Properties of Poincaré half-maps for planar linear systems and some direct applications to periodic orbits of piecewise systems

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Abstract. This paper deals with fundamental properties of Poincaré half-maps defined on a straight line for planar linear systems. Concretely, we focus on the analyticity of the Poincaré half-maps, their series expansions (Taylor and Newton–Puiseux) at the tangency point and at infinity, the relative position between the graph of Poincaré half-maps and the bisector of the fourth quadrant, and the sign of their second derivatives. All these properties are essential to understand the dynamic behavior of planar piecewise linear systems. Accordingly, we also provide some of their most immediate, but non-trivial, consequences regarding periodic orbits.

Keywords: piecewise planar linear systems, Poincaré half-maps, Taylor series expansion, Newton–Puiseux series expansion.

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1 Introduction

The study of the qualitative properties of distinguished solutions of piecewise linear differential systems rests mainly on the analysis of the features of Poincaré maps, which are defined as composition of transition maps between the separation manifolds. Sometimes these transition maps are called Poincaré half-maps. The linearity of the system in each zone invites to its integration, which automatically causes the emergence of a wide range of cases due to the nature

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of the different spectra of the matrices of the linear systems and the relative position between the equilibria, if any, and the separation manifolds. The number of cases to be studied is high even for planar systems with two zones of linearity. Moreover, the direct integration leads to different nonlinear equations where the flight time appears as a new variable.

Since the publication of the seminal work by Freire et al. [9], a large number of interesting papers have appeared in order to establish the dynamical behavior in planar piecewise linear systems with two zones of linearity and, in particular, to give conditions for the existence and stability of limit cycles and to provide an optimal bound for the number of coexisting limit cycles (see for instance, [11, 15–21]). None of these papers considers all the possible cases. Moreover, they are forced to use individualized approaches to study the different kind of functions that arise due to the distinct spectra of the matrices. This causes that a same result is usually expressed in different terms and, sometimes, it may be a hard task to obtain a common and brief statement for it. Thus, the use of individualized techniques for each case does not allow a unified view of several properties of the Poincaré half-maps and, when it does, more effort is required to complete the case-by-case study and to achieve independent statements of these cases.

This paper relies on a novel characterization of Poincaré half-maps for planar linear systems [2] which allows us to see the properties of these maps from a common point of view and to prove the results in a simple way, without the need of making particularized case-by-case studies. Accordingly, we will not have any of the disadvantages mentioned in the previous paragraphs because this novel characterization does not require integration of the systems and, therefore, the distinction of the spectra of the matrices is not needed. The strength of this approach can be seen in [3], where the uniqueness of limit cycles for continuous piecewise linear systems was provided in a simple and synthesized way.

In the framework of the study of Poincaré half-maps for planar linear systems, the most relevant properties are those related to the local behavior at tangency points between the flow and the Poincaré section, the behavior at infinity (obviously, in the case of the focus or center), and the sign of the derivatives. Some of these properties have been proven just for concrete cases. Even those which are valid for all situations have been proven in a large case-by-case study. This manuscript is primarily devoted to simplifying and unifying the proofs of these properties by considering all possible scenarios simultaneously. In addition, it will be stated an interesting fact about the relative position between the graphs of Poincaré half-maps and the bisector of the fourth quadrant. Among other things, from this property it is direct that Poincaré half-maps inherit the expansion/compression behavior of the flow of the planar linear system. Additionally, it is proven here that this relative position is also related to the (constant) sign of its second derivative.

As might be expected from the first two paragraphs of this introduction, all the previously commented properties have direct applications to planar piecewise linear systems; from the analysis of stability and bifurcations of equilibria, singularities, or the infinity, to the existence and characterization of periodic orbits and the obtention of uniform bounds to the number of limit cycles. In this work, some straightforward conclusions concerning the periodic behavior are obtained.

The paper is organized as follows. Section 2 presents the integral characterization of Poincaré half-maps for planar linear systems given in [2]. Two basic consequences of this characterization for the Poincaré half-maps are their analyticity and their understanding as solutions of a differential equation. In Section 3, we summarize the results on analyticity of the Poincaré half-maps given in [2] and obtain the Taylor and Newton–Puiseux series expansions

at tangency points and infinity by means of the differential equation. Section 4 studies the relationship between the graphs of Poincaré half-maps and the bisector of the fourth quadrant, which is used to establish, in Section 5, the sign of the second derivatives of the Poincaré half-maps. Finally, Section 6 addresses the analysis of the periodic behavior of planar piecewise linear systems with two zones separated by a straight line. There, some direct consequences of the properties of Poincaré half-maps obtained in previous sections are stated.

2 Integral characterization for the Poincaré half-maps

Let us consider, for $\mathbf{x} = (x_1, x_2)^T$, the autonomous linear system

$$\dot{\mathbf{x}} = A\,\mathbf{x} + \mathbf{b} \tag{2.1}$$

where $A = (a_{ij})_{i,j=1,2}$ is a real matrix and $\mathbf{b} = (b_1, b_2)^T \in \mathbb{R}^2$. Let us choose, without loss of generality, the Poincaré section $\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$.

Notice that if the coefficient a_{12} vanishes, system (2.1) is uncoupled and a Poincaré half-map on section Σ cannot be defined. Hence, let us assume in this work that $a_{12} \neq 0$ (observability condition [6]). On the one hand, observe that, among other configurations, this condition removes the possibility of star-nodes to appear. On the other hand, under the assumption $a_{12} \neq 0$, the linear change of variable $x = x_1$, $y = a_{22}x_1 - a_{12}x_2 - b_1$, with $a = a_{12}b_2 - a_{22}b_1$, allows to write system (2.1) into the generalized Liénard form,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} T & -1 \\ D & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ a \end{pmatrix}, \tag{2.2}$$

where T and D stand for the trace and the determinant of matrix A, respectively. In the new coordinates, since $x_1 = x$, Poincaré section Σ remains the same.

The first equation of system (2.2) evaluated on the section Σ is reduced to $\dot{x}|_\Sigma = -y$. Therefore, the flow of the system crosses Σ from the half-plane $\Sigma^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ to the half-plane $\Sigma^- = \{(x, y) \in \mathbb{R}^2 : x < 0\}$ when $y > 0$, from Σ^- to Σ^+ when $y < 0$, and it is tangent to Σ at the origin.

Since this work is devoted to Poincaré half-maps of system (2.2) corresponding to the section Σ and due to the fact that there is no possible return to section Σ when $a = D = 0$, we assume that $a^2 + D^2 \neq 0$ throughout this work. Note that this condition avoids the existence of a continuum of equilibria.

We are going to focus on the left Poincaré half-map (the one defined by the flow in the closed half-plane $\Sigma^- \cup \Sigma$ and the intersection points of its orbits with the Poincaré section Σ). Notice that the definition of the right Poincaré half-map and their corresponding results may be immediately obtained by the invariance of system (2.2) under the change $(x, y, a) \leftrightarrow (-x, -y, -a)$.

The left Poincaré half-map is usually defined in the following way. Let us consider $(0, y_0) \in \Sigma$ with $y_0 \geqslant 0$ and let $\Phi(t; y_0) = (\Phi_1(t; y_0), \Phi_2(t; y_0))$ the solution of system (2.2) that satisfies the initial condition $\Phi(0; y_0) = (0, y_0)$. The existence of a value $\tau(y_0) > 0$ such that $\Phi_1(\tau(y_0); y_0) = 0$ and $\Phi_1(t; y_0) < 0$ for every $t \in (0, \tau(y_0))$ allows to define the image of y_0 by the left Poincaré half-map as $P(y_0) = \Phi_2(\tau(y_0); y_0) \leqslant 0$. Moreover, the value $\tau(y_0)$ is called the left flight time.

Regarding the definition of the left Poincaré half-map at the origin, $P(0)$ cannot be defined as above when for every $\tau > 0$ there exists $t \in (0, \tau)$ such that $\Phi_1(t; 0) > 0$. However, it can

be continuously extended as $P(0) = 0$ provided that for every $\varepsilon > 0$ there exist $y_0 \in (0, \varepsilon)$ and $y_1 \in (-\varepsilon, 0)$ such that $P(y_0) = y_1$. This finishes the usual definition of the left Poincaré half-map.

According to the above definition, it is natural to compute the flow of the system by means of explicit integrations. This leads to many case-by-case studies and forces the nonlinear appearance of the flight time. Here, we will use a characterization that avoids the computation of the flow, as it is done in [2]. For the sake of completeness, we give a brief summary of the main results and ideas of [2] that are going to be used in this paper.

The left Poincaré half-map P and its definition interval I are given in Theorem 19 and Corollary 21 of [2]. By using the quadratic polynomial function

$$W(y) = Dy^2 - aTy + a^2, \tag{2.3}$$

the *left Poincaré half-map* is the unique function $P : I \subset [0, +\infty) \longrightarrow (-\infty, 0]$ that, for every $y_0 \in I$, satisfies

$$\mathrm{PV}\left\{ \int_{P(y_0)}^{y_0} \frac{-y}{W(y)} dy \right\} = cT, \tag{2.4}$$

where c is given, in terms of the parameters, as follows: (i) $c = 0$ if $a > 0$, (ii) $c = \pi \left(D\sqrt{4D - T^2} \right)^{-1} \in \mathbb{R}$ if $a = 0$, and (iii) $c = 2\pi \left(D\sqrt{4D - T^2} \right)^{-1} \in \mathbb{R}$ if $a < 0$. Here, $\mathrm{PV}\{\cdot\}$ stands for the *Cauchy Principal Value* at the origin (see, for instance, [14]), which is defined as

$$\mathrm{PV}\left\{ \int_{y_1}^{y_0} \frac{-y}{W(y)} dy \right\} = \lim_{\varepsilon \searrow 0} \left(\int_{y_1}^{-\varepsilon} \frac{-y}{W(y)} dy + \int_{\varepsilon}^{y_0} \frac{-y}{W(y)} dy \right),$$

for $y_1 < 0 < y_0$.

As emphasized in [2], the interval I is essentially related with the roots of the quadratic polynomial function W. In the next remark, we shall briefly comment some of those relationships and other interesting properties of P which are proven in [2].

Remark 2.1. System (2.2), under the assumed condition $a^2 + D^2 \neq 0$, has invariant straight lines for several values of the parameters. These straight lines are either the invariant eigenspaces of equilibria (saddles, degenerate nodes or non-degenerate nodes) or the straight line $y = Tx + a/T$ in the case $T \neq 0$, $D = 0$ (what implies $a \neq 0$). In those cases, every invariant straight line intersects the Poincaré section Σ in a point $(0, \mu)$, where μ is a root of the quadratic polynomial function W given in (2.3). Moreover, when $I \subset [0, +\infty)$ is bounded, then the right endpoint of I is a real root of W and, in the same way, if $P(I)$ is bounded, then the left endpoint of $P(I)$ is also a real root of W. In Fig. 2.1 (a) and Fig. 2.1(b), we show two examples of bounded intervals I and/or $P(I)$, corresponding respectively to saddle and non-degenerate node configurations.

The interval I can be unbounded. For instance, if $4D - T^2 > 0$, then the equilibrium point of system (2.2) is a focus or a center, the intervals I and $P(I)$ are unbounded, and, obviously, $P(y_0)$ tends to $-\infty$ as $y_0 \to +\infty$. In this case, the intervals are $I = [0, +\infty)$ and $P(I) = (-\infty, 0]$, except when the equilibrium is a focus (i.e. $T \neq 0$) and it is located in the left half-plane $\{(x, y) \in \mathbb{R}^2 : x < 0\}$ (i.e. $a < 0$). In fact, when $T > 0$, the interval $P(I)$ is reduced to $(-\infty, \hat{y}_1]$, where $\hat{y}_1 = P(0)$ (see Fig. 2.1 (c)). Analogously, for $T < 0$, $I = [\hat{y}_0, +\infty)$ with $\hat{y}_0 = P^{-1}(0)$ (see Fig. 2.1 (d)).

Finally, the polynomial function W is strictly positive in each set $[P(y_0), 0) \cup (0, y_0]$, with $y_0 \in I$. Besides that, since $W(0) = a^2$, then $W(0) > 0$ for $a \neq 0$ and $W(0) = 0$ for $a = 0$.
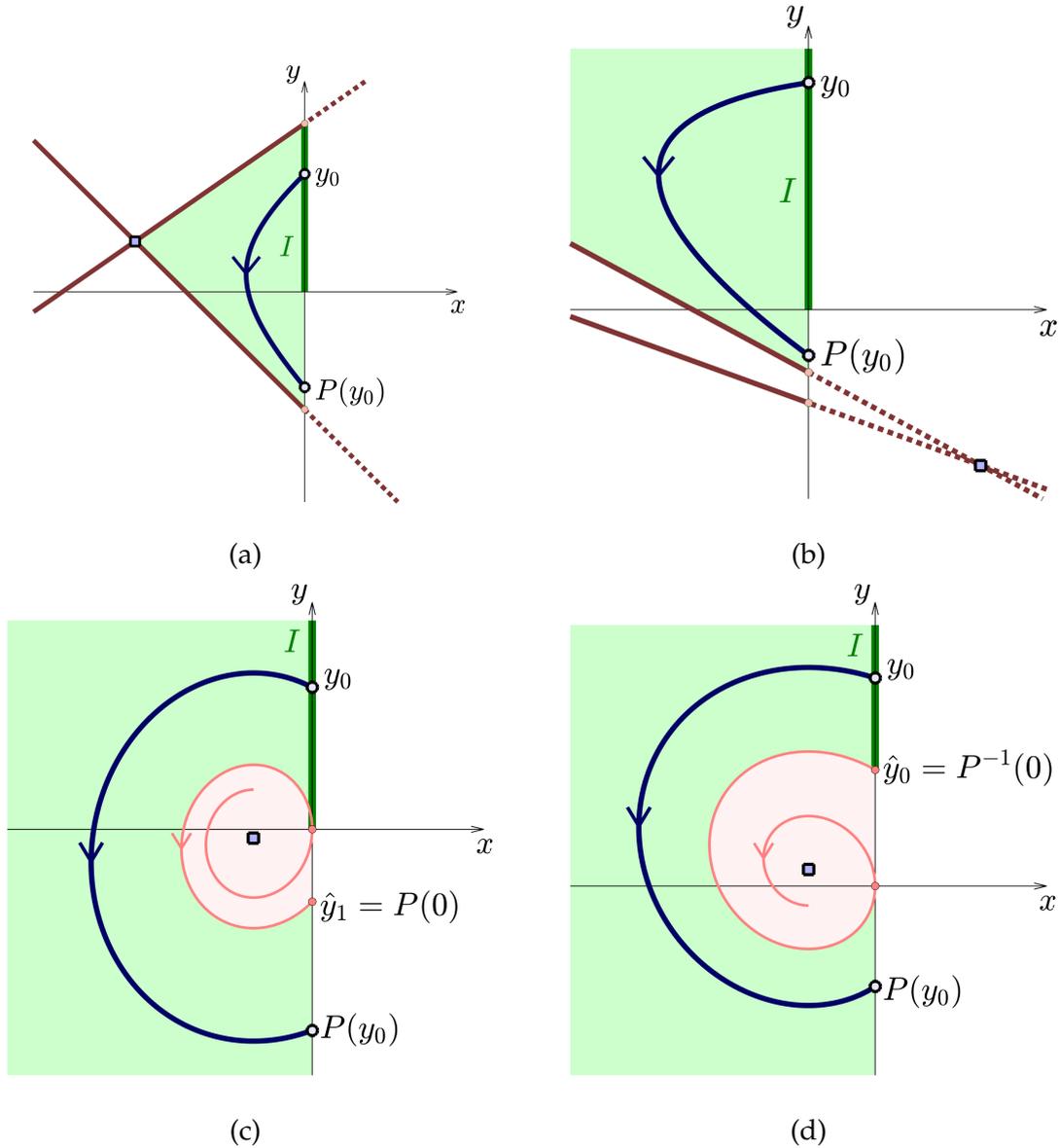
(a)

(b)

(c)

(d)

Figure 2.1: The left Poincaré half-map P and its interval of definition I for the cases: (a) saddle, (b) non-degenerate node, (c) unstable focus, and (d) stable focus.

It is worth mentioning that the integral given in (2.4) diverges when $a = 0$ and the Cauchy principal value is necessary to overcome this difficulty. Moreover, in this case, for $y_1 < 0 < y_0$, the Cauchy principal value at the origin is given by

$$\text{PV}\left\{ \int_{y_1}^{y_0} \frac{-y}{Dy^2}dy \right\} = \lim_{\varepsilon \searrow 0} \left(\int_{y_1}^{-\varepsilon} \frac{-y}{Dy^2}dy + \int_{\varepsilon}^{y_0} \frac{-y}{Dy^2}dy \right) = \frac{1}{D}\log\left|\frac{y_1}{y_0}\right|. \tag{2.5}$$

When $a \neq 0$, the integrating function $h(y) = -y/W(y)$ is continuous and, consequently, the Cauchy principal value just takes the value of the integral.

3 Analiticity and series expansions of Poincaré half-maps at the tangency point and its preimage, and at infinity

In this section, by means of the integral characterization and a subsequent differential equation, we shall compute the first coefficients of the Taylor expansion of the left Poincaré half-map P. Obviously, the used method does not depend on the spectrum of the matrix of the system. Before obtaining these coefficients it is necessary to determine the analyticity of the left Poincaré half-map P.

When $P(y_0) \neq 0$, it is well-known (see, for example, [7]) that the transversality between the flow of the system and the separation line Σ ensures the analyticity of P at y_0. The analyticity for the tangency point between the flow and Σ (that is, the origin) is more intricate and, in the literature, it has been approached with a case-by-case study (see some partial results at [8]). However, as follows from Corollary 24 of [2], the maps P and P^{-1} are real analytic functions in the open intervals $\text{int}(I)$ and $P(\text{int}(I))$, respectively, and at least one of the following statements is true:

(i) the map P is a real analytic function at the left endpoint of its domain,

(ii) the map P^{-1} is a real analytic function at the right endpoint of its domain.

When the equilibrium of system (2.2) is a center or a focus, the left Poincaré half-map P can be considered also at infinity. In addition, we shall obtain the first coefficients of the Taylor expansion of P around the infinity.

A first consequence from the definition of the left Poincaré half-map given in the integral form (2.4) is easily deduced by computing the derivative with respect to variable y_0 (see Remark 16 of [2]). Hence, one can see that the graph of the left Poincaré half-map P and its inverse function P^{-1}, oriented according to increasing y_0, are particular orbits of the cubic vector field

$$
\begin{aligned}
X(y_0, y_1) &= -\big(y_1 W(y_0), y_0 W(y_1)\big) \\
&= -\big(y_1 (Dy_0^2 - aTy_0 + a^2), y_0 (Dy_1^2 - aTy_1 + a^2)\big).
\end{aligned}
$$

In fact, the left Poincaré half-map P and its inverse function P^{-1} are solutions of the differential equation

$$
y_1 W(y_0) dy_1 - y_0 W(y_1) dy_0 = 0. \tag{3.1}
$$

The next proposition is a direct consequence of the results in [2] and allows to obtain the Taylor expansion of P around the origin when $a \neq 0$ and $P(0) = 0$. Notice that for $a = 0$, the existence of the left Poincaré half-map P implies $4D - T^2 > 0$. From Remark 2.1, the interval of definition of P is $I = [0, +\infty)$ and, for $y_0 \geqslant 0$, expression (2.4) can be written as

$$
\text{PV}\left\{ \int_{P(y_0)}^{y_0} \frac{-y}{Dy^2} dy \right\} = \frac{\pi T}{D\sqrt{4D - T^2}}.
$$

Hence, by using the value for PV given in (2.5), the left Poincaré half-map P is given by

$$
P(y_0) = -\exp\left(\frac{\pi T}{\sqrt{4D - T^2}}\right) y_0, \quad \text{for } y_0 \geqslant 0. \tag{3.2}
$$

When $a \neq 0$, by denoting $\mathcal{I} = \{y \in \mathbb{R} : W(y) > 0\}$, from Theorem 14 of [2], it is deduced that the set

$$C_0 = \left\{ (y_1, y_0) \in \mathcal{I}^2 : \int_{y_1}^{y_0} -y/W(y) dy = 0 \right\}$$

can be written in the form

$$C_0 = \left\{ (y_1, y_0) \in \mathcal{I}^2 : (y_1 - y_0)(y_1 - \varphi_0(y_0)) \right\},$$

where φ_0 is a real analytic function in $\mathcal{I}$ which is also an involution, that is, $(\varphi_0(\varphi_0(y_0))) = y_0$ for all $y_0 \in \mathcal{I}$. Now, by means of Corollary 21 of [2], it follows that the left Poincaré half-map P coincides with the function φ_0 restricted to the interval $\mathcal{I} \cap [0, \infty]$, provided $0 \in I$ and $P(0) = 0$. By an abuse of notation, we say that the Poincaré half-map P is an involution when $a \neq 0$, $0 \in I$, and $P(0) = 0$.

Proposition 3.1. *Assume that $a \neq 0$ and $0 \in I$. If $P(0) = 0$, then left Poincaré half-map P is a real analytic function in I, it is an involution and its Taylor expansion around the origin writes as*

$$\begin{aligned}
P(y_0) = -y_0 &- \frac{2Ty_0^2}{3a} - \frac{4T^2y_0^3}{9a^2} + \frac{2\left(9DT - 22T^3\right)y_0^4}{135a^3} \\
&+ \frac{4\left(27DT^2 - 26T^4\right)y_0^5}{405a^4} - \frac{2\left(27D^2T - 176DT^3 + 100T^5\right)y_0^6}{945a^5} + \mathcal{O}\left(y_0^7\right).
\end{aligned}$$
(3.3)

Proof. From the hypotheses of the proposition and by means of Theorem 14 and Corollary 24 of [2], it is deduced that left Poincaré half-map P is a real analytic function in I and it is an involution. Hence, the derivative of P at the origin is $P'(0) = -1$.

Now, taking into account that P is a solution of the differential equation given in (3.1), it is easy to obtain, via undetermined coefficients, the Taylor expansion given in (3.3) and so the proof is concluded. $\square$

Notice that the Taylor expansion around the origin given in Proposition 3.1 was already obtained in [23]. Although the calculations are not fully detailed in that work, the authors rely on the results given in [8], where the study requires different techniques depending on the situations. Before [23], the same series expansion was obtained in [10], by means of an inversion of the flight time, but only for the focus case.

When $0 \in I$ and $P(0) \neq 0$, the function P is a real analytic function at the origin and it is possible to obtain its Taylor expansion of P around the origin.

Proposition 3.2. *Assume that $0 \in I$. If $P(0) = \hat{y}_1 < 0$, then $a < 0, T > 0, 4D - T^2 > 0$, $\hat{y}_1$ is the right endpoint of the interval $P(I)$, and the left Poincaré half-map P is a real analytic function in I and its Taylor expansion around the origin writes as*

$$\begin{aligned}
P(y_0) = \hat{y}_1 &+ \frac{W(\hat{y}_1)y_0^2}{2a^2\hat{y}_1} + \frac{TW(\hat{y}_1)y_0^3}{3a^3\hat{y}_1} - \frac{\left(a^2 + \left(D - 2T^2\right)\hat{y}_1^2\right)W(\hat{y}_1)y_0^4}{8a^4\hat{y}_1^3} \\
&- \frac{T\left(5a^2 + \left(7D - 6T^2\right)\hat{y}_1^2W(\hat{y}_1)\right)y_0^5}{30a^5\hat{y}_1^3} \\
&+ \frac{\left(9a^4 - 6a^3T\hat{y}_1 + 2a^2\left(9D - 13T^2\right)\hat{y}_1^2 + \left(9D^2 - 46DT^2 + 24T^4\right)\hat{y}_1^4\right)W(\hat{y}_1)y_0^6}{144a^6\hat{y}_1^5} \\
&+ \mathcal{O}\left(y_0^7\right).
\end{aligned}$$
(3.4)

Proof. The expression given in (3.2) provides the left Poincaré half-map for the case $a = 0$. From there, one obtains $P(0) = 0$ when $a = 0$.

Suppose that $0 \in I$ and $P(0) = \hat{y}_1 < 0$. Then $a \neq 0$ and expression (2.4) leads us to

$$\int_{\hat{y}_1}^{0} \frac{-y}{W(y)} dy = cT.$$

From Remark 2.1, the polynomial W is strictly positive and, therefore, the left-hand term of the last expression is also strictly positive. If $a > 0$, from expression (2.4), $c = 0$ and this is impossible. Thus, it is deduced that $a < 0, T > 0$, and $4D - T^2 > 0$. Now, from Remark 2.1 again, the intervals I and $P(I)$ are unbounded and $P(y_0)$ tends to $-\infty$ as $y_0 \to +\infty$.

Next, let us prove that $\hat{y}_1$ is the right endpoint of the interval I. Let us consider $y_0 \geqslant 0$ and $y_1 \in (\hat{y}_1, 0]$. From the inequalities

$$\int_{y_1}^{y_0} \frac{-y}{W(y)} dy < \int_{\hat{y}_1}^{y_0} \frac{-y}{W(y)} dy \leqslant \int_{\hat{y}_1}^{0} \frac{-y}{W(y)} dy = cT$$

one can see that no point in the interval $y_1 \in (\hat{y}_1, 0]$ belongs to the interval $P(I)$ and so the right endpoint of interval $P(I)$ is $\hat{y}_1$.

The analyticity of P is a direct consequence of Theorem 14 and Corollary 21 of [2] and the Taylor expansion around the origin given in (3.4) follows from the method of undetermined coefficients applied to the differential equation (3.1). □

Remark 3.3. Note that the condition $P(0) = \hat{y}_1 < 0$ together with the linearity of the system implies that there exists an unstable focus equilibrium in the left half-plane Σ^- (see Fig. 2.1(c)) and so it is immediate that $a < 0, T > 0$, and $4D - T^2 > 0$. This is an alternative proof for the inequalities of Proposition 3.2. On the other hand, the endpoints of intervals I and $P(I)$ were also determined in Corollary 21 of [2] in a more generic way. For the sake of completeness, in the previous proof, we have included a different and specific reasoning for this case.

When there exists a point $\hat{y}_0 > 0$ such that $P(\hat{y}_0) = 0$, then left Poincaré half-map P is a non-analytic function at $\hat{y}_0$. However, in [2] it is proven that the inverse function P^{-1} is analytic at the origin and so it is possible, by means of an inversion, to get a Newton–Puiseux series expansion for the left Poincaré half-map P around $\hat{y}_0$. Some results about series inversion and Newton–Puiseux series can be found in [12] and the references therein. Also of interest are the results included in [1] concerning the expression of the solutions of differential equations as Newton–Puiseux series expansion and its convergence.

Proposition 3.4. *Assume that there exists a value $\hat{y}_0 > 0$ with $P(\hat{y}_0) = 0$. Then, $a < 0, T < 0$, $4D - T^2 > 0$, $\hat{y}_0$ is the left endpoint of the interval I, the inverse function P^{-1} is a real analytic function, and the left Poincaré half-map P admits the Newton–Puiseux serie expansion around the point $\hat{y}_0$ given by*

$$P(y_0) = a\sqrt{\frac{2\hat{y}_0}{W(\hat{y}_0)}} (y_0 - \hat{y}_0)^{1/2} - \frac{aT}{3} \frac{2\hat{y}_0}{W(\hat{y}_0)} (y_0 - \hat{y}_0)$$

$$+ \frac{a^3}{72} \left(\frac{9D + 2T^2}{a^2} + \frac{9}{\hat{y}_0^2} \right) \left(\sqrt{\frac{2\hat{y}_0}{W(\hat{y}_0)}} \right)^3 (y_0 - \hat{y}_0)^{3/2} + \mathcal{O}\left((y_0 - \hat{y}_0)^2\right),$$

(3.5)

which is valid for $y_0 \geqslant \hat{y}_0$.

Proof. Suppose that there exists a point $\hat{y}_0 > 0$ such that $P(\hat{y}_0) = 0$. An analogous reasoning to the first part of the proof of Proposition 3.2 leads to the inequalities $a < 0, T < 0$, and $4D - T^2 > 0$ and to the fact that $\hat{y}_0$ is the left endpoint of the interval I.

The inverse function P^{-1} satisfies $P^{-1}(0) = \hat{y}_0$ and, from differential equation (3.1), it follows that its derivative at the origin vanishes. This implies that P is a non-analytic function at $\hat{y}_0$. From Theorem 14 and Corollary 21 of [2], it follows that the inverse function P^{-1} is an analytic function at the origin and P^{-1} admits the Taylor expansion (3.4) by changing $\hat{y}_1$ by $\hat{y}_0$.

Now, the Newton–Puiseux series expansion of P is obtained by the inversion of the Taylor expansion of P^{-1}. Note that the direct inversion provides two possible series expansions but, since $P(y_0) \leqslant 0$ for all $y_0 \in I$, the valid one is that given in (3.5) and the proof is finished. $\quad\square$

An analogous comment to Remark 3.3 can be made about the inequalities of Proposition 3.4 and the left endpoint of I. The scenario described by the hypothesis stated in Proposition 3.4 is illustrated in Fig. 2.1(d).

Remark 3.5. The inversion used to obtain the Newton–Puiseux series expansion of P is equivalent to the computation of the Taylor expansion of $Q(z_0) := P(\hat{y}_0 + z_0^2)$ around $z_0 = 0$ and the subsequent change of z_0 by $\sqrt{y_0 - \hat{y}_0}$. In order to get this Taylor expansion it is enough to make the change of variable $y_0 \to \hat{y}_0 + z_0^2$ in the differential equation (3.1) to achieve a differential equation for the function Q.

Let us recall from Remark 2.1 that when $4D - T^2 > 0$ the domain I is unbounded with $P(y_0)$ tending to $-\infty$ as $y_0 \to +\infty$. Thus, the study of the left Poincaré half-map around the infinity is feasible. In fact, the first two terms of the Taylor expansion of left Poincaré half-map P around the infinity were already obtained in [9] by means of an expression of P parameterized by the flight time. In the following proposition, we present a simple method to get these and others terms.

Proposition 3.6. *Assume that $4D - T^2 > 0$. Then, the Taylor expansion of left Poincaré half-map P around the infinity writes as*

$$P(y_0) = -\exp\left(\frac{\pi T}{\sqrt{4D - T^2}}\right) y_0 + \frac{aT}{D}\left(1 + \exp\left(\frac{\pi T}{\sqrt{4D - T^2}}\right)\right)$$
$$- \frac{a^2}{D}\sinh\left(\frac{\pi T}{\sqrt{4D - T^2}}\right)\cdot\frac{1}{y_0}$$
$$- \frac{a^3 e^{-\frac{2\pi T}{\sqrt{4D - T^2}}}\left(-2 + e^{\frac{\pi T}{\sqrt{4D - T^2}}}\right)\left(1 + e^{\frac{\pi T}{\sqrt{4D - T^2}}}\right)^2 T}{6D^2}\cdot\frac{1}{y_0^2} + \mathcal{O}\left(\frac{1}{y_0^3}\right).$$

Proof. Firstly, we shall prove the equality

$$\lim_{y_0 \to +\infty} \frac{P(y_0)}{y_0} = -\exp\left(\frac{\pi T}{\sqrt{4D - T^2}}\right). \tag{3.6}$$

If $a = 0$, then expression (3.2) leads us directly to equality (3.6).

If $a \neq 0$, taking into account that $W(y) > 0$ for $y \in \mathbb{R}$ (see Remark 2.1), then relationship (2.4) can be written as

$$\int_{P(y_0)}^{-y_0} \frac{-y}{W(y)}dy + \int_{-y_0}^{y_0} \frac{-y}{W(y)}dy = cT, \tag{3.7}$$

for $y_0 \in I$, being $c = 0$ for $a > 0$ and $c = 2\pi \left(D\sqrt{4D - T^2} \right)^{-1}$ for $a < 0$.

The change of variable $Y = 1/y$ applied to the first integral in expression (3.7) transforms it into

$$\int_{1/P(y_0)}^{-1/y_0} \frac{1}{a^2Y^3 - aTY^2 + DY} dY + \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy = cT$$

or, equivalently, into the expression

$$\int_{1/P(y_0)}^{-1/y_0} \frac{1}{DY} dY + \int_{1/P(y_0)}^{-1/y_0} \frac{a(T - aY)}{D \left(a^2Y^2 - aTY + D \right)} dY + \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy = cT.$$

That is,

$$\frac{P(y_0)}{y_0} = -\exp \left(DcT - D \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy - \int_{1/P(y_0)}^{-1/y_0} \frac{a(T - aY)}{a^2Y^2 - aTY + D} dY \right).$$

Now, a direct integration provides

$$\lim_{y_0 \to +\infty} \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy = -\frac{\pi T \text{sign}(a)}{D\sqrt{4D - T^2}} \tag{3.8}$$

and taking into account that

$$\lim_{y_0 \to +\infty} \int_{1/P(y_0)}^{-1/y_0} \frac{a(T - aY)}{a^2Y^2 - aTY + D} dY = 0,$$

the equality (3.6) follows.

Thus, the function $\tilde{P}$, defined by

$$\tilde{P}(Y_0) = \begin{cases} \dfrac{1}{P(1/Y_0)} & \text{if } Y_0 \neq 0 \text{ and } 1/Y_0 \in I, \\ 0 & \text{if } Y_0 = 0, \end{cases}$$

has derivative on the right at the origin and its value is

$$\alpha_1 := \frac{d\tilde{P}}{dY_0} (0^+) = -\exp \left(\frac{-\pi T}{\sqrt{4D - T^2}} \right). \tag{3.9}$$

Moreover, it is immediate to see that the function $\tilde{P}$ is a solution of differential equation

$$\left(a^2Y_0^2 - aTY_0 + D \right) Y_0 dY_1 - \left(a^2Y_1^2 - aTY_1 + D \right) Y_1 dY_0 = 0,$$

obtained from the differential equation (3.1) by means of the change of variables $(Y_0, Y_1) = (1/y_0, 1/y_1)$ (defined for $y_0 y_1 \neq 0$). From here, it is deduced that the function $\tilde{P}$ has derivatives on the right of all orders at $Y_0 = 0$ and, after a direct computation, one finds

$$\alpha_2 := \frac{d^2\tilde{P}}{dY_0^2} (0^+) = -\frac{2aT}{D} e^{-\frac{2\pi T}{\sqrt{4D - T^2}}} \left(e^{\frac{\pi T}{\sqrt{4D - T^2}}} + 1 \right),$$

$$\alpha_3 := \frac{d^3\tilde{P}}{dY_0^3} (0^+) = \frac{3a^2}{D^2} e^{-\frac{3\pi T}{\sqrt{4D - T^2}}} \left(e^{\frac{\pi T}{\sqrt{4D - T^2}}} + 1 \right) \left(-2T^2 e^{\frac{\pi T}{\sqrt{4D - T^2}}} + De^{\frac{\pi T}{\sqrt{4D - T^2}}} - D - 2T^2 \right),$$

and

$$
\begin{aligned}
\alpha_4 &:= \frac{d^4 \tilde{P}}{dY_0^4}\left(0^+\right) \\
&= \frac{4a^3 T}{D^3} e^{-\frac{4\pi T}{\sqrt{4D-T^2}}} \left(1 + e^{\frac{\pi T}{\sqrt{4D-T^2}}}\right)^2 \left(-8D + 7De^{\frac{\pi T}{\sqrt{4D-T^2}}} - 6T^2 - 6T^2 e^{\frac{\pi T}{\sqrt{4D-T^2}}}\right).
\end{aligned}
\tag{3.10}
$$

Since the Taylor expansion of left Poincaré half-map P around the infinity is given by

$$
P(y_0) = \frac{1}{\alpha_1} y_0 - \frac{\alpha_2}{2\alpha_1^2} + \frac{3\alpha_2^2 - 2\alpha_1\alpha_3}{12\alpha_1^3} \cdot \frac{1}{y_0} - \frac{3\alpha_2^3 - 4\alpha_1\alpha_2\alpha_3 + \alpha_1^2\alpha_4}{24\alpha_1^4} \cdot \frac{1}{y_0^2} + \mathcal{O}\left(\frac{1}{y_0^3}\right),
\tag{3.11}
$$

the proof concludes by substituting expressions (3.9)–(3.10) into (3.11). $\qquad\square$

4 The relative position between the graph of Poincaré half-maps and the bisector of the fourth quadrant

To study the relative position between the graph of the left Poincaré half-map and the bisector of the fourth quadrant, it is natural to analyze the sign of the difference $y_0 - (-P(y_0))$. In the next proposition, we show the relationship between this difference and the trace T. Notice that this relationship has been addressed via a case-by-case treatment (by distinguishing the spectrum of the matrix of the system) in the main results of chapter 4 of [22]. Here, we provide a concise proof by using the integral characterization of the left Poincaré half-map.

Proposition 4.1. *The left Poincaré half-map P satisfies the relationship*

$$
\operatorname{sign}\left(y_0 + P(y_0)\right) = -\operatorname{sign}(T) \quad \text{for } y_0 \in I \setminus \{0\}.
$$

In addition, when $0 \in I$ and $P(0) \neq 0$ or when $T = 0$, the relationship also holds for $y_0 = 0$.

Proof. We will prove this proposition by distinguishing the cases $T = 0$ and $T \neq 0$.

For $T = 0$, the integral equation given in (2.4) is reduced to

$$
PV\left\{\int_{P(y_0)}^{y_0} \frac{-y}{Dy^2 + a^2} dy\right\} = 0, \quad \text{for } y_0 \in I.
$$

By taking into account that the integrating function is an odd function, it is direct to see that $P(y_0) = -y_0$ for all $y_0 \in I$ and so the proposition is true for $T = 0$.

Now, we focus on the proof for the case $T \neq 0$ and we will consider the situations $a = 0$ and $a \neq 0$.

When $a = 0$, the left Poincaré half-map P is given by expression (3.2) and so the equality $\operatorname{sign}\left(y_0 + P(y_0)\right) = -\operatorname{sign}(T)$ holds for every $y_0 \in I$.

When $a \neq 0$, let us consider the interval

$$
J = \{u \in \mathbb{R} : W(y) > 0, \quad \forall y \in [-|u|, |u|]\}
$$

and function $g : J \longrightarrow \mathbb{R}$ defined by

$$
g(u) = \int_{-u}^{u} \frac{-y}{W(y)} dy,
$$

where W is the polynomial function defined in (2.3).

Notice that function g satisfies $g(0) = 0$, its derivative is

$$g'(u) = \frac{-2aTu^2}{W(u)W(-u)}$$

and so $\text{sign}(g'(u)) = -\text{sign}(aT)$ for every $u \in J \setminus \{0\}$. Thus, $\text{sign}(g(u)) = -\text{sign}(aT)$ for every $u \in J \cap (0, +\infty)$ and $\text{sign}(g(u)) = \text{sign}(aT)$ for every $u \in J \cap (-\infty, 0)$.

Moreover, if $J = \mathbb{R}$ (i.e., when $4D - T^2 > 0$), then, from (3.8),

$$\lim_{u \to +\infty} g(u) = -\frac{\pi T \text{sign}(a)}{D\sqrt{4D - T^2}}.$$

The existence of the left Poincaré half-map P for the case $a \neq 0$ implies $a > 0$ and $c = 0$ or $a < 0$ and $c = 2\pi \left(D\sqrt{4D - T^2} \right)^{-1} \in \mathbb{R}$. It is straightforward to see that these conditions together with the properties of function g lead to the equality

$$\text{sign}(cT - g(u)) = \text{sign}(T). \tag{4.1}$$

Let us consider $y_0 \in \text{int}(I) \cap J$. From equality (2.4), one gets

$$cT = \int_{P(y_0)}^{y_0} \frac{-y}{W(y)} dy = \int_{P(y_0)}^{-y_0} \frac{-y}{W(y)} dy + \int_{-y_0}^{y_0} \frac{-y}{W(y)} dy,$$

that is,

$$\int_{P(y_0)}^{-y_0} \frac{-y}{W(y)} dy = cT - g(y_0).$$

Thus, from (4.1),

$$\text{sign} \left(\int_{P(y_0)}^{-y_0} \frac{-y}{W(y)} dy \right) = \text{sign}(T) \neq 0$$

and, taking into account that $-y_0 \cdot P(y_0) \geqslant 0$, equality $\text{sign}(y_0 + P(y_0)) = -\text{sign}(T)$ holds for every $y_0 \in \text{int}(I) \cap J$. Therefore, the conclusion follows by using the continuity of the left Poincaré half-map and the function $y_1(y_0) = -y_0$. $\qquad \square$

The next result establishes, as a direct consequence of Proposition 4.1, the relationship between the graph of the left Poincaré half-map and the bisector of the fourth quadrant.

Corollary 4.2. *The following items are true.*

1. *If $T = 0$, then the graph of the left Poincaré half-map P of system (2.2) associated to section $\Sigma \equiv \{x = 0\}$, if it exists, is included in the bisector of the fourth quadrant.*

2. *If $T > 0$ (resp. $T < 0$), then the graph of left Poincaré half-map P of system (2.2) associated to section $\Sigma \equiv \{x = 0\}$, if it exists, is located below (resp. above) the bisector of the fourth quadrant except perhaps at the origin.*

5   The sign of the second derivative of Poincaré half-maps

From the differential equation given in (3.1), it is easy to obtain explicit expressions for the derivatives of P with respect to y_0. The first and second derivatives are shown in the next result. Its proof is a simple computation and so it is omitted.

Proposition 5.1. *The first and second derivatives of the left Poincaré half-map P with respect to y_0, in the interval* $\text{int}(I)$, *are given by*

$$\frac{dP}{dy_0}(y_0) = \frac{y_0 W(P(y_0))}{P(y_0)W(y_0)}$$

and

$$\frac{d^2 P}{dy_0^2}(y_0) = -\frac{a^2\left(y_0^2 - (P(y_0))^2\right)W(P(y_0))}{(P(y_0))^3\,(W(y_0))^2}\,. \tag{5.1}$$

As will be stated in the next section, some interesting applications to periodic behavior of piecewise linear systems come out from the signs of the first and the second derivatives of P. Note that the sign of the first derivative is obvious from (5.1), because $y_0 P(y_0) < 0$ for $y_0 \in \text{int}(I)$ and the polynomial W is positive (see Remark 2.1). Besides that, the sign of the second derivative of left Poincaré half-map P is an immediate consequence of expression given in (5.1) and Proposition 4.1.

Proposition 5.2. *The sign of the second derivative of left Poincaré half-map P is given by*

$$\text{sign}\left(\frac{d^2 P}{dy_0^2}(y_0)\right) = -\text{sign}(a^2 T) \quad \text{for } y_0 \in \text{int}(I).$$

Note that a^2 is written in the previous expression to include the case $a = 0$.

In previous works, the sign of the second derivative of the Poincaré half-maps has been addressed via case-by-case studies (see, for instance [22]), where distinguished analyses must be employed for different values of the parameters. Nevertheless, in Proposition 5.2, the integral characterization has allowed to obtain a closed expression for such a sign regardless the cases. As far as we know, this common expression has not been previously obtained in the literature.

6   Some immediate consequences in piecewise linear systems

The previous results established some fundamental properties of Poincaré half-maps defined on a straight line for planar linear systems. These properties are essential to understand the dynamic behavior of planar piecewise linear systems. This section is devoted to provide some immediate consequences regarding periodic behavior in piecewise linear systems with two zones separated by a straight line.

From Freire et. al in [10, Proposition 3.1], we known that any piecewise linear system with two zones separated by a straight line Σ for which a Poincaré map is well defined can be written in the following Liénard canonical form

$$\begin{cases} \dot{x} = T_L x - y \\ \dot{y} = D_L x - a_L \end{cases} \quad \text{for } x < 0, \qquad \begin{cases} \dot{x} = T_R x - y + b \\ \dot{y} = D_R x - a_R \end{cases} \quad \text{for } x > 0. \tag{6.1}$$

Note that the points $(0,0)$ and $(0,b)$ are the tangency points between Σ and, respectively, the flow of the left and right systems. When $b = 0$ these points coincide and the flow of system (6.1) crosses the separation line transversally except at the origin. In this case, the system is called *sewing*. On the contrary, for $b \neq 0$ the flow of system (6.1) does not cross the separation line along the segment

$$\Sigma_s = \{(0, \mu + (1 - \mu)b) \in \Sigma : \mu \in (0,1)\},$$

which is usually called the *sliding region*.

In order to analyze the behaviour of system (6.1) we consider two Poincaré half-maps associated to Σ, to wit, the *Forward Poincaré half-map* $y_L : I_L \subset [0, +\infty) \longrightarrow (-\infty, 0]$ and the *Backward Poincaré half-map* $y_R : I_R \subset [b, +\infty) \to (-\infty, b]$. The forward one goes in the positive direction of the flow and maps a point $(0, y_0)$, with $y_0 \geqslant 0$, to a point $(0, y_L(y_0))$. Analogously, the backward one goes in the negative direction of the flow and maps a point $(0, y_0)$, with $y_0 \geqslant b$, to $(0, y_R(y_0))$. Notice that y_L is defined by the left system and y_R is defined by the right system. Naturally, $y_L = P$ by taking $T = T_L, D = D_L$, and $a = a_L$ in system (2.2). In addition, taking into account the change $(t, x) \to -(t, x)$, one has $y_R(y_0) = P(y_0 - b) + b$ by taking $T = -T_R, D = D_R$, and $a = -a_R$.

Evidently, the intersections between the curves $y_1 = y_L(y_0)$ and $y_1 = y_R(y_0)$, for $y_0 \in \text{Int}(I_L \cap I_R)$, are in bijective correspondence to crossing periodic solutions of (6.1).

From Proposition 4.1,

$$T_L = 0 \text{ (resp. } T_R = 0) \Rightarrow y_L(y_0) = -y_0 \text{ (resp. } y_R(y_0) = -y_0 + 2b), \tag{6.2}$$

when, of course, the map y_L (resp. y_R) exists. Therefore, the following result follows immediately.

Corollary 6.1. *Assume that $T_L^2 + T_R^2 = 0$. If $b \neq 0$, the system (6.1) does not have crossing periodic orbits. If $b = 0$ and $\text{Int}(I_L \cap I_R) \neq \emptyset$, then it has a continuum of crossing periodic orbits.*

It is also possible to give some results for the case $T_L^2 + T_R^2 > 0$. From Corollary 4.2, if $T_L > 0$ (resp. $T_L < 0$), then the curve $y_1 = y_L(y_0)$, if it exists, is located below (resp. above) the straight line $y_1 = -y_0$ except perhaps at the origin. Analogously, if $T_R > 0$ (resp. $T_R < 0$), the curve $y_1 = y_R(y_0)$, if it exists, is located above (resp. below) the straight line $y_1 = -y_0 + 2b$ except perhaps at the point (b, b). Hence, also taking (6.2) into account, if $T_L > 0$, $T_R \geqslant 0$, and $b \geqslant 0$, then

$$y_L(y_0) < -y_0 \leqslant -y_0 + 2b \leqslant y_R(y_0).$$

Therefore, the graphs of y_L and y_R have no intersection points and so system (6.1) has no crossing periodic orbits. The following result about non-existence of periodic orbits follows immediately via a similar reasoning.

Corollary 6.2. *Assume that $T_L T_R \geqslant 0$ and that one of the following two non-exclusive hypotheses holds:*

1) $T_L \neq 0$ and $T_L b \geqslant 0$;

2) $T_R \neq 0$ and $T_R b \geqslant 0$.

Then, system (6.1) does not have crossing periodic orbits.

By merging the information of Corollaries 6.1 and 6.2, we get that if $T_L T_R \geqslant 0$ and $T_L b \geqslant 0$, then system (6.1) either does not have crossing periodic orbits or has a continuum of crossing periodic orbits. In other words, it does not have limit cycles.

Let us add some lines regarding the condition $T_L T_R \geqslant 0$ added in Corollary 6.2. Note that for the case $b = 0$ it is well known that $T_L T_R \leqslant 0$ is a necessary condition for the existence of crossing periodic orbits (see, for instance, [10]). However, when $b \neq 0$ they could exist even for $T_L T_R > 0$. Thus, the previous result allows to remove some cases where crossing periodic solutions do not exist.

The obtention of the previous results relies only on the relative location of the graphs of the Poincaré half-maps, which is easily determined in terms of the basic parameters T_L, T_R, and b, by using Proposition 4.1. Now, information about the shape of their graphs, revealed by Proposition 5.2, can be used to bound the number of limit cycles of piecewise linear systems (isolated crossing periodic solutions) in some generic cases. In fact, from Proposition 5.2, a simple expression is obtained for the sign of the second derivatives of the Poincaré half-maps,

$$\text{sign}\left(\frac{d^2 y_L}{dy_0^2}(y_0)\right) = -\text{sign}(a_L^2 T_L) \quad \text{and} \quad \text{sign}\left(\frac{d^2 y_R}{dy_0^2}(y_0)\right) = \text{sign}(a_R^2 T_R).$$

Therefore, the concavity of the functions y_L and y_R is established by $a_L^2 T_L$ and $a_R^2 T_R$, respectively. Thus, the following result for limit cycles follows immediately.

Corollary 6.3. *If $T_L T_R > 0$, then system (6.1) has at most two limit cycles.*

The upper bound given by the corollary above is reachable. Indeed, the last example provided by Han and Zhang in [13] satisfies $T_L T_R > 0$ and has two limit cycles near the origin.

Concerning the series expansions of Poincaré half-maps provided by Propositions 3.1-3.6, a natural application could consist in obtaining stability properties of some singular invariant sets of piecewise linear systems under suitable assumptions. For instance, Proposition 3.1 can provide whether the monodromic singularity at the separation line is attracting, repelling, or a center; analogously, Proposition 3.6 can provide whether the infinity is attracting, repelling, or a center in the case it is monodromic; finally, Propositions 3.2 and 3.4 can be used to study the stability of some fold-fold connections. Mixing the stability properties above, one can immediately get sufficient conditions for the existence of a limit cycle (forcing, for instance, the mondromic singularity at the discontinuity line and the infinity, in the monodromic case, to have the same stability).

7 Conclusions

In this paper we provided fundamental properties of Poincaré half-maps defined on a straight line for planar linear systems. Our analysis was based on a novel characterization of Poincaré half-maps [2], presented in Section 2. This characterization has proven to be an effective method to study these maps from a common point of view and to obtain results in a simple way, without the need of making particularized case-by-case studies.

We have focused on the analyticity of the Poincaré half-maps, their series expansions, the relative position between the graph of Poincaré half-maps and the bisector of the fourth quadrant, and the sign of their second derivatives. In what follows, we summarize the obtained results. In Section 3, we addressed the series expansion of a Poincaré half-map,

$P : I \subset [0, +\infty) \longrightarrow (-\infty, 0]$, around the extrema of its interval of definition I, namely: Propositions 3.1 and 3.2 provided the Taylor expansion of P around the origin when $0 \in I$; Proposition 3.4 provided the Newton–Puiseux series expansion of P around $\hat{y}_0 \in I$, where $\hat{y}_0 > 0$ satisfies $P(\hat{y}_0) = 0$; and Proposition 3.6 provided the Taylor series expansions of P around the infinity when $4D - T^2 > 0$. In Section 4, Proposition 4.1 and Corollary 4.2 established a relationship between the graphs of Poincaré half-maps and the bisector of the fourth quadrant depending only on the sign of the trace T. Finally, in Section 5, Proposition 5.1 provided expressions for the first and second derivative of the Poincaré half-maps and Proposition 5.2 determined the sign of the second derivative of the Poincaré half-maps.

All these properties are essential to understand the dynamic behavior of planar piecewise linear systems with two zones separated by a straight line (PPWLS, for short). Thus, in Section 6 we provided some immediate consequences regarding periodic behavior of such systems, namely: Corollary 6.1 established non-generic conditions for a PPWLS either not having periodic orbits and having a continuum of crossing periodic orbits; Corollary 6.2 gives generic conditions for a PPWLS not having periodic orbits; finally, Corollary 6.3 provided generic conditions for a PPWLS having at most two limit cycles.

The results obtained in this paper also allow deeper insights regarding periodic solutions for piecewise linear systems. For instance, in [4], the present results among others were of assistance in proving that PPWLS without sliding region (that is, $b = 0$) has at most one limit cycles. This result was obtained without unnecessary distinctions of spectra of the matrices. In addition, it is proven that this limit cycle, if exists, is hyperbolic and its stability is determined by a simple condition in terms of the parameters. Also, in [5], it was provide the existence of a uniform upper bound, L^*, for the maximum number of limit cycles of PPWLS. The present Proposition 4.1 helped to show that $L^* \le 8$.

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