

Theorem 2.11. *Let $p \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$ be such that*

$$1 + p(t)\Delta^+ g(t) \neq 0, \quad \textit{for all } t \in [t_0, t_0 + T) \cap D_g. \tag{2.5}$$

Then, the set $T_p^- = \{t \in [t_0, t_0+T) \cap D_g : 1 + p(t)\Delta^+ g(t) < 0\}$ *has finite cardinality. Furthermore, if* $T_p^- = \{t_1, \dots, t_k\}$*,* $t_0 \le t_1 < t_2 < \cdots < t_k < t_{k+1} = t_0 + T$*, then the map* $\widehat{p} : [t_0, t_0+T) \to \mathbb{R}$ *defined as*

$$\widehat{p}(t) = \begin{cases} p(t), & \text{if } t \in [t_0, t_0+T) \setminus D_g, \\ \dfrac{\log(1 + p(t)\Delta^+ g(t))}{\Delta^+ g(t)}, & \text{if } t \in [t_0, t_0+T) \cap D_g, \end{cases}$$

belongs to $\mathcal{L}_g^1([t_0, t_0+T), \mathbb{R})$*; the map* $\exp_g(p, \cdot) : [t_0, t_0+T] \to (0, +\infty)$ *given by*

$$\exp_g(p,t) = \begin{cases} \exp\left(\displaystyle\int_{[t_0,t)} \widehat{p}(s)\, \mathrm{d}\mu_g(s)\right), & \text{if } t_0 \le t \le t_1, \\ (-1)^j \exp\left(\displaystyle\int_{[t_0,t)} \widehat{p}(s)\, \mathrm{d}\mu_g(s)\right), & \text{if } t_j < t \le t_{j+1},\ j = 1, \dots, k, \end{cases}$$

is g-absolutely continuous on $[t_0, t_0+T]$*; and the function* $u(t) = u_0 \exp_g(p,t)$*,* $t \in [t_0, t_0+T]$*, is the unique solution of*

$$u_g'(t) = p(t)u(t), \quad u(t_0) = u_0.$$

Finally, in [11, Theorem 3.5] and [7, Proposition 4.12], using the method of variation of constants, the authors obtained the explicit expression of the unique solution of the nonhomogeneous linear equation, which we present in the next theorem.

Theorem 2.12. *Let* $p, f \in \mathcal{L}_g^1([t_0, t_0+T), \mathbb{R})$ *and suppose that* (2.5) *holds. Then, the function* $u : [t_0, t_0+T] \to \mathbb{R}$ *defined as*

$$u(t) = x_0 \exp_g(p,t) + \exp_g(p,t) \int_{[t_0,t)} \frac{f(s)}{1 + p(s)\Delta^+ g(s)} \exp_g(p,s)^{-1} \mathrm{d}\mu_g(s), \quad t \in [a,b], \tag{2.6}$$

is the unique solution of

$$u_g'(t) = p(t)u(t) + f(t), \quad u(t_0) = u_0. \tag{2.7}$$

3 The logistic equation in the context of Stieltjes derivatives

In the setting of ordinary differential equations and dynamic equations on time scales, one way of defining the logistic equation is to consider it as the equation for which a change of variables of the form $u(t) = (x(t))^{-1}$ yields a linear equation in the corresponding setting. Hence, following the reasonings in [5, Section 2.4], we will obtain the form of the logistic equation in the context of Stieltjes derivatives through the mentioned change of variables.

In what follows we assume that $x_0, t_0, T \in \mathbb{R}$, $T > 0$. Let us start by looking at the change of variables above. Suppose u is a function which is a solution of Eq. (2.7). If $u(t) = (x(t))^{-1}$, provided that the corresponding hypotheses are satisfied, we can compute the g-derivative of x using Proposition 2.3. Indeed, clearly, the function 1 is g-differentiable everywhere (except on C_g) and has null g-derivative so, under suitable conditions, (2.1) ensures that

$$\begin{aligned} x_g'(t) &= -\frac{u_g'(t)}{u(t)(u(t) + u_g'(t)\Delta^+ g(t))} = -\frac{p(t)u(t) + f(t)}{u(t)(u(t) + (p(t)u(t) + f(t))\Delta^+ g(t))} \\ &= -\frac{p(t) + f(t)(u(t))^{-1}}{1 + (p(t) + f(t)(u(t))^{-1})\Delta^+ g(t)} \frac{1}{u(t)} = -\frac{p(t) + f(t)x(t)}{1 + \Delta^+ g(t)(p(t) + f(t)x(t))} x(t). \end{aligned} \tag{3.1}$$

At this point, one might be inclined to define the logistic equation with Stieltjes derivatives on $[t_0, t_0 + T]$ as Eq. (3.1) as it is in the form of Eq. (2.4). However, in doing so, one needs to require that any solution on $[t_0, t_0 + \tau)$ in the sense of Definition 2.10 must also satisfy that $1 + \Delta^+ g(t)(p(t) + f(t)x(t)) \neq 0$ for every $t \in [t_0, t_0 + \tau) \cap D_g$. Alternatively, instead of Eq. (3.1), we can consider the more general equation

$$x'_g(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \tag{3.2}$$

which no longer requires such consideration at the cost of moving away from problems of the form (2.4). Observe that when the Stieltjes derivative coincides with the usual derivative (namely, when $g = \mathrm{Id}$), Eq. (3.2) yields the usual logistic equation.

After these considerations, we define the *logistic equation with Stieltjes derivatives* as the initial value problem

$$x'_g(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad x(t_0) = x_0, \tag{3.3}$$

with $p, f \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$. Naturally, since Eq. (3.3) is no longer in the framework of Eq. (2.4), we need to define the concept of solution for this problem in a similar manner.

Definition 3.1. Given $\tau \in (0, T]$, a *solution* of Eq. (3.3) on $[t_0, t_0 + \tau]$ is a function $x \in \mathcal{AC}_g([t_0, t_0 + \tau], \mathbb{R})$ such that $x(t_0) = x_0$ and

$$x'_g(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad g\text{-a.a. } t \in [t_0, t_0 + \tau).$$

Remark 3.2. Observe that, if $x_0 = 0$, the map $x(t) = 0$, $t \in [t_0, t_0 + T]$, is a solution of Eq. (3.3) so, without loss of generality, we shall assume that $x_0 \neq 0$ for the remaining of the section.

Remark 3.3. Remark 2.2 and Proposition 2.9 imply that, for any $x \in \mathcal{AC}_g([t_0, t_0 + \tau], \mathbb{R})$,

$$x'_g(t)\Delta^+ g(t) = x(t+) - x(t), \quad t \in [t_0, t_0 + \tau].$$

Hence, it is clear that x is a solution of Eq. (3.3) if and only if it is a solution of

$$x'_g(t) = -(p(t) + f(t)x(t))x(t+), \quad x(t_0) = x_0. \tag{3.4}$$

The following result provides an explicit expression for a solution of Eq. (3.3), which is obtained through the solution of the nonhomogeneous linear equation, Eq. (2.6).

Theorem 3.4. *Let $p, f \in \mathcal{L}^1_g([t_0, t_0 + T), \mathbb{R})$ be such that (2.5) holds and define*

$$\phi(t) = \frac{1}{x_0} + \int_{[t_0,t)} \frac{f(s)}{1 + p(s)\Delta^+ g(s)} \exp_g(p, s)^{-1} \mathrm{d}\mu_g(s), \quad t \in [t_0, t_0 + T).$$

If there exists $\tau \in (0, T]$ such that $\phi(t) \neq 0$ for $t \in [t_0, t_0 + \tau]$ and

$$\phi(t) \neq -\frac{f(t)\Delta^+ g(t)}{1 + p(t)\Delta^+ g(t)} \exp_g(p, t)^{-1}, \quad t \in [t_0, t_0 + \tau] \cap D_g, \tag{3.5}$$

then, the map $x : [t_0, t_0 + \tau] \to \mathbb{R}$ defined as

$$x(t) = \frac{1}{\exp_g(p, t)\phi(t)}, \quad t \in [t_0, t_0 + \tau] \tag{3.6}$$

is a solution of Eq. (3.3) on $[t_0, t_0 + \tau]$.

Proof. Let us denote
$$\Phi(t) = \exp_g(p,t)\phi(t), \quad t \in [t_0, t_0 + T).$$
Observe that Theorem 2.12 ensures that $\Phi \in \mathcal{AC}_g([t_0, t_0 + T], \mathbb{R})$. Also, there exists $N \subset [t_0, t_0 + \tau)$ such that $\mu_g(N) = 0$ and
$$\Phi'_g(t) = p(t)\Phi(t) + f(t), \quad t \in [t_0, t_0 + \tau) \setminus N.$$
Furthermore, for $t \in [t_0, t_0 + \tau)$, since $\phi(t) \neq 0$ by hypothesis and $\exp_g(p, t) \neq 0$ by definition, we have that $\Phi(t) \neq 0$, which ensures that x is well-defined. Hence, in order to prove that $x \in \mathcal{AC}_g([t_0, t_0 + T], \mathbb{R})$ it is enough to show that
$$\exists \lim_{s \to t^-} \Phi(s) \neq 0, \quad t \in (t_0, t_0 + \tau] \tag{3.7}$$
$$\exists \lim_{s \to t^+} \Phi(s) \neq 0, \quad t \in [t_0, t_0 + \tau) \tag{3.8}$$
as in that case, Lemma 2.7 ensures the g-absolute continuity.

Since $\Phi \in \mathcal{AC}_g([t_0, t_0 + T], \mathbb{R})$, Φ is left-continuous at every $t \in (t_0, t_0 + \tau]$ (see Proposition 2.9), so for each $t \in (t_0, t_0 + \tau]$, $\Phi(t-) = \Phi(t) \neq 0$, which proves (3.7). Similarly, if $t \in [t_0, t_0 + \tau) \setminus D_g$, Proposition 2.9 ensures that Φ is continuous at t, so $\Phi(t+) = \Phi(t) \neq 0$. Finally, if $t \in [t_0, t_0 + \tau) \cap D_g$, then $t \notin N$, so it follows from Remark 2.2 and (3.5) that
$$\begin{aligned}
\Phi(t+) &= \Phi(t) + \Phi'_g(t)\Delta^+ g(t) \\
&= \Phi(t) + (p(t)\Phi(t) + f(t))\Delta^+ g(t) \\
&= (1 + p(t)\Delta^+ g(t))\Phi(t) + f(t)\Delta^+ g(t) \\
&= (1 + p(t)\Delta^+ g(t)) \exp_g(p,t)\phi(t) + f(t)\Delta^+ g(t) \neq 0,
\end{aligned}$$
which shows that (3.8) holds.

Finally, we prove that x satisfies the equation g-a.e. in $[t_0, t_0 + \tau]$. Note that the reasoning above and the fact that $\Phi \neq 0$ ensure that $\Phi(t) + \Phi'_g(t)\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0 + \tau) \setminus N$. Hence, given that the map $h(t) = 1$, $t \in [t_0, t_0 + \tau)$, is g-differentiable on $[t_0, t_0 + \tau)$ with null g-derivative, Proposition 2.3 guarantees that x is g-differentiable for each $t \in [t_0, t_0 + \tau) \setminus N$ and
$$\begin{aligned}
x'_g(t) &= -\frac{\Phi'_g(t)}{\Phi(t)(\Phi(t) + \Phi'_g(t)\Delta^+ g(t))} \\
&= -\frac{p(t)\Phi(t) + f(t)}{\Phi(t)(1 + (p(t)\Phi(t) + f(t))\Delta^+ g(t))} = -\frac{p(t) + f(t)x(t)}{1 + (p(t) + f(t)x(t))\Delta^+ g(t)} x(t),
\end{aligned} \tag{3.9}$$
so we have that, for $t \in [t_0, t_0 + \tau) \setminus N$,
$$\begin{aligned}
&x'_g(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 \\
&\quad = -(p(t) + f(t)x(t))x(t) + p(t)x(t) + f(t)x(t)^2 = 0,
\end{aligned}$$
which finishes the proof. □

Remark 3.5. Let us briefly reflect on the conditions that we are requiring on the map ϕ in the hypotheses of Theorem 3.4. When we ask for ϕ to not vanish on the interval, we are essentially asking for the solution of the nonhomogeneous linear equation to be different from zero on the whole interval, which allows us to properly define the map x on that set. Observe that this condition is also necessary in the ODE setting. Condition (3.5), on the other hand, is a condition that is only relevant in this context (as $D_g = \emptyset$ when $g = \mathrm{Id}$) and it is equivalent to the requirements for the quotient rule in Proposition 2.3, guaranteeing that the derivative of x exists wherever the derivative of the solution of the nonhomogeneous linear equation exists.

A careful reader might have noticed that in the proof of Theorem 3.4, we obtained (3.9). In other words, we showed that the map x in (3.6) satisfies Eq. (3.1). This might be a bit surprising since Eq. (3.2) is the more general equation. However, as we show in the next result, under the assumption that (2.5) holds, Eq. (3.1) and Eq. (3.2) are equivalent problems in the sense that a solution of one of the problems is a solution of the other one.

Proposition 3.6. *Let $\tau \in (0, T]$ and assume that $1 + p(t)\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0 + \tau) \cap D_g$.*
If $x : [t_0, t_0 + \tau] \to \mathbb{R}$ is such that

$$x_g'(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad g\text{-a.a. } t \in [t_0, t_0 + \tau), \tag{3.10}$$

then, $1 + (p(t) + f(t)x(t))\Delta^+ g(t) \neq 0$ for g-a.a. $t \in [t_0, t_0 + \tau)$ and

$$x_g'(t) = -\frac{p(t) + f(t)x(t)}{1 + \Delta^+ g(t)(p(t) + f(t)x(t))} x(t), \quad g\text{-a.a. } t \in [t_0, t_0 + \tau). \tag{3.11}$$

Conversely, if $x : [t_0, t_0 + \tau] \to \mathbb{R}$ is such that (3.11) holds (in which case, we are implicitly assuming that $1 + (p(t) + f(t)x(t))\Delta^+ g(t) \neq 0$ for g-a.a. $t \in [t_0, t_0 + \tau)$), then x satisfies (3.10).

Proof. First, let $x : [t_0, t_0 + \tau] \to \mathbb{R}$ be such that (3.10) holds. In that case, there exists $N \subset [t_0, t_0 + \tau)$ such that $\mu_g(N) = 0$ and

$$x_g'(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad t \in [t_0, t_0 + \tau) \setminus N. \tag{3.12}$$

Let us first show that

$$1 + (p(t) + f(t)x(t))\Delta^+ g(t) \neq 0, \quad t \in [t_0, t_0 + \tau) \setminus N. \tag{3.13}$$

Observe that this is clear for $t \in [t_0, t_0 + \tau) \setminus (N \cup D_g)$ as $\Delta^+ g(t) = 0$ in that case. Thus, in order to prove (3.13) we need to show that $1 + (p(t) + f(t)x(t))\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0 + \tau) \cap D_g$.

Choose an arbitrary $t \in [t_0, t_0 + \tau) \cap D_g$ and suppose for the sake of contradiction that $1 + (p(t) + f(t)x(t))\Delta^+ g(t) = 0$. Then, since $t \in D_g$, we have $\Delta^+ g(t) > 0$, so we can write $p(t) + f(t)x(t) = -1/\Delta^+ g(t)$. In that case, (3.12) yields

$$0 = p(t)x(t) + f(t)x(t)^2 = (p(t) + f(t)x(t))x(t) = -\frac{x(t)}{\Delta^+ g(t)},$$

which means that $x(t) = 0$. Thus $0 = 1 + (p(t) + f(t)x(t))\Delta^+ g(t) = 1 + p(t)\Delta^+ g(t)$, which contradicts the assumption of the result. Thus, (3.13) must hold.

Now, (3.11) is a direct consequence of (3.12) and (3.13), which finishes the proof of the first part of the result. The second part of the result is trivial since we are implicitly assuming that $1 + (p(t) + f(t)x(t))\Delta^+ g(t) \neq 0$ for g-a.a. $t \in [t_0, t_0 + \tau)$. □

In [13, Section 3], the authors introduced the adjoint linear equation of Eq. (2.7) as the equation

$$y_g'(t) = -\frac{p(t)}{1 + p(t)\Delta^+ g(t)} y(t) + \frac{f(t)}{1 + p(t)\Delta^+ g(t)}, \qquad y(t_0) = y_0, \tag{3.14}$$

with $y_0 \in \mathbb{R}$ and $p, f \in \mathcal{L}^1_g([t_0, t_0+T], \mathbb{R})$ such that (2.5) holds. Observe that if we define

$$P(t) = -\frac{p(t)}{1+p(t)\Delta^+ g(t)}, \quad F(t) = \frac{f(t)}{1+p(t)\Delta^+ g(t)}, \quad t \in [t_0, t_0+T], \tag{3.15}$$

then Eq. (3.14) can be rewritten as

$$y'_g(t) = P(t)y(t) + F(t), \qquad y(t_0) = y_0,$$

i.e., it can be regarded as a particular case of Eq. (2.7) since [13, Lemma 3.4, statement (iii)] ensures that $P, F \in \mathcal{L}^1_g([t_0, t_0+T], \mathbb{R})$ and, furthermore,

$$1 + P(t)\Delta^+ g(t) = 1 - \frac{p(t)}{1+p(t)\Delta^+ g(t)}\Delta^+ g(t) = \frac{1}{1+p(t)\Delta^+ g(t)} \neq 0, \quad t \in [t_0, t_0+T) \cap D_g.$$

Hence, we have a logistic equation associated with Eq. (3.14), which is determined by

$$\begin{aligned}
0 &= y'_g(t)(1 + (P(t) + F(t)y(t))\Delta^+ g(t)) + P(t)y(t) + F(t)y(t)^2 \\
&= y'_g(t)\left(1 - \frac{p(t)\Delta^+ g(t)}{1+p(t)\Delta^+ g(t)} + \frac{f(t)\Delta^+ g(t)}{1+p(t)\Delta^+ g(t)}y(t)\right) - \frac{p(t)}{1+p(t)\Delta^+ g(t)}y(t) \\
&\quad + \frac{f(t)}{1+p(t)\Delta^+ g(t)}y(t)^2 \\
&= \frac{1}{1+p(t)\Delta^+ g(t)}(y'_g(t)(1 + \Delta^+ g(t) f(t) y(t)) - p(t)y(t) + f(t)y(t)^2).
\end{aligned}$$

Therefore, we define the *adjoint logistic equation with Stieltjes derivatives* – that is, the logistic equation associated with the adjoint equation (3.14) – as the initial value problem

$$y'_g(t)(1 + \Delta^+ g(t) f(t) y(t)) - p(t)y(t) + f(t)y(t)^2 = 0, \quad y(t_0) = y_0, \tag{3.16}$$

with $p, f \in \mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$ such that (3.13) holds. This equation turns out to be a much simpler version of Eq. (3.3).

Remark 3.7. In a similar fashion to Remark 3.3, we can see that Eq. (3.3) is equivalent to

$$y'_g(t) = (p(t) - f(t)y(t+))y(t), \quad y(t_0) = y_0. \tag{3.17}$$

As a direct consequence of Theorem 3.4, we have the following result providing an explicit solution for (3.16).

Theorem 3.8. *Let $p, f \in \mathcal{L}^1_g([t_0, t_0+T), \mathbb{R})$ be such that (2.5) holds and define*

$$\varphi(t) = \frac{1}{y_0} + \int_{[t_0,t)} f(s) \exp_g(p, s) \, d\mu_g(s), \quad t \in [t_0, t_0+T).$$

If there exists $\tau \in (0, T]$ such that $\varphi(t) \neq 0$ for $t \in [t_0, t_0+\tau]$ and

$$\varphi(t) \neq -f(t) \exp_g(p, t), \quad t \in [t_0, t_0+\tau] \cap D_g, \tag{3.18}$$

then, the map $y : [t_0, t_0+\tau] \to \mathbb{R}$ defined as

$$y(t) = \frac{\exp_g(p, t)}{\varphi(t)}, \quad t \in [t_0, t_0+\tau] \tag{3.19}$$

is a solution of Eq. (3.16) on $[t_0, t_0+\tau]$.

Proof. First, observe that, given that (3.13) holds, y is a solution of Eq. (3.16) if and only if x solves

$$y_g'(t)(1 + (P(t) + F(t)y(t))\Delta^+ g(t)) + P(t)y(t) + F(t)y(t)^2 = 0, \quad y(t_0) = y_0, \tag{3.20}$$

for P, F as in (3.15). Let us check that P, F satisfy the conditions of Theorem 3.4. Since we have already shown that $P, F \in \mathcal{L}_g^1([t_0, t_0 + T], \mathbb{R})$ and $1 + P(t)\Delta^+ g(t) \neq 0$, $t \in [t_0, t_0 + T) \cap D_g$, all that is left to do is check that the map ϕ in Theorem 3.4 satisfies the corresponding conditions under our hypotheses.

First, observe that

$$\frac{F(t)}{1 + P(t)\Delta^+ g(t)} = \frac{\dfrac{f(t)}{1 + p(t)\Delta^+ g(t)}}{1 - \dfrac{p(t)}{1 + p(t)\Delta^+ g(t)}\Delta^+ g(t)} = \frac{\dfrac{f(t)}{1 + p(t)\Delta^+ g(t)}}{\dfrac{1}{1 + p(t)\Delta^+ g(t)}\Delta^+ g(t)} = f(t)$$

for all $t \in [t_0, t_0 + T]$. Now, by definition,

$$\begin{aligned}
\phi(t) &= \frac{1}{y_0} + \int_{[t_0,t)} \frac{F(s)}{1 + P(s)\Delta^+ g(s)} \exp_g(P, s)^{-1} \mathrm{d}\mu_g(s) \\
&= \frac{1}{y_0} + \int_{[t_0,t)} f(s) \exp_g(p, s) \mathrm{d}\mu_g(s) = \varphi(t),
\end{aligned}$$

where we have used the identity $\exp_g(P, \cdot)^{-1} = \exp_g(p, \cdot)$, see [7, Proposition 4.6]. Therefore, $\Phi(t) = \varphi(t) \neq 0$ for $t \in [t_0, t_0 + \tau]$ and, using the identity $\exp_g(P, \cdot)^{-1} = \exp_g(p, \cdot)$ once again,

$$\phi(t) = \varphi(t) \neq -f(t) \exp_g(p, t) = -\frac{F(t)\Delta^+ g(t)}{1 + P(t)\Delta^+ g(t)} \exp_g(P, t)^{-1}, \quad t \in [t_0, t_0 + \tau] \cap D_g.$$

Therefore, ϕ satisfies the conditions in Theorem 3.4 so the map

$$y(t) = \frac{1}{\exp_g(P, t)\phi(t)} = \frac{\exp_g(p, t)}{\varphi(t)}, \quad t \in [t_0, t_0 + \tau],$$

is a solution of Eq. (3.20) and, thus, a solution of Eq. (3.16) as we wanted to show. □

Finally, note that it is possible to adapt Proposition 3.6 for (3.16) in a similar way to Theorem 3.8, which yields the following result. We leave the proof to the reader.

Proposition 3.9. *Let $\tau \in (0, T]$ and assume that $1 + p(t)\Delta^+ g(t) \neq 0$ for all $t \in [t_0, t_0 + \tau) \cap D_g$.*
If $y : [t_0, t_0 + \tau] \to \mathbb{R}$ is such that

$$y_g'(t)(1 + \Delta^+ g(t) f(t) y(t)) - p(t)y(t) + f(t)y(t)^2 = 0, \quad g\text{-a.a. } t \in [t_0, t_0 + \tau), \tag{3.21}$$

then, $1 + \Delta^+ g(t) f(t) y(t) \neq 0$ for g-a.a. $t \in [t_0, t_0 + \tau)$ and

$$y_g'(t) = \frac{p(t) - f(t)y(t)}{1 + \Delta^+ g(t) f(t) y(t)} y(t), \quad g\text{-a.a. } t \in [t_0, t_0 + \tau). \tag{3.22}$$

Conversely, if $y : [t_0, t_0 + \tau] \to \mathbb{R}$ is such that (3.22) holds (in which case, we are implicitly assuming that $1 + \Delta^+ g(t) f(t) y(t) \neq 0$ for g-a.a. $t \in [t_0, t_0 + \tau)$), then y satisfies (3.21).

4 Relations between Stieltjes differential equations and dynamic equations

Throughout this section, we assume that the reader is familiar with time scale calculus and dynamic equations. For more information on these topics, see [4,5].

Let $\mathbb{T}$ be a time scale, $t_0, t_0+T \in \mathbb{T}$, $T>0$, and denote $[t_0, t_0+T)_{\mathbb{T}} = [t_0, t_0+T) \cap \mathbb{T}$. The aim of this section is the study of possible relations between the logistic equation with Stieltjes derivatives, Eq. (3.3), and its corresponding counterpart in the context of dynamic equations as described in [5],

$$x^{\Delta}(t) = -(p(t)+f(t)x(t))x(\sigma(t)), \quad t \in [t_0, t_0+T)_{\mathbb{T}}, \tag{4.1}$$

where x^{Δ} denotes the Δ-derivative of x and $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator. Here, we assume that p and f are defined on the whole $[t_0, t_0+T)$ despite the fact that we only need them to be defined on $[t_0, t_0+T)_{\mathbb{T}}$ for Eq. (4.1). We do this so that we can easily compare Eq. (3.3) and Eq. (4.1). Similarly, we also want to consider the relations that might take place between the adjoint logistic equation, Eq. (3.16), and the corresponding logistic equation that can be deduced from the adjoint linear equation in [5], namely

$$y^{\Delta}(t) = (p(t)-f(t)y(\sigma(t)))y(t), \quad t \in [t_0, t_0+T)_{\mathbb{T}}. \tag{4.2}$$

In order to discuss the possible relations between the different logistic equations, we need to consider a context in which we can compare the two types of differential problems. In [8, Section 8.3] and [12, Section 3.3.3], it is shown that equations on time scales can be regarded as a particular case of Stieltjes differential equations when we consider the nondecreasing and left-continuous map $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(t) = \begin{cases} t_0, & t \le t_0, \\ \inf\{s \in \mathbb{T} : s \ge t\}, & t_0 < t \le t_0+T, \\ t_0+T, & t > t_0+T. \end{cases} \tag{4.3}$$

As pointed out in [8, Section 8.3], $g(t) = t$ for all $t \in [t_0, t_0+T)_{\mathbb{T}}$, from which it follows that

$$\Delta^+ g(t) = g(t+) - g(t) = \inf\{s \in \mathbb{T} : s > t\} - t = \sigma(t) - t = \mu(t), \quad t \in [t_0, t_0+T)_{\mathbb{T}}, \tag{4.4}$$

where $\mu : \mathbb{T} \to \mathbb{T}$ denotes the graininess function.

Theorems 3.49 and 3.51 in [12] establish the mentioned relation between Stieltjes differential problems and dynamic equations on time scales. Furthermore, a closer look at the proofs of these results shows that, in fact, the equivalence is between the Stieltjes derivative and the Δ-derivative. We gathered this information in the following result. Observe that, unlike [12, Theorem 3.49] we do not require continuity from the left at right-scattered points as such condition is always satisfied for Δ-differentiable maps, see [4, Theorem 1.16 (i)].

Theorem 4.1. *If $u : [t_0, t_0+T)_{\mathbb{T}} \to \mathbb{R}$ is Δ-differentiable for each $t \in [t_0, t_0+T)_{\mathbb{T}}$, then the map $\widetilde{u} = u \circ g$ for g as in (4.3) is g-differentiable for g-a.a. $t \in [t_0, t_0+T)$ and, furthermore,*

$$\widetilde{u}(t) = u(t), \quad \widetilde{u}'_g(t) = u^{\Delta}(t), \quad g\text{-a.a. } t \in [t_0, t_0+T).$$

Conversely, if $\widetilde{u} : [t_0, t_0+T] \to \mathbb{R}$ is a g-continuous function which is g-differentiable for each $t \in [t_0, t_0+T)_{\mathbb{T}}$, then $u = \widetilde{u}|_{[t_0, t_0+T]_{\mathbb{T}}}$ is Δ-differentiable on $[t_0, t_0+T)_{\mathbb{T}}$ and, furthermore,

$$u^{\Delta}(t) = \widetilde{u}'_g(t), \quad t \in [t_0, t_0+T)_{\mathbb{T}}.$$

Now, given Theorem 4.1, the equivalence between Eq. (3.3) and Eq. (4.1) should be clear. Indeed, if x satisfies Eq. (4.1), [4, Theorem 1.16 (iv)] ensures that

$$x^{\Delta}(t) = -(p(t) + f(t)x(t))(x(t) + \mu(t)x^{\Delta}(t)), \quad t \in [t_0, t_0 + T)_{\mathbb{T}},$$

or, equivalently,

$$x^{\Delta}(t)(1 + (p(t) + f(t)x(t))\mu(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad t \in [t_0, t_0 + T)_{\mathbb{T}}. \tag{4.5}$$

Hence, Theorem 4.1 ensures that if $\widetilde{x} = x \circ g$ with g as in (4.3), then for g-a.a. $t \in [t_0, t_0 + T)$,

$$\begin{aligned} 0 &= \widetilde{x}'_g(t)(1 + (p(t) + f(t)\widetilde{x}(t))\mu(t)) + p(t)\widetilde{x}(t) + f(t)\widetilde{x}(t)^2 \\ &= \widetilde{x}'_g(t)(1 + (p(t) + f(t)\widetilde{x}(t))\Delta^+ g(t)) + p(t)\widetilde{x}(t) + f(t)\widetilde{x}(t)^2, \end{aligned}$$

where the last equality follows from (4.4). Hence, $\widetilde{x}$ satisfies Eq. (3.3).

Conversely, if $\widetilde{x}$ is a g-continuous function satisfying Eq. (3.3), then $x = \widetilde{x}|_{[t_0, t_0+T]_{\mathbb{T}}}$ is such that

$$x^{\Delta}(t)(1 + (p(t) + f(t)x(t))\Delta^+ g(t)) + p(t)x(t) + f(t)x(t)^2 = 0, \quad t \in [t_0, t_0 + T)_{\mathbb{T}},$$

so, once again, given (4.4), we see that (4.5) holds. Now [4, Theorem 1.16 (iv)] is enough to guarantee that x satisfies Eq. (4.1).

The equivalence between (3.16) and (4.2) is done in an analogous manner and we leave it to the reader.

5 Applications to population models

Impulsive differential equations and equations on time scales can be regarded as particular cases of differential equations with Stieltjes derivatives, see [8, Section 8]. This fact was taken into account in [8, Section 9], where the authors showed that some real-life phenomena can be modelled in the context of Stieltjes calculus. Similarly, in [10, Sections 5 and 6], the authors used these relations to show that Stieltjes differential equations can be a better tool than ODEs for population models of species that exhibit very short periods of reproductions or are subject to dormant states in which the population size is unlikely to change in a noticeable manner. With these ideas in mind, and bearing the applications of the usual logistic equation for population models, we want to show that the logistic equations with Stieltjes derivative introduced above can be an adequate tool to describe the behavior of certain species.

During the winter and early spring months, the grizzly bears, like many other bears, enter a stupor stage, during which they reduce their activity as much as possible in order to survive that time of the year. This is possible because, in the months prior to the hibernation stage, they build a layer of fat that they will use to sustain themself during this dormant state. Naturally, this might cause a population of grizzly bears to compete for resources during the months leading to winter. Interestingly, the mating of the grizzly bear occurs during this period of time when the grizzly bear is preparing itself for the winter. However, the development of the embryos goes on hold until the hibernation stage, which eventually leads to the introduction of newborn cubs towards the end of the stupor stage.

We claim that a logistic equation with Stieltjes derivatives can be used to represent the evolution of a population of grizzly bears. To that end, we shall divide years into the four different seasons and we shall assume that one unit of time, denoted by t, represents a full season, which leads to the following classification of time intervals:

Season	Time intervals
Winter	$(4k, 4k+1]$, $k = 0, 1, 2, \ldots$
Spring	$(4k+1, 4k+2]$, $k = 0, 1, 2, \ldots$
Summer	$(4k+2, 4k+3]$, $k = 0, 1, 2, \ldots$
Fall	$(4k+3, 4k+4]$, $k = 0, 1, 2, \ldots$

With this notation, the intervals $(4k, 4k+\frac{3}{2}]$, $k = 0, 1, 2, \ldots$, represent the hibernation periods of the population and, for simplicity, we shall assume that the remaining times of the year, namely, $(4k+\frac{3}{2}, 4k+4]$, $k = 0, 1, 2, \ldots$, represent the period of time when the bears prepare for the next winter.

The next step is to select an adequate nondecreasing and left-continuous map $g : \mathbb{R} \to \mathbb{R}$ which reflects the behavior explained above, keeping in mind the information in [10, Section 5]: "[the map g] can be regarded as a time modulator. Discontinuities correspond to sudden changes ... while constancy intervals correspond to dormant states ... The greater the slope, the more influence the corresponding times have in the process". Hence, we would like the map g to exhibit the following properties:

(a) On intervals of the form $(4k, 4k+\frac{3}{2}]$, $k = 0, 1, 2, \ldots$, the map g should remain constant as during these times, the population is hibernating and, thus, very unlikely to change drastically.

(b) At times of the form $4k+\frac{3}{2}$, $k = 0, 1, 2, \ldots$, the map g should possess a jump discontinuity, representing the introduction of newborns into the population, which we shall assume to happen simultaneously so that they can be represented by impulses. The map g must be continuous everywhere else as there are no other sudden changes in the population.

(c) In the months directly after new individuals are born, g must have a greater slope as newborns are weaker and, therefore, the population size is more volatile. As time progresses, the slope of the function should flatten as new individuals get stronger. In the times immediately prior to the hibernation periods we would want g to have a less steep slope, representing the slowing down of the population as they approach their dormant state.

Since we will be assuming that the evolution of the population starts at $t = 0$, for simplicity, we shall assume that g is constant on $(-\infty, 0]$. Furthermore, given the cyclical nature of the previously described annual phenomena, we will assume that there exists $c \in \mathbb{R}$ such that

$$g(t) - g(t-4) = c, \quad t \geq 4. \tag{5.1}$$

Observe that, in particular, this implies that $\Delta^+ g(t) = \Delta^+ g(\frac{3}{2})$ for all $t \in D_g$.

An example of a map $g : \mathbb{R} \to \mathbb{R}$ satisfying conditions (a)–(c) and the extra assumptions is

$$g(t) = \begin{cases} 0, & t \in \left(-\infty, \frac{3}{2}\right], \\ 1 + 5\sin\left(\frac{\pi}{5}\left(t - \frac{3}{2}\right)\right), & t \in \left(\frac{3}{2}, 4\right], \end{cases} \tag{5.2}$$

and $g(t) = g(4) + g(t-4)$ for $t > 4$, see Figure 5.1.

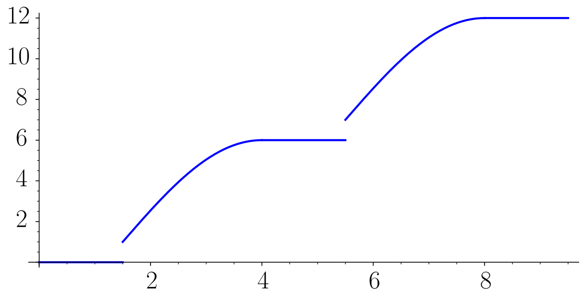

Figure 5.1: Graph of the map g in (5.2).

We now consider the initial value problem

$$x_g'(t) = F(t, x(t)), \quad x(0) = x_0, \tag{5.3}$$

where $x_0 > 0$ and $F : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is defined as

$$F(t,x) = \begin{cases} -\beta x, & \text{if } t \in \bigcup_{k=0}^{\infty} \left[4k, 4k + \dfrac{3}{2}\right), \ x \in \mathbb{R}, \\ \alpha x, & \text{if } t = 4k + \dfrac{3}{2}, k = 0, 1, 2, \ldots, \ x \in \mathbb{R}, \\ -\beta x\,(1 + \gamma x)\,, & \text{if } t \in \bigcup_{k=0}^{\infty} \left(4k + \dfrac{3}{2}, 4k + 4\right), \ x \in \mathbb{R}, \end{cases}$$

where $\beta > 0$ represents the death rate of the population; $\alpha > 0$, the reproduction rate; and $\gamma > 0$ represents the competition strength. Naturally, (5.3) only represents the evolution of a population as long as $x(t) \geq 0$, which will be the case for our solution as we will show later. Furthermore, observe that for $t \neq 4k + \frac{3}{2}$, $k = 0, 1, 2, \ldots$, and $x(t) > 0$, we have $x_g'(t) \leq 0$, which shows that the population is bound to decay over time; while $x_g'(t) \geq 0$ for $t = 4k + \frac{3}{2}$, $k = 0, 1, 2, \ldots$ and $x(t) > 0$, which is consistent with the fact that only new members of the population are introduced at such times. Furthermore, during intervals of the form $(4k + 3, 4k + 4]$, $k = 0, 1, 2, \ldots$, the population decays faster as the population increases. The competition term is not present in the equation on the intervals $(4k, 4k + \frac{3}{2}]$, $k = 0, 1, 2, \ldots$, as during hibernation, there is no competition for resources. Of course, given our choice of g, this is not relevant for $(4k, 4k + \frac{3}{2})$, $k = 0, 1, 2, \ldots$, as they belong to C_g and, thus, have measure zero. Nevertheless, for other choices of g this might be relevant.

Consider the maps $p, f : [0, +\infty) \to \mathbb{R}$ defined as

$$p(t) = \begin{cases} -\beta, & \text{if } t \neq 4k + \dfrac{3}{2}, \ k = 0, 1, 2, \ldots, \\ \alpha, & \text{if } t = 4k + \dfrac{3}{2}, \ k = 0, 1, 2, \ldots, \end{cases} \tag{5.4}$$

$$f(t) = \begin{cases} \beta\gamma, & \text{if } t \in \bigcup_{k=0}^{\infty} \left(4k + \dfrac{3}{2}, 4k + 4\right), \\ 0, & \text{otherwise.} \end{cases} \tag{5.5}$$

We claim that (5.3) can be rewritten as

$$x_g'(t)(1 + \Delta^+ g(t) f(t) x(t)) - p(t)x(t) + f(t)x(t)^2 = 0, \quad x(0) = x_0,$$

that is, it is an adjoint logistic equation with Stieltjes derivatives of the form (3.16). Indeed, given that $f(t)=0$ for $t\notin\bigcup_{k=0}^{\infty}(4k+\frac{3}{2},4k+4)$, it follows that

$$\begin{aligned}&x_g'(t)(1+\Delta^+g(t)f(t)x(t))-p(t)x(t)+f(t)x(t)^2\\&\quad=F(t,x(t))-p(t)x(t),\quad t\notin\bigcup_{k=0}^{\infty}\left(4k+\frac{3}{2},4k+4\right).\end{aligned}$$

Observe that if $t=4k+\frac{3}{2}, k=0,1,2,\dots$, then

$$F(t,x(t))-p(t)x(t)=\alpha x(t)-\alpha x(t)=0;$$

while for $t\in\bigcup_{k=0}^{\infty}\left[4k,4k+\frac{3}{2}\right)$,

$$F(t,x(t))-p(t)x(t)=-\beta x(t)-(-\beta)x(t)=0.$$

Now, if $t\in\bigcup_{k=0}^{\infty}(4k+\frac{3}{2},4k+4)$, then $t\notin D_g=\{4k+\frac{3}{2}:k=0,1,2,\dots\}$, so $\Delta^+g(t)=0$. Thus, for $t\in\bigcup_{k=0}^{\infty}(4k+\frac{3}{2},4k+4)$,

$$\begin{aligned}x_g'(t)(1+\Delta^+g(t)f(t)x(t))-p(t)x(t)+f(t)x(t)^2&=F(t,x(t))-p(t)x(t)+f(t)x(t)^2\\&=-\beta x(t)\left(1+\gamma x(t)\right)-(-\beta x(t))+\beta\gamma x(t)^2=0.\end{aligned}$$

Thus, we can apply Theorem 3.8 on an interval $[0,T]$, $T>0$, to obtain a solution of (5.3). To that end, we need to check that p and f in (5.4) and (5.5) satisfy the corresponding hypotheses.

Let $T>0$. First, observe that p and f are Borel-measurable maps which guarantees that they are g-measurable. Hence, since they are bounded, it follows that $p,f\in\mathcal{L}_g^1([0,T),\mathbb{R})$. Furthermore, observe that (2.5) holds since

$$1+p(t)\Delta^+g(t)=1+\alpha\Delta^+g(t)>0,\quad t\in[0,T]\cap D_g.$$

Observe that, in particular, this implies that $\exp_g(p,t)>0$ for all $t\in[0,T]$, see Theorem 2.11.

Consider

$$\varphi(t)=\frac{1}{x_0}+\int_{[0,t)}f(s)\exp_g(p,s)\,\mathrm{d}\mu_g(s),\quad t\in[0,T).$$

Given that $f(t)\ge 0$ for all $t\in[0,T)$, it follows that φ is nondecreasing. Therefore, $\varphi(t)\ge\varphi(0)=x_0^{-1}>0$ for all $t\in[0,T]$. In particular, this proves that $\varphi(t)\neq 0$ on $[0,T]$, which also shows that (3.18) holds since $f(t)=0$ for $t\in D_g$. Therefore, since the conditions of Theorem 3.8 are satisfied on the whole $[0,T]$, we know that the map

$$x(t)=\frac{\exp_g(p,t)}{\varphi(t)},\quad t\in[0,T],$$

is a solution of (5.3). Since $\exp_g(p,t),\varphi(t)>0$ for $t\in[0,T]$, it follows that $x(t)>0$ for all $t\in[0,T]$ as we claimed before. Given that Theorem 3.8 can be applied for each $T>0$, we can obtain a solution on $[0,+\infty)$. The following result provides a recursive expression for such map.

Theorem 5.1. *The solution of* (5.3) *on* $[0,+\infty)$ *given by* Theorem 3.8 *is the map* $x:[0,+\infty)\to\mathbb{R}$ *defined as* $x(0)=x_0$ *and, for* $k=0,1,2,\dots$,

$$x(t)=\begin{cases}x(4k), & 4k<t\le 4k+\dfrac{3}{2}\\[2ex] \dfrac{x(4k)\left(1+\widetilde{\alpha}\right)}{e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)}+x(4k)\gamma\left(1+\widetilde{\alpha}\right)\left(e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)}-1\right)}, & 4k+\dfrac{3}{2}<t\le 4(k+1),\end{cases}$$

with $\widetilde{\alpha}=\alpha\Delta^+g(\frac{3}{2})$.

Proof. First, observe that by definition, $\exp_g(p,0) = 1$ and $\varphi(0) = x_0^{-1}$, and so $x(0) = x_0$. Next, note that since g is constant on each interval of the form $[4k, 4k+\frac{3}{2}]$, $k = 0,1,2,\ldots$, and $\exp_g(p,\cdot), \varphi$ are g-absolutely continuous maps, they are also constant on the same interval, see Proposition 2.9. Therefore,

$$x(t) = \frac{\exp_g(p,t)}{\varphi(t)} = \frac{\exp_g(p,4k)}{\varphi(4k)} = x(4k), \quad t \in \left[4k, 4k+\frac{3}{2}\right],\ k = 0,1,2,\ldots.$$

Hence, all that is left to do is to show that, for $k = 0,1,2,\ldots,$

$$x(t) = \frac{x(4k)\,(1+\alpha_k)}{e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)} + x(4k)\gamma\,(1+\alpha_k)\left(e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)}-1\right)}, \quad t \in \left[4k+\frac{3}{2}, 4(k+1)\right].$$

Let $k \in \{0,1,2,\ldots\}$. Observe that, by definition, for $t \in \left(4k+\frac{3}{2}, 4(k+1)\right]$,

$$\begin{aligned}
\exp_g(p,t) &= \exp\left(\int_{[0,t)} \widehat{p}(s)\mathrm{d}\mu_g(s)\right) \\
&= \exp_g\left(p, 4k+\frac{3}{2}\right)\exp\left(\int_{[4k+\frac{3}{2},t)} \widehat{p}(s)\mathrm{d}\mu_g(s)\right) \\
&= \exp_g\left(p,4k\right)\exp\left(\int_{[4k+\frac{3}{2},t)} \widehat{p}(s)\mathrm{d}\mu_g(s)\right), \\
\varphi(t) &= \frac{1}{x_0} + \int_{[0,t)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s) \\
&= \varphi\left(4k+\frac{3}{2}\right) + \int_{[4k+\frac{3}{2},t)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s) \\
&= \varphi\left(4k\right) + \int_{[4k+\frac{3}{2},t)} f(s)\exp_g(p,s)\mathrm{d}\mu_g(s).
\end{aligned}$$

Now, for $t \in (4k+\frac{3}{2}, 4(k+1)]$,

$$\begin{aligned}
\int_{[4k+\frac{3}{2},t)} \widehat{p}(s)\mathrm{d}\mu_g(s) &= \int_{\{4k+\frac{3}{2}\}} \widehat{p}(s)\mathrm{d}\mu_g(s) + \int_{(4k+\frac{3}{2},t)} \widehat{p}(s)\mathrm{d}\mu_g(s) \\
&= \widehat{p}\left(4k+\frac{3}{2}\right)\Delta^+ g\left(4k+\frac{3}{2}\right) + \int_{(4k+\frac{3}{2},t)} p(s)\mathrm{d}\mu_g(s) \\
&= \log\left(1+p\left(4k+\frac{3}{2}\right)\Delta^+ g\left(4k+\frac{3}{2}\right)\right) - \int_{(4k+\frac{3}{2},t)} \beta\mathrm{d}\mu_g(s) \\
&= \log\left(1+\alpha\Delta^+ g\left(\frac{3}{2}\right)\right) + \beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right) \\
&= \log\left(1+\widetilde{\alpha}\right) + \beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right).
\end{aligned}$$

Hence, for $t \in (4k+\frac{3}{2}, 4(k+1)]$, we have

$$\exp\left(\int_{[4k+\frac{3}{2},t)} \widehat{p}(s)\mathrm{d}\mu_g(s)\right) = (1+\widetilde{\alpha})\,e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}.$$

On the other hand, for $t \in (4k+\frac{3}{2}, 4(k+1)]$, since $f(4k+\frac{3}{2})=0$, we have

$$\begin{aligned}
\int_{[4k+\frac{3}{2},t)} f(s)\exp_g(p,s)\,\mathrm{d}\mu_g(s) &= \int_{\left(4k+\frac{3}{2},t\right)} f(s)\exp_g(p,s)\,\mathrm{d}\mu_g(s) \\
&= \int_{\left(4k+\frac{3}{2},t\right)} \beta\gamma\exp_g(p,s)\,\mathrm{d}\mu_g(s) = -\gamma\int_{\left(4k+\frac{3}{2},t\right)} -\beta\exp_g(p,s)\,\mathrm{d}\mu_g(s) \\
&= -\gamma\int_{\left(4k+\frac{3}{2},t\right)} p(s)\exp_g(p,s)\,\mathrm{d}\mu_g(s) = -\gamma\int_{\left(4k+\frac{3}{2},t\right)} (\exp_g(p,\cdot))'_g(s)\,\mathrm{d}\mu_g(s).
\end{aligned}$$

Now, using the Fundamental Theorem of Calculus, Theorem 2.6, it follows that

$$\begin{aligned}
\int_{[4k+\frac{3}{2},t)} f(s)\exp_g(p,s)\,\mathrm{d}\mu_g(s) &= -\gamma\left(\exp_g(p,t) - \exp_g\left(p, 4k+\frac{3}{2}+\right)\right) \\
&= -\gamma\exp_g(p,4k)\,(1+\widetilde{\alpha})\left(e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}-1\right) \\
&= \gamma\exp_g(p,4k)\,(1+\widetilde{\alpha})\left(1-e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}\right).
\end{aligned}$$

Therefore, for $t \in (4k+\frac{3}{2}, 4(k+1)]$,

$$\begin{aligned}
x(t) &= \frac{\exp_g(p,4k)\exp\left(\int_{[4k+\frac{3}{2},t)} \widehat{p}(s)\,\mathrm{d}\mu_g(s)\right)}{\varphi(4k) + \int_{[4k+\frac{3}{2},t)} f(s)\exp_g(p,s)\,\mathrm{d}\mu_g(s)} \\
&= \frac{\exp_g(p,4k)\,(1+\widetilde{\alpha})\,e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}}{\varphi(4k)+\gamma\exp_g(p,4k)\,(1+\widetilde{\alpha})\left(1-e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}\right)} \\
&= \frac{\dfrac{\exp_g(p,4k)}{\varphi(4k)}\,(1+\widetilde{\alpha})\,e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}}{1+\dfrac{\exp_g(p,4k)}{\varphi(4k)}\gamma\,(1+\widetilde{\alpha})\left(1-e^{\beta\left(g\left(4k+\frac{3}{2}+\right)-g(t)\right)}\right)} \\
&= \frac{x(4k)\,(1+\widetilde{\alpha})}{e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)}+x(4k)\gamma\,(1+\widetilde{\alpha})\left(e^{\beta\left(g(t)-g\left(4k+\frac{3}{2}+\right)\right)}-1\right)},
\end{aligned}$$

as we needed to show. □

In Figure 5.2 we have plotted the solution above for different values of γ. Observe that the population presents the behavior we expected. Indeed, first note that the population remains constant during the hibernation periods. Furthermore, the population decays between generations, and the rate of this decay depends on the competition strength, γ. This can be easily observed by noting that $x(\frac{3}{2}+) = (1+\alpha)x_0 = \frac{17}{10}$ in all the graphs in Figure 5.2, however, the population levels at $t=4$ are lower for higher values of γ.

In order to study the asymptotic behavior of the solution of (5.3), we will look at the sequences $\{P_k\}_{k=0}^\infty = \{x(4k+\frac{3}{2})\}_{k=0}^\infty$ and $\{\widetilde{P}_k\}_{k=0}^\infty = \{x(4k+\frac{3}{2}+)\}_{k=0}^\infty$ representing the population at the end of the hibernation period and the population after newborns are introduced, respectively. Using the expression for x obtained in Theorem 5.1, we see that $\{P_k\}_{k=0}^\infty$ satisfies

$$P_0 = x_0, \quad P_{k+1} = \frac{P_k(1+\widetilde{\alpha})}{e^{\beta(g(4k)-g(4(k-1)+\frac{3}{2}+))}+P_k\gamma(1+\widetilde{\alpha})(e^{\beta(g(4k)-g(4(k-1)+\frac{3}{2}+)}-1)}, \quad k=0,1,\dots,$$

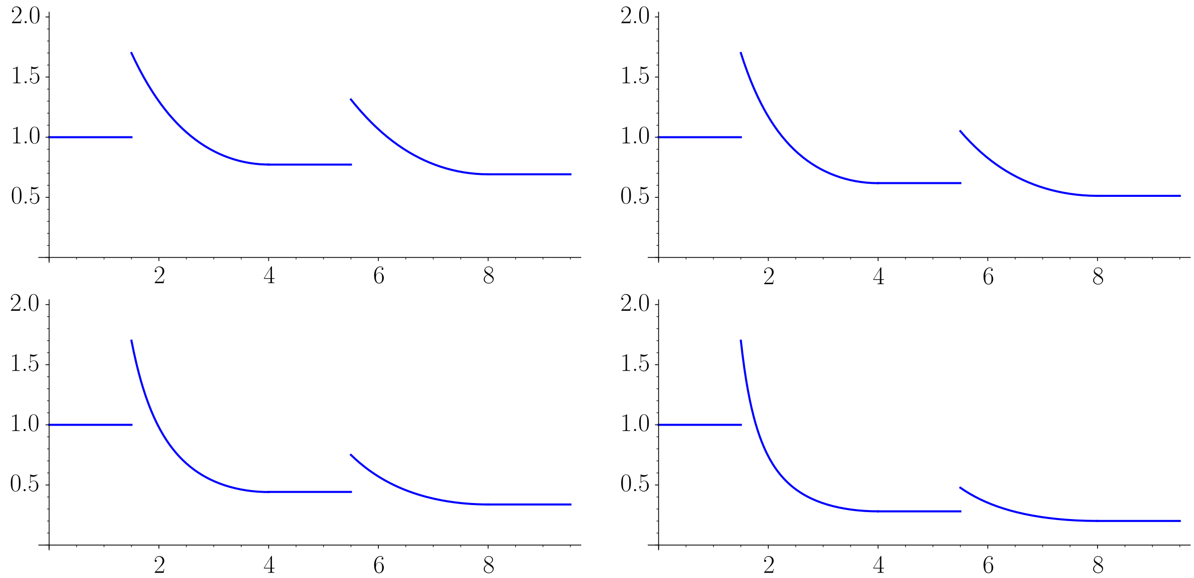

Figure 5.2: Graphs of the solution of (5.3) for g as in (5.2), $x_0 = 1$, $\alpha = \frac{7}{10}$, $\beta = \frac{1}{10}$ and different values of γ. In order, $\gamma = \frac{1}{2}$, $\gamma = 1$, $\gamma = 2$ and $\gamma = 4$.

which, thanks to (5.1), simplifies to

$$P_{k+1} = \frac{P_k(1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)}, \quad k = 0, 1, \ldots, \tag{5.6}$$

with $\widetilde{\beta} = \beta(g(4) - g(\frac{3}{2}+))$. Furthermore, $\widetilde{P}_k = (1+\widetilde{\alpha})P_k$ for $k = 0, 1, 2, \ldots$. Let us rewrite Eq. (5.6) in the form

$$P_{k+1} = H(P_k), \quad k = 0, 1, \ldots,$$

where

$$H(t) = \frac{t(1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + t\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)}, \quad t \in [0, \infty).$$

A simple calculation shows that the map H has, in general, two fixed points, namely zero and

$$L = \frac{1+\widetilde{\alpha} - e^{\widetilde{\beta}}}{\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)}.$$

The next result shows that the asymptotic behavior of the sequences $\{P_k\}_{k=0}^{\infty}$ and $\{\widetilde{P}_k\}_{k=0}^{\infty}$ (and therefore of the whole solution x) depends on whether L is positive (i.e., $e^{\widetilde{\beta}} < 1+\widetilde{\alpha}$) or nonpositive (i.e., $e^{\widetilde{\beta}} \geq 1+\widetilde{\alpha}$).

Theorem 5.2. *Denote* $\widetilde{\alpha} = \alpha\Delta^+ g(\frac{3}{2}) > 0$, $\widetilde{\beta} = \beta(g(4) - g(\frac{3}{2}+)) > 0$.

(a) *If* $e^{\widetilde{\beta}} \geq 1+\widetilde{\alpha}$, *the sequence* $\{P_k\}_{k=0}^{\infty}$ *is nonincreasing and converges to* 0. *As a consequence,* $\{\widetilde{P}_k\}_{k=0}^{\infty}$ *has the same behavior, and* $\lim_{t\to\infty} x(t) = 0$.

(b) *If* $e^{\widetilde{\beta}} < 1+\widetilde{\alpha}$, *we distinguish two cases:*

  (i) *If* $x_0 \geq L$, *the sequence* $\{P_k\}_{k=0}^{\infty}$ *is nonincreasing and converges to* L. *As a consequence,* $\{\widetilde{P}_k\}_{k=0}^{\infty}$ *is also nonincreasing and converges to* $(1+\widetilde{\alpha})L$.

(ii) *If* $x_0 \leq L$, *the sequence* $\{P_k\}_{k=0}^{\infty}$ *is nondecreasing and converges to* L. *As a consequence,* $\{\widetilde{P}_k\}_{k=0}^{\infty}$ *is also nondecreasing and converges to* $(1+\widetilde{\alpha})L$.

Proof. We shall only prove the result for $\{P_k\}_{k=0}^{\infty}$ as the properties for $\{\widetilde{P}_k\}_{k=0}^{\infty}$ follow from the relation $\widetilde{P}_k = (1+\widetilde{\alpha})P_k$ for $k = 0, 1, 2, \ldots$.

First, assume that $e^{\widetilde{\beta}} \geq 1+\widetilde{\alpha}$. Observe that, for $k = 0, 1, 2, \ldots$,

$$P_{k+1} = H(P_k) = \frac{P_k(1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} \leq P_k \frac{1+\widetilde{\alpha}}{e^{\widetilde{\beta}}} \leq P_k,$$

which proves that the sequence is nonincreasing. Furthermore, by definition, we have that $P_k > 0$ for $k = 0, 1, 2, \ldots$. Hence, the sequence $\{P_k\}_{k=0}^{\infty}$ is nonincreasing and bounded from below, so it is convergent. Since the only nonnegative fixed point of H is zero, it follows that $\{P_k\}_{k=0}^{\infty}$ converges to 0.

Next, we assume that $e^{\widetilde{\beta}} < 1+\widetilde{\alpha}$. Standard computations show that

$$H'(t) = \frac{e^{\widetilde{\beta}}(1+\widetilde{\alpha})}{(e^{\widetilde{\beta}} + t\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1))^2}, \quad t \geq 0,$$

so it follows that H is nondecreasing on $[0, +\infty)$. Recalling that $H(L) = L$ and $P_{k+1} = H(P_k)$, it follows that if $x_0 \geq L$, then $P_k \geq L$, $k = 0, 1, 2, \ldots$; and if $x_0 \leq L$, then $P_k \leq L$, $k = 0, 1, 2, \ldots$.

Now, suppose that $x_0 \geq L$. In that case, for $k = 0, 1, 2, \ldots$

$$\begin{aligned} P_k - P_{k+1} &= P_k - \frac{P_k(1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} = P_k \frac{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1) - (1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} \\ &\geq P_k \frac{e^{\widetilde{\beta}} + L\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1) - (1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} = P_k \frac{e^{\widetilde{\beta}} + 1 + \widetilde{\alpha} - e^{\widetilde{\beta}} - (1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} = 0. \end{aligned}$$

Hence, the sequence $\{P_k\}_{k=0}^{\infty}$ is nonincreasing and bounded from below by the unique positive fixed point L, so it is convergent to L.

On the other hand, if $x_0 \leq L$ then, for $k = 0, 1, 2, \ldots$,

$$P_k - P_{k+1} = P_k \frac{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1) - (1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} \leq P_k \frac{e^{\widetilde{\beta}} + L\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1) - (1+\widetilde{\alpha})}{e^{\widetilde{\beta}} + P_k\gamma(1+\widetilde{\alpha})(e^{\widetilde{\beta}}-1)} = 0$$

In this case, the sequence $\{P_k\}_{k=0}^{\infty}$ is nondecreasing and bounded from above by the unique positive fixed point L, so it is convergent to L. □

Remark 5.3. Observe that, if $e^{\widetilde{\beta}} < 1+\widetilde{\alpha}$ and $x_0 = L$, then the sequences $\{P_k\}_{k=0}^{\infty}$ and $\{\widetilde{P}_k\}_{k=0}^{\infty}$ are constant and equal to x_0 and $(1+\widetilde{\alpha})x_0$, respectively. Hence, it follows from Theorem 5.1 that the solution is 4-periodic in this case.

In Figure 5.3 we can observe the different asymptotic behaviors that we can expect from the solution of Eq. (5.3) as described by Theorem 5.2. In particular, we can see that when $e^{\widetilde{\beta}} \geq 1+\widetilde{\alpha}$ (i.e., when the death rate is high enough) the population is bound to extinction as presented in the first of the graphs. On the other hand, the second and third plot show that if $e^{\widetilde{\beta}} < 1+\widetilde{\alpha}$ (i.e., when the reproduction rate is high enough), we can expect the population to approach an equilibrium state corresponding to a 4-periodic solution shown in the fourth plot.

As a final note, observe that the example here provided is relatively simple. More complicated models can be obtained if we consider the parameters α, β and K to be functions instead, or if we relax the condition (5.1).

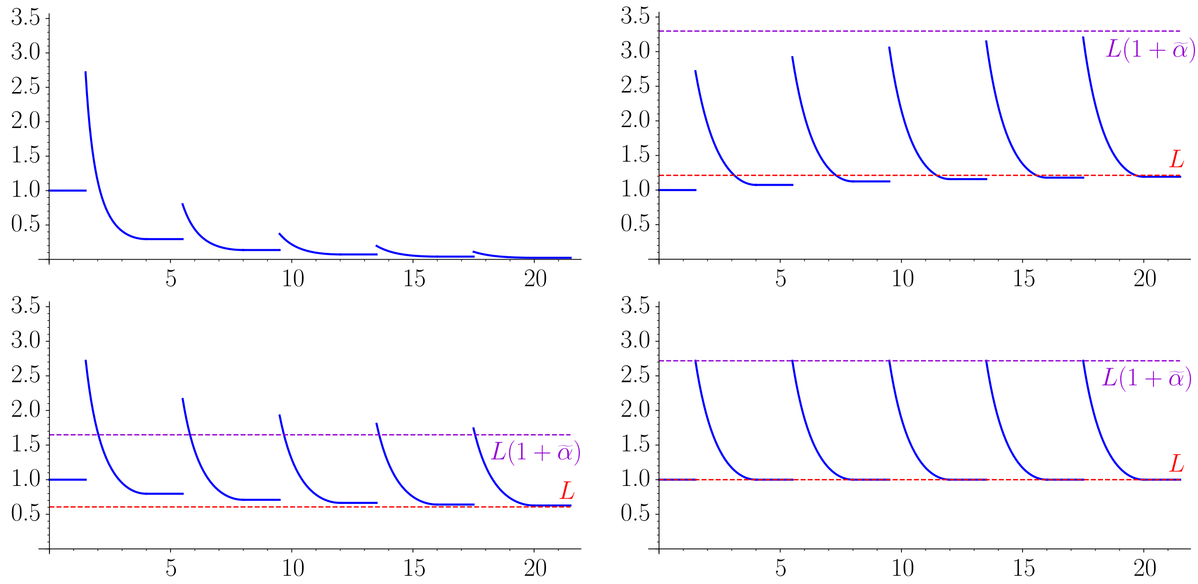

Figure 5.3: Graphs of the solution of (5.3) for g as in (5.2) showing the different asymptotic behaviors for $x_0 = 1$, $\alpha = e - 1$ and different values of the parameters (β, γ). In order, $(\frac{3}{10}, \frac{1}{2})$, $(\frac{1}{10}, \frac{1}{2})$, $(\frac{1}{10}, 1)$ and $(\frac{1}{10}, \frac{1}{\sqrt{e}})$.

6 Preliminaries on Stieltjes integrals

In the rest of the paper, we focus on the logistic equation in the context of Stieltjes integral equations. We will work with Kurzweil–Stieltjes integrals (also known as Perron–Stieltjes integrals), but we only need some basic properties of these integrals, which are summarized in the present section. A much more comprehensive treatment is available in [15]. Alternatively, it would be possible to work with the Young integral, which coincides with the Kurzweil–Stieltjes integral if the integrand and integrator are regulated and one of them has bounded variation (cf. [15, Theorem 6.13.1]).

We need the substitution theorem for the Kurzweil–Stieltjes integral (see [15, Theorem 6.6.1]).

Theorem 6.1. *Assume that $h : [a, b] \to \mathbb{R}$ is bounded and $f, g : [a, b] \to \mathbb{R}$ are such that $\int_a^b f \, \mathrm{d}g$ exists. Then*

$$\int_a^b h(t) \, \mathrm{d}\left(\int_a^t f(s) \, \mathrm{d}g(s)\right) = \int_a^b h(t) f(t) \, \mathrm{d}g(t),$$

whenever either side of the equation exists.

The next result describes the properties of indefinite Kurzweil–Stieltjes integrals (see [15, Corollary 6.5.5]).

Theorem 6.2. *Let $f, g : [a, b] \to \mathbb{R}$ be such that g is regulated and $\int_a^b f \, \mathrm{d}g$ exists. Then, for every $t_0 \in [a, b]$, the function*

$$h(t) = \int_{t_0}^t f \, \mathrm{d}g, \quad t \in [a, b]$$

is regulated and satisfies

$$h(t+) = h(t) + f(t)\Delta^+ g(t), \quad t \in [a, b),$$
$$h(t-) = h(t) - f(t)\Delta^- g(t), \quad t \in (a, b].$$

Moreover, if f is regulated and g has bounded variation, then h has bounded variation.

For the next result, see [15, Exercise 6.3.5].

Lemma 6.3. *If $f : [\alpha, \beta] \to \mathbb{R}$ is an arbitrary function and $g : [\alpha, \beta] \to \mathbb{R}$ is such that $g(t) = c$ for each $t \in (\alpha, \beta)$, then*

$$\int_\alpha^\beta f(t)\,\mathrm{d}g(t) = f(\beta)g(\beta) - f(\alpha)g(\alpha) - c(f(\beta) - f(\alpha)).$$

Our next goal is to obtain an integral version of the formula

$$\left(\frac{1}{g(t)}\right)' = -\frac{g'(t)}{g(t)^2}.$$

We begin with the case when g is a step function.

Lemma 6.4. *If $g : [a, b] \to \mathbb{R}$ is a step function, which is nonzero on $[a, b]$, then*

$$\int_a^b \mathrm{d}\left(\frac{1}{g(t)}\right) = \frac{1}{g(b)} - \frac{1}{g(a)} = -\int_a^b \frac{1}{g(t-)g(t+)}\,\mathrm{d}g(t),$$

with the convention that $g(a-) = g(a)$ and $g(b+) = g(b)$.

Proof. The first equality is obvious from the definition of the integral; let us verify the second one.

Since g is a step function, there exists a partition $a = \alpha_0 < \alpha_1 < \cdots < \alpha_m = b$ and constants $c_1, \ldots, c_m \in \mathbb{R}$ such that $g(t) = c_j$ for each $t \in (\alpha_{j-1}, \alpha_j)$. Let us also denote $c_0 = g(a)$, $c_{m+1} = g(b)$. Then $g(\alpha_{j-1}-) = c_{j-1}$ and $g(\alpha_{j-1}+) = c_j$ for each $j \in \{1, \ldots, m+1\}$. Applying Lemma 6.3 to each interval $[\alpha_{j-1}, \alpha_j]$, $j \in \{1, \ldots, m\}$, we calculate

$$\begin{aligned}
\int_a^b \frac{1}{g(t-)g(t+)}\,\mathrm{d}g(t) &= \sum_{j=1}^m c_j \left(\frac{1}{g(\alpha_{j-1}-)g(\alpha_{j-1}+)} - \frac{1}{g(\alpha_j-)g(\alpha_j+)}\right) \\
&\quad + \sum_{j=1}^m \left(\frac{g(\alpha_j)}{g(\alpha_j-)g(\alpha_j+)} - \frac{g(\alpha_{j-1})}{g(\alpha_{j-1}-)g(\alpha_{j-1}+)}\right) \\
&= \sum_{j=1}^m c_j \left(\frac{1}{c_{j-1}c_j} - \frac{1}{c_j c_{j+1}}\right) + \frac{g(b)}{g(b-)g(b+)} - \frac{g(a)}{g(a-)g(a+)} \\
&= \sum_{j=1}^m \frac{1}{c_{j-1}} - \sum_{j=1}^m \frac{1}{c_{j+1}} + \frac{1}{g(b-)} - \frac{1}{g(a+)} \\
&= \sum_{j=1}^m \frac{1}{c_{j-1}} - \sum_{j=1}^m \frac{1}{c_{j+1}} + \frac{1}{c_m} - \frac{1}{c_1} \\
&= \frac{1}{c_0} - \frac{1}{c_{m+1}} = \frac{1}{g(a)} - \frac{1}{g(b)}.
\end{aligned}$$

□

We now generalize Lemma 6.4 to functions of bounded variation.

Theorem 6.5. *If $g : [a, b] \to \mathbb{R}$ has bounded variation and for each $t \in [a, b]$, we have $g(t) \neq 0$, $g(t-) \neq 0$, and $g(t+) \neq 0$, then*

$$\int_a^b \mathrm{d}\left(\frac{1}{g(t)}\right) = \frac{1}{g(b)} - \frac{1}{g(a)} = -\int_a^b \frac{1}{g(t-)g(t+)}\,\mathrm{d}g(t),$$

with the convention that $g(a-) = g(a)$ and $g(b+) = g(b)$.

Proof. It suffices to prove the second equality. Since g has bounded variation, there exist nondecreasing functions $g^1, g^2 : [a,b] \to \mathbb{R}$ such that $g = g^1 - g^2$. Also, for each $i \in \{1,2\}$, there exists a sequence of nondecreasing step functions $\{g_n^i\}_{n=1}^{\infty}$ which is uniformly convergent to g^i. Without loss of generality, we can assume that these sequences are such that

$$g^i(a) \le g_n^i(a) \le g_n^i(b) \le g^i(b)$$

for all $n \in \mathbb{N}$ and $i \in \{1,2\}$. Therefore,

$$\operatorname{var}(g_n^i, [a,b]) = g_n^i(b) - g_n^i(a) \le g^i(b) - g^i(a), \quad n \in \mathbb{N}, \quad i \in \{1,2\}.$$

Consequently, by letting $g_n = g_n^1 - g_n^2$ for all $n \in \mathbb{N}$, we obtain a sequence of finite step functions $\{g_n\}_{n=1}^{\infty}$, which is uniformly convergent to g, and its members have uniformly bounded variation.

Let us again use the convention that $g_n(a-) = g_n(a)$ and $g_n(b+) = g_n(b)$ for each $n \in \mathbb{N}$. Note that $g_n(t-) \rightrightarrows g(t-)$ and $g_n(t+) \rightrightarrows g(t+)$ with respect to $t \in [a,b]$ (see [15, Lemma 4.2.3]).

Also, there exists an $M > 0$ such that

$$|g(t-)| \ge M, \quad t \in [a,b]$$

(apply Lemma 2.7 to $f(t) = g(t-)$). Hence, for sufficiently large $n \in \mathbb{N}$, we have

$$|g_n(t-)| \ge M/2, \quad t \in [a,b],$$

and therefore

$$\left| \frac{1}{g_n(t-)} - \frac{1}{g(t-)} \right| = \left| \frac{g(t-) - g_n(t-)}{g_n(t-)g(t-)} \right| \le \frac{2}{M^2} |g(t-) - g_n(t-)|,$$

which shows that $1/g_n(t-) \rightrightarrows 1/g(t-)$ with respect to $t \in [a,b]$. In a similar way, one can show that $1/g_n(t+) \rightrightarrows 1/g(t+)$ with respect to $t \in [a,b]$. Consequently,

$$\frac{1}{g_n(t-)g_n(t+)} \rightrightarrows \frac{1}{g(t-)g(t+)}$$

with respect to $t \in [a,b]$. Thus, we conclude that

$$\begin{aligned} \frac{1}{g(b)} - \frac{1}{g(a)} = \lim_{n\to\infty} \left(\frac{1}{g_n(b)} - \frac{1}{g_n(a)} \right) &= - \lim_{n\to\infty} \int_a^b \frac{1}{g_n(t-)g_n(t+)} \, \mathrm{d}g_n(t) \\ &= - \int_a^b \frac{1}{g(t-)g(t+)} \, \mathrm{d}g(t), \end{aligned}$$

where the second equality follows from Lemma 6.3 and the third from the uniform convergence theorem for integrals whose integrators have uniformly bounded variation (see [15, Theorem 6.8.8]). □

Once we have Theorem 6.5, it is not difficult to obtain the following integral version of the quotient rule, i.e., of the classical formula

$$\left(\frac{f(t)}{g(t)} \right)' = \frac{f'(t)}{g(t)} - \frac{f(t)g'(t)}{g(t)^2}.$$

Theorem 6.6. *If $f, g : [a,b] \to \mathbb{R}$ have bounded variation and for each $t \in [a,b]$, we have $g(t) \neq 0$, $g(t-) \neq 0$, and $g(t+) \neq 0$, then*

$$\int_a^b \mathrm{d}\left(\frac{f(t)}{g(t)}\right) = \frac{f(b)}{g(b)} - \frac{f(a)}{g(a)} = \int_a^b \frac{\mathrm{d}f(t)}{g(t+)} - \int_a^b \frac{f(t-)\,\mathrm{d}g(t)}{g(t-)g(t+)},$$

with the convention that $g(a-) = g(a)$ and $g(b+) = g(b)$.

Proof. It suffices to prove the second equality. Lemma 2.7 implies that $1/g$ has bounded variation. Using the integration by parts formula in the form presented in [13, Theorem B.6], we get

$$\int_a^b \frac{\mathrm{d}f(t)}{g(t+)} = \frac{f(b)}{g(b)} - \frac{f(a)}{g(a)} - \int_a^b f(t-)\,\mathrm{d}\left(\frac{1}{g(t)}\right).$$

The definition of the integral, Theorem 6.5 and Theorem 6.1 imply

$$\begin{aligned}\int_a^b f(t-)\,\mathrm{d}\left(\frac{1}{g(t)}\right) &= \int_a^b f(t-)\,\mathrm{d}\left(\frac{1}{g(t)} - \frac{1}{g(a)}\right) \\ &= -\int_a^b f(t-)\,\mathrm{d}\left(\int_a^t \frac{\mathrm{d}g(s)}{g(s-)g(s+)}\right) = -\int_a^b \frac{f(t-)\,\mathrm{d}g(t)}{g(t-)g(t+)},\end{aligned}$$

which completes the proof. □

Theorem 6.6 is not needed in the rest of this paper, but we hope it might be useful for subsequent research.

7 Stieltjes-integral versions of the logistic equation

We are now ready to deal with Stieltjes integral equations. In this section, we always assume that $g : [a,b] \to \mathbb{R}$ has bounded variation (left-continuity is no longer required). We begin with the linear nonhomogeneous equation

$$x(t) = x(t_0) + \int_{t_0}^t (p(s)x(s) + f(s))\,\mathrm{d}g(s), \quad t \in [a,b], \tag{7.1}$$

and try to obtain the corresponding logistic equation as an integral equation whose solution is the function $y(t) = x(t)^{-1}$.

Theorem 7.1. *Suppose that $g : [a,b] \to \mathbb{R}$ has bounded variation, $p : [a,b] \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ are regulated, and $x : [a,b] \to \mathbb{R}$ satisfies Eq. (7.1). If $x(t) \neq 0$, $x(t-) \neq 0$, and $x(t+) \neq 0$ for all $t \in [a,b]$, then the function $y(t) = x(t)^{-1}$ satisfies*

$$y(t) = y(t_0) - \int_{t_0}^t \frac{(p(s) + f(s)y(s))y(s)}{(1 - (p(s) + f(s)y(s))\Delta^- g(s))(1 + (p(s) + f(s)y(s))\Delta^+ g(s))}\,\mathrm{d}g(s) \tag{7.2}$$

for all $t \in [a,b]$, with the convention that $\Delta^+ g(s) = 0$ if $s = \max(t, t_0)$, and $\Delta^- g(s) = 0$ if $s = \min(t, t_0)$.

Proof. According to Theorem 6.5, we have

$$y(t) - y(t_0) = \frac{1}{x(t)} - \frac{1}{x(t_0)} = -\int_{t_0}^t \frac{1}{x(s-)x(s+)}\,\mathrm{d}x(s),$$

with the convention that $x(s-) = x(s)$ if $s = \min(t, t_0)$, and $x(s+) = x(s)$ if $s = \max(t, t_0)$. Using Eq. (7.1) and Theorem 6.1, we get

$$y(t) - y(t_0) = -\int_{t_0}^{t} \frac{p(s)x(s) + f(s)}{x(s-)x(s+)}\, \mathrm{d}g(s).$$

Theorem 6.2 yields

$$x(s+) = x(s) + (p(s)x(s) + f(s))\Delta^+ g(s) = x(s)(1 + (p(s) + f(s)y(s))\Delta^+ g(s)), \tag{7.3}$$
$$x(s-) = x(s) - (p(s)x(s) + f(s))\Delta^- g(s) = x(s)(1 - (p(s) + f(s)y(s))\Delta^- g(s)). \tag{7.4}$$

Therefore,

$$y(t) - y(t_0) = -\int_{t_0}^{t} \frac{(p(s) + f(s)y(s))y(s)}{(1 - (p(s) + f(s)y(s))\Delta^- g(s))(1 + (p(s) + f(s)y(s))\Delta^+ g(s))}\, \mathrm{d}g(s),$$

with the convention that $\Delta^+ g(s) = 0$ if $s = \max(t, t_0)$, and $\Delta^- g(s) = 0$ if $s = \min(t, t_0)$. □

Remark 7.2. Theorem 7.1 requires that $x(t) \neq 0$, $x(t-) \neq 0$, and $x(t+) \neq 0$ for all $t \in [a, b]$. The first condition is obviously necessary, for otherwise the definition of y would not make sense. If this condition is satisfied, then Eq. (7.3) and (7.4) show that the latter two conditions are equivalent to

$$1 + (p(t) + f(t)y(t))\Delta^+ g(t) \neq 0, \tag{7.5}$$
$$1 - (p(t) + f(t)y(t))\Delta^- g(t) \neq 0 \tag{7.6}$$

for all $t \in [a, b]$. Since these terms appear in the denominator on the right-hand side of the logistic equation, it is clear that the two conditions are necessary as well. Recalling that $y(t) = 1/x(t)$, we can rewrite the conditions (7.5) and (7.6) as

$$x(t) \neq -\frac{f(t)\Delta^+ g(t)}{1 + p(t)\Delta^+ g(t)}, \tag{7.7}$$
$$x(t) \neq \frac{f(t)\Delta^- g(t)}{1 - p(t)\Delta^- g(t)} \tag{7.8}$$

whenever the denominators are nonzero.

Remark 7.3. In the theory of Stieltjes differential equations, it is always assumed that g is a left-continuous nondecreasing function. In this case, Eq. (7.1) is the integral version of the Stieltjes differential equation $x_g'(t) = p(t)x(t) + f(t)$, and Eq. (7.2) simplifies to

$$y(t) = y(t_0) - \int_{t_0}^{t} \frac{(p(s) + f(s)y(s))y(s)}{1 + (p(s) + f(s)y(s))\Delta^+ g(s)}\, \mathrm{d}g(s), \quad t \in [a, b],$$

which is the integral version of the Stieltjes differential equation (3.1). Thus, we see that the form of the logistic equation (7.2) is consistent with the form obtained in Section 3. Condition (7.7) corresponds to the earlier condition (3.5), and condition (7.8) reduces to $x(t) \neq 0$.

Note that Eq. (7.1) is a special case of a generalized linear differential equation, whose solution can be explicitly expressed using the variation of constants formula (see e.g. [15, Theorems 7.8.4 and 7.8.5]). Thus, the reciprocal of this solution is a solution of the logistic equation given in Theorem 7.1.

Besides Eq. (7.1), one can also investigate the linear nonhomogeneous Stieltjes equations

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s-) + f(s))\, \mathrm{d}g(s), \quad t \in [a,b], \tag{7.9}$$

$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(s+) + f(s))\, \mathrm{d}g(s), \quad t \in [a,b], \tag{7.10}$$

which were studied in [13, 20], and which are dual to each other. Note that the one-sided limits $x(s-)$ and $x(s+)$ in the integrands have to be interpreted as $x(s)$ when s coincides with the lower or upper limit of the integral, respectively.

Starting with a solution x of Eq. (7.9) or Eq. (7.10), let us find the corresponding integral equation for the function $y(t) = x(t)^{-1}$. Interestingly, we will see that the resulting logistic equations are simpler than the logistic equation obtained in Theorem 7.1. We need the following modification of Theorem 6.1.

Lemma 7.4. *Assume that $g, h : [a,b] \to \mathbb{R}$ have bounded variation and $k, x : [a,b] \to \mathbb{R}$ are regulated.*

1. *If*

$$y(t) = \int_{t_0}^{t} k(s)x(s+)\, \mathrm{d}g(s), \quad t \in [a,b],$$

with the convention that $x(s+)$ means $x(s)$ if $s = \max(t, t_0)$, then for each $t \in [a,b]$, we have

$$\int_{t_0}^{t} h(s)\, \mathrm{d}y(s) = \int_{t_0}^{t} h(s)k(s)x(s+)\, \mathrm{d}g(s)$$

with the convention that $x(s+)$ means $x(s)$ if $s = \max(t, t_0)$.

2. *If*

$$y(t) = \int_{t_0}^{t} k(s)x(s-)\, \mathrm{d}g(s), \quad t \in [a,b],$$

with the convention that $x(s-)$ means $x(s)$ if $s = \min(t, t_0)$, then for each $t \in [a,b]$, we have

$$\int_{t_0}^{t} h(s)\, \mathrm{d}y(s) = \int_{t_0}^{t} h(s)k(s)x(s-)\, \mathrm{d}g(s)$$

with the convention that $x(s-)$ means $x(s)$ if $s = \min(t, t_0)$.

Proof. Let us prove the first statement. We will use the symbol χ_A to denote the characteristic (indicator) function of a set $A \subset \mathbb{R}$. Suppose first that $t > t_0$. Using Theorem 6.1 and the formula $\int_{t_0}^{t} p(s)\chi_{\{t\}}(s)\, \mathrm{d}q(s) = p(t)\Delta^{-}q(t)$, which holds for each $t > t_0$ and all functions $p, q : [a,b] \to \mathbb{R}$, we get

$$\begin{aligned}
\int_{t_0}^{t} h(s)\, \mathrm{d}y(s) &= \int_{t_0}^{t} h(s)\, \mathrm{d}\left(\int_{t_0}^{s} k(\tau)(x(\tau+)\chi_{[t_0,s)}(\tau) + x(\tau)\chi_{\{s\}}(\tau))\, \mathrm{d}g(\tau)\right) \\
&= \int_{t_0}^{t} h(s)\, \mathrm{d}\left(\int_{t_0}^{s} k(\tau)x(\tau+)\, \mathrm{d}g(\tau)\right) - \int_{t_0}^{t} h(s)\, \mathrm{d}\left(\int_{t_0}^{s} k(\tau)\chi_{\{s\}}(\tau)\Delta^{+}x(\tau)\, \mathrm{d}g(\tau)\right) \\
&= \int_{t_0}^{t} h(s)k(s)x(s+)\, \mathrm{d}g(s) - \int_{t_0}^{t} h(s)\, \mathrm{d}\left(\chi_{(t_0,t]}(s)k(s)\Delta^{+}x(s)\Delta^{-}g(s)\right) \\
&= \int_{t_0}^{t} h(s)k(s)(x(s+)\chi_{[t_0,t)}(s) + x(s)\chi_{\{t\}}(s))\, \mathrm{d}g(s) \\
&\quad + \int_{t_0}^{t} h(s)k(s)\Delta^{+}x(s)\chi_{\{t\}}(s)\, \mathrm{d}g(s) - \int_{t_0}^{t} h(s)\, \mathrm{d}\left(\chi_{(t_0,t]}(s)k(s)\Delta^{+}x(s)\Delta^{-}g(s)\right).
\end{aligned}$$

The last two integrals cancel each other out, since both have the value $h(t)k(t)\Delta^+x(t)\Delta^-g(t)$; for the latter integral, this follows from [15, Lemma 6.3.16] (note that the integrand has bounded variation, the integrator is regulated and vanishes in all points with at most countably many exceptions). This settles the case $t > t_0$. Similarly, if $t < t_0$, we have

$$\begin{aligned}
\int_{t_0}^t h(s)\,\mathrm{d}y(s) &= \int_{t_0}^t h(s)\,\mathrm{d}\left(\int_{t_0}^s k(\tau)(x(\tau+)\chi_{[s,t_0)}(\tau) + x(\tau)\chi_{\{t_0\}}(\tau))\,\mathrm{d}g(\tau)\right)\\
&= \int_{t_0}^t h(s)\,\mathrm{d}\left(\int_{t_0}^s k(\tau)x(\tau+)\,\mathrm{d}g(\tau)\right) + \int_{t_0}^t h(s)\,\mathrm{d}\left(\int_s^{t_0} k(\tau)\chi_{\{t_0\}}(\tau)\Delta^+x(\tau)\,\mathrm{d}g(\tau)\right)\\
&= \int_{t_0}^t h(s)k(s)x(s+)\,\mathrm{d}g(s) + \int_{t_0}^t h(s)\,\mathrm{d}\left(\chi_{[t,t_0)}(s)k(t_0)\Delta^+x(t_0)\Delta^-g(t_0)\right)\\
&= \int_{t_0}^t h(s)k(s)(x(s+)\chi_{[t,t_0)}(s) + x(s)\chi_{\{t_0\}}(s))\,\mathrm{d}g(s)\\
&\quad - \int_t^{t_0} h(s)k(s)\Delta^+x(s)\chi_{\{t_0\}}(s)\,\mathrm{d}g(s) - \int_t^{t_0} h(s)\,\mathrm{d}\left(\chi_{[t,t_0)}(s)k(t_0)\Delta^+x(t_0)\Delta^-g(t_0)\right).
\end{aligned}$$

The last two integrals cancel each other out, since the former equals $h(t_0)k(t_0)\Delta^+x(t_0)\Delta^-g(t_0)$, while the latter has the opposite value. This completes the proof of the first statement.

The second statement can be proved in a similar way. □

We can now obtain the logistic equations corresponding to Eq. (7.9) and Eq. (7.10).

Theorem 7.5. *Suppose that $g : [a,b] \to \mathbb{R}$ has bounded variation, $p : [a,b] \to \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ are regulated.*

1. *Suppose that $x : [a,b] \to \mathbb{R}$ satisfies Eq. (7.9). If $x(t) \neq 0$, $x(t-) \neq 0$, and $x(t+) \neq 0$ for all $t \in [a,b]$, then the function $y(t) = x(t)^{-1}$ satisfies*
$$y(t) = y(t_0) - \int_{t_0}^t (p(s) + f(s)y(s-))y(s+)\,\mathrm{d}g(s), \quad t \in [a,b]. \tag{7.11}$$
2. *Suppose that $x : [a,b] \to \mathbb{R}$ satisfies Eq. (7.10). If $x(t) \neq 0$, $x(t-) \neq 0$, and $x(t+) \neq 0$ for all $t \in [a,b]$, then the function $y(t) = x(t)^{-1}$ satisfies*
$$y(t) = y(t_0) - \int_{t_0}^t (-p(s) + f(s)y(s+))y(s-)\,\mathrm{d}g(s), \quad t \in [a,b]. \tag{7.12}$$

In both cases, $y(s-)$ or $y(s+)$ in the integrands should be understood as $y(s)$ when s coincides with the lower or upper limit of the integral, respectively.

Proof. Let us prove the first statement. According to Theorem 6.5 and Eq. (7.9), we have

$$\begin{aligned}
y(t) - y(t_0) = \frac{1}{x(t)} - \frac{1}{x(t_0)} &= -\int_{t_0}^t \frac{\mathrm{d}x(s)}{x(s-)x(s+)}\\
&= -\int_{t_0}^t \frac{1}{x(s-)x(s+)}\,\mathrm{d}\left(\int_{t_0}^t (p(s)x(s-) + f(s))\,\mathrm{d}g(s)\right).
\end{aligned}$$

Note that Eq. (7.9) implies that x has bounded variation, and by Lemma 2.7, the function $1/x$ has the same property. Hence, the functions $s \mapsto 1/x(s-)$ and $s \mapsto 1/x(s+)$ as well as their product have bounded variation. Using Lemma 7.4 and Theorem 6.1, we get

$$y(t) - y(t_0) = -\int_{t_0}^t \frac{p(s)x(s-) + f(s)}{x(s-)x(s+)}\,\mathrm{d}g(s) = -\int_{t_0}^t (p(s) + f(s)y(s-))y(s+)\,\mathrm{d}g(s),$$

where the second equality follows from the fact that $x(s+)^{-1} = y(s+)$ and $x(s-)^{-1} = y(s-)$.

The proof of the second statement is similar. $\square$

Remark 7.6. If g is left-continuous, then a function x satisfying (7.9) or (7.10) is also left-continuous, i.e., $x(t-) = x(t)$ for all t. In this case, Eq. (7.9) coincides with Eq. (7.1), i.e., we have the following pair of equations:

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + f(s))\,\mathrm{d}g(s), \quad t \in [a,b], \tag{7.13}$$

$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(s+) + f(s))\,\mathrm{d}g(s), \quad t \in [a,b]. \tag{7.14}$$

According to Theorem 7.5, if $x(t) \neq 0$ and $x(t+) \neq 0$ for all t, then $y = 1/x$ satisfies one of the following equations:

$$y(t) = y(t_0) - \int_{t_0}^{t} (p(s) + f(s)y(s))y(s+)\,\mathrm{d}g(s), \quad t \in [a,b], \tag{7.15}$$

$$y(t) = y(t_0) - \int_{t_0}^{t} (-p(s) + f(s)y(s+))y(s)\,\mathrm{d}g(s), \quad t \in [a,b]. \tag{7.16}$$

These are integral versions of the Stieltjes differential equations of Eq. (3.4) and Eq. (3.17), respectively, which are equivalent to the two logistic equations presented in Section 3.

Remark 7.7. General solution formulas for Eq. (7.9) and (7.10) were recently published in [20]. They resemble the well-known variation of constants formula, and involve solutions of the corresponding homogenous Stieltjes integral equations. According to Theorem 7.5, explicit solutions of Eq. (7.9) and (7.10) immediately give rise to explicit solution formulas for the two versions of the logistic equation.

Remark 7.8. Theorem 7.5 again requires that $x(t) \neq 0$, $x(t-) \neq 0$, and $x(t+) \neq 0$ for all $t \in [a,b]$. The first condition is obviously necessary, for otherwise the definition of y would not make sense. Let us have a closer look on the latter two conditions, trying to avoid $x(t-)$ and $x(t+)$, and express both conditions in terms of $x(t)$.

Suppose first that $x : [a,b] \to \mathbb{R}$ satisfies Eq. (7.9). Using the properties of the Kurzweil–Stieltjes integral and performing similar calculations as in the proof of [13, Lemma 6.5] (which corresponds to the homogenous case $f = 0$), we find that

$$x(t-)(1 + p(t)\Delta g(t)) = x(t)(1 + p(t)\Delta^+ g(t)) - f(t)\Delta^- g(t), \qquad t \in (a, t_0), \tag{7.17}$$

$$x(t-)(1 + p(t)\Delta^- g(t)) = x(t) - f(t)\Delta^- g(t), \qquad t \in [t_0, b], \tag{7.18}$$

$$x(t+) = x(t)(1 + p(t)\Delta^+ g(t)) + f(t)\Delta^+ g(t), \qquad t \in [a, t_0], \tag{7.19}$$

$$x(t+) = x(t) + x(t-)p(t)\Delta^+ g(t) + f(t)\Delta^+ g(t), \quad t \in (t_0, b). \tag{7.20}$$

First, we deal with $x(t+)$. Taking $t \in [a, t_0]$, Eq. (7.19) implies that $x(t+) \neq 0$ if and only if $x(t)(1 + p(t)\Delta^+ g(t)) + f(t)\Delta^+ g(t) \neq 0$; assuming that $1 + p(t)\Delta^+ g(t) \neq 0$, this is equivalent to

$$x(t) \neq -\frac{f(t)\Delta^+ g(t)}{1 + p(t)\Delta^+ g(t)}, \quad t \in [a, t_0]. \tag{7.21}$$

For $t \in (t_0, b)$, if $1 + p(t)\Delta^- g(t) \neq 0$, we can express $x(t-)$ from Eq. (7.18) and substitute to Eq. (7.20) to obtain

$$x(t+) = x(t) + \frac{x(t) - f(t)\Delta^- g(t)}{1 + p(t)\Delta^- g(t)} p(t)\Delta^+ g(t) + f(t)\Delta^+ g(t), \quad t \in [a, t_0].$$

Hence, to ensure that $x(t+) \neq 0$, we need

$$x(t)(1+p(t)\Delta^- g(t)) + (x(t) - f(t)\Delta^- g(t))p(t)\Delta^+ g(t) + f(t)\Delta^+ g(t)(1+p(t)\Delta^- g(t)) \neq 0,$$

which simplifies to

$$x(t)(1+p(t)\Delta g(t)) + f(t)\Delta^+ g(t) \neq 0,$$

and if $1+p(t)\Delta g(t) \neq 0$, this is equivalent to

$$x(t) \neq -\frac{f(t)\Delta^+ g(t)}{1+p(t)\Delta g(t)}, \quad t \in (t_0, b). \tag{7.22}$$

Next, we focus on $x(t-)$. If $t \in (a, t_0)$ and $1+p(t)\Delta g(t) \neq 0$, then Eq. (7.17) implies that $x(t-) \neq 0$ if and only if

$$x(t)(1+p(t)\Delta^+ g(t)) - f(t)\Delta^- g(t) \neq 0,$$

and if $1+p(t)\Delta^+ g(t) \neq 0$, this is equivalent to

$$x(t) \neq \frac{f(t)\Delta^- g(t)}{1+p(t)\Delta^+ g(t)}, \quad t \in (a, t_0). \tag{7.23}$$

Similarly, if $t \in [t_0, b]$ and $1+p(t)\Delta^- g(t) \neq 0$, then Eq. (7.18) implies that $x(t-) \neq 0$ if and only if

$$x(t) - f(t)\Delta^- g(t) \neq 0,$$

or equivalently

$$x(t) \neq f(t)\Delta^- g(t), \quad t \in [t_0, b]. \tag{7.24}$$

Thus, we have shown how to reformulate the conditions $x(t+) \neq 0$ and $x(t-) \neq 0$ in terms of $x(t)$. Note that if g is left-continuous, then the conditions in (7.22) and (7.21) coincide, and the conditions in (7.23) and (7.24) reduce to $x(t) \neq 0$.

A similar analysis can be performed for Eq. (7.10). However, it is easier to observe that $x : [a, b] \to \mathbb{R}$ satisfies

$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(s+) + f(s))\,\mathrm{d}g(s), \quad t \in [a, b],$$

if and only if the function $y : [-b, -a] \to \mathbb{R}$ given by $y(t) = x(-t)$ satisfies

$$y(t) = y(-t_0) + \int_{-t_0}^{t} (\tilde{p}(s)y(s-) + \tilde{f}(s))\,\mathrm{d}\tilde{g}(s), \quad t \in [-b, -a],$$

where $\tilde{p}(s) = p(-s)$, $\tilde{f}(s) = -f(-s)$, $\tilde{g}(s) = -g(-s)$. The proof of the fact is similar to the proof in [13, Remark 6.4] (which corresponds to the case $f = 0$). Notice that we have $x(t+) = y(-t-)$, $x(t-) = y(-t+)$, $\Delta^+ g(t) = \Delta^- g(-t)$, and $\Delta^- g(t) = \Delta^+ g(-t)$. Using these relations, it is clear that the conditions guaranteeing that $x(t+) \neq 0$ and $x(t-) \neq 0$ for Eq. (7.10) can obtained from the conditions derived earlier for Eq. (7.9) by interchanging $\Delta^+ g$ and $\Delta^- g$, f and $-f$, and a and b. In this way, we obtain the following counterparts to conditions (7.21)–(7.24):

$$x(t) \neq \frac{f(t)\Delta^- g(t)}{1+p(t)\Delta^- g(t)}, \quad t \in [t_0, b], \tag{7.25}$$

$$x(t) \neq \frac{f(t)\Delta^- g(t)}{1+p(t)\Delta g(t)}, \quad t \in (a, t_0), \tag{7.26}$$

$$x(t) \neq -\frac{f(t)\Delta^+ g(t)}{1+p(t)\Delta^- g(t)}, \quad t \in (t_0, b), \tag{7.27}$$

$$x(t) \neq -f(t)\Delta^+ g(t), \quad t \in [a, t_0]. \tag{7.28}$$

8 Relations between Stieltjes integral equations and dynamic equations

It has been known for a long time that dynamic equations on time scales represent a special case of Stieltjes integral equations (also known as measure differential equations), see [19]. Hence, it is interesting to check whether the logistic equations obtained in the previous section are consistent with logistic dynamic equations on time scales. In comparison with Section 4, we will discuss both Δ- and ∇-dynamic equations.

Let $\mathbb{T}$ be a time scale. It is convenient to work with a fixed time scale interval $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$, where $a, b \in \mathbb{T}$, $a < b$. We need the functions

$$g(t) = \inf\{s \in \mathbb{T} : s \geq t\}, \quad t \in [a,b], \tag{8.1}$$

$$h(t) = \sup\{s \in \mathbb{T} : s \leq t\}, \quad t \in [a,b]. \tag{8.2}$$

The function g is left-continuous, and h is right-continuous.

The relations between Stieltjes integral equations and dynamic equations are described in [15, Section 8.7]. They are based on the following relation (see [15, Corollary 8.6.9]) between Kurzweil–Stieltjes integrals and Henstock–Kurzweil Δ- and ∇-integrals, which were introduced in [16].

Theorem 8.1. *Consider a function $f : [a,b] \to \mathbb{R}$. Then the following statements hold:*

1. *The Henstock–Kurzweil Δ-integral $\int_a^b f(t)\,\Delta t$ exists if and only if the Kurzweil–Stieltjes integral $\int_a^b f(t)\,\mathrm{d}g(t)$ exists; in this case, both integrals have the same value.*
2. *The Henstock-Kurzweil ∇-integral $\int_a^b f(t)\,\nabla t$ exists if and only if the Kurzweil-Stieltjes integral $\int_a^b f(t)\,\mathrm{d}h(t)$ exists; in this case, both integrals have the same value.*

Hence, Δ-dynamic equations on time scales are special cases of Stieltjes integral equations with the integrator g given by Eq. (8.1). In particular, the Δ-dynamic equation

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(s) + f(s))\Delta s$$

is a special case of Eq. (7.9); note that a solution x of Eq. (7.9) satisfies $x(s-) = x(s)$ for all s, because g is left-continuous, and therefore x has the same property. The corresponding logistic equation (7.11) given by Theorem 7.5 is then equivalent to the Δ-dynamic equation

$$y(t) = y(t_0) - \int_{t_0}^{t} (p(s) + f(s)y(s))y(\sigma(s))\Delta s, \tag{8.3}$$

where σ is the forward jump operator. Indeed, if y is a solution of Eq. (7.11), then $y(s-) = y(s)$ (because g is left-continuous). Moreover, g is constant on each interval $(\alpha,\beta] \subset [a,b]$ such that $(\alpha,\beta) \cap \mathbb{T} = \emptyset$. Thus, y has the same property, and $y(s+) = y(\sigma(s))$ for each $s \in [a,b)_{\mathbb{T}}$.

Similarly, the Δ-dynamic equation

$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(\sigma(s)) + f(s))\Delta s$$

is a special case of Eq. (7.10). The corresponding logistic equation (7.12) given by Theorem 7.5 is then equivalent to the Δ-dynamic equation

$$y(t) = y(t_0) - \int_{t_0}^{t} (-p(s) + f(s)y(\sigma(s)))y(s)\Delta s. \tag{8.4}$$

Equations (8.3) and (8.4) are integral forms of the two versions of Δ-dynamic logistic equation described in [3] and mentioned in the introduction of the present paper.

To deal with ∇-dynamic equations, we replace g by the integrator h given by Eq. (8.2). The ∇-dynamic equation

$$x(t) = x(t_0) + \int_{t_0}^{t} (-p(s)x(s) + f(s))\nabla s$$

is then a special case of Eq. (7.10) with g replaced by h; note that a solution x of Eq. (7.10) satisfies $x(s+) = x(s)$ for all s, because h is right-continuous, and therefore x has the same property. The corresponding logistic equation (7.12) given by Theorem 7.5 is then equivalent to the ∇-dynamic equation

$$y(t) = y(t_0) - \int_{t_0}^{t} (-p(s) + f(s)y(s))y(\rho(s))\nabla s, \tag{8.5}$$

where ρ is the backward jump operator. Indeed, if y is a solution of Eq. (7.12), then $y(s+) = y(s)$ (because h is left-continuous). Moreover, h is constant on each interval $[\alpha, \beta) \subset [a, b]$ such that $(\alpha, \beta) \cap \mathbb{T} = \emptyset$. Thus, y has the same property, and $y(s-) = y(\rho(s))$ for each $s \in (a, b]_{\mathbb{T}}$.

Similarly, the ∇-dynamic equation

$$x(t) = x(t_0) + \int_{t_0}^{t} (p(s)x(\rho(s)) + f(s))\nabla s$$

is a special case of Eq. (7.9). The corresponding logistic equation (7.11) given by Theorem 7.5 is then equivalent to the ∇-dynamic equation

$$y(t) = y(t_0) - \int_{t_0}^{t} (p(s) + f(s)y(\rho(s)))y(s)\nabla s. \tag{8.6}$$

As far as we are aware, the ∇-dynamic logistic equations (8.5) and (8.6) did not appear in the literature yet.

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