Electronic Journal of Qualitative Theory of Differential Equations
2023, No. **49**, 1–26; https://doi.org/10.14232/ejqtde.2023.1.49 www.math.u-szeged.hu/ejqtde/

Limit cycles in piecewise smooth perturbations of a class of cubic differential systems

Dan Sun[1], **Yunfei Gao**[1,2], **Linping Peng**[✉1] and **Li Fu**[3]

[1]School of Mathematical Sciences, LIMB of the Ministry of Education,
Beihang University, Beijing, 100191, China
[2]Beijing Etown Academy, Beijing, 100176, China
[3]National (Virtual Simulation) Experimental Teaching Demonstration Center of Mechanical and Control Engineering, Beihang University, Beijing, 100191, China

Received 24 November 2022, appeared 8 December 2023
Communicated by Armengol Gasull

Abstract. In this paper, we study the bifurcation of limit cycles from a class of cubic integrable non-Hamiltonian systems under arbitrarily small piecewise smooth perturbations of degree n. By using the averaging theory and complex method, the lower and upper bounds for the maximum number of limit cycles bifurcating from the period annulus of the unperturbed systems are given at first order in ε. It is also shown that in this case, the maximum number of limit cycles produced by piecewise smooth perturbations is almost twice the upper bound of the maximum number of limit cycles produced by smooth perturbations for the considered systems.

Keywords: bifurcation of limit cycles, piecewise smooth perturbation, cubic differential system, averaging theory, complex method.

2020 Mathematics Subject Classification: 34C07, 37G15, 34C05.

1 Introduction

Non-smooth phenomena widely exists in real world and scientific fields, such as dynamic compensation of inertial element error in autonomous navigation of high dynamic aircraft, the non-smooth switching between modules in multi-source information fusion, electronic relays, mechanical impact, neuronal networks and etc., see for instance [1, 14, 16, 27]. Generally, it can be modeled by non-smooth differential systems. Piecewise smooth differential systems, served as one of the most important non-smooth dynamical systems, attracts many researcher's interest. In recent years, more attention focuses on studying dynamical behaviors, especially the bifurcation theory of limit cycles in piecewise smooth systems, see [5, 7, 11, 17, 18, 34, 39, 42, 43]. There are quite a few innovative methods which have been proposed and some theoretical results were established. For example, the conjecture that a class of piecewise Liénard equations with $n+1$ intervals has up to $2n$ limit cycles was proved in

✉Corresponding author. Email: penglp@buaa.edu.cn

[44]. Through analyzing the Lyapunov constants, Hopf bifurcation of non-smooth systems was presented in [10,12,23]. The Melnikov method for Hopf and homoclinic bifurcations was applied to non-smooth systems [2,19,25,31,32]. In addition, the first order Melnikov function for planar piecewise smooth Hamiltonian systems was derived to study Poincaré bifurcation [33], while the averaging theory of discontinuous dynamical systems was developed to find limit cycles of piecewise continuous dynamical systems [35].

It is well known that the simplest piecewise smooth systems are the piecewise linear ones with two zones separated by a straight line. Lum and Chua [40, 41] conjectured that such a continuous piecewise linear differential system in the plane has at most one limit cycle, which was proved by Freire et al. [21]. While for the planar discontinuous piecewise linear differential systems with two zones separated by a straight line, Han and Zhang [25] showed that such systems may have two limit cycles. Huan and Yang [26] provided a numerical example which possesses three limit cycles. Llibre and Ponce [38] presented three nested limit cycles in discontinuous piecewise linear differential systems with two zones. Some results on other discontinuous piecewise linear differential systems with two zones separated by a straight line exhibiting three limit cycles can also be seen in [7, 8, 28] etc. There are also some works concerning the limit cycles bifurcation from a linear center under piecewise smooth perturbations. For example, the nth degree piecewise polynomial perturbations of a linear center were considered in [6], and an upper bound of no more than $Nn-1$ limit cycles appearing up to a study of order N was presented. Cen et al. [9] studied quadratic isochronous centers $S1, S2, S3$ and $S4$ under the piecewise polynomial perturbations of degree n by applying the first order averaging theory, and found the sharp upper bound for the first three isochronous centers and an upper bound for the last center. More results on this topic can be found in [15,20,22,24,29,30,36,37] and the references therein.

In the present paper, we focus our attention on the study of limit cycle bifurcation from a class of planar cubic integrable non-Hamilton differential system.

$$(\dot{x},\dot{y}) = (-y(x+a)(y+b), x(x+a)(y+b)), \tag{1.1}$$

which has

$$H(x,y) = x^2+y^2 = h, \quad h \in (0, \min\{a^2, b^2\})$$

as its first integral with the integrating factor $\mu = 2/((x+a)(y+b))$, and $(0,0)$ is the unique center.

Consider arbitrarily small piecewise smooth perturbations of system (1.1)

$$(\dot{x},\dot{y}) = \begin{cases} (-y(x+a)(y+b)+\varepsilon f^+(x,y), x(x+a)(y+b)+\varepsilon g^+(x,y)), & x>0, \\ (-y(x+a)(y+b)+\varepsilon f^-(x,y), x(x+a)(y+b)+\varepsilon g^-(x,y)), & x<0, \end{cases} \tag{1.2}$$

where the polynomials $f^{\pm}(x,y), g^{\pm}(x,y), i=1,2$ are given by

$$f^+(x,y) = \sum_{i+j=0}^{n} a_{i,j}x^iy^j, \quad g^+(x,y) = \sum_{i+j=0}^{n} b_{i,j}x^iy^j,$$
$$f^-(x,y) = \sum_{i+j=0}^{n} c_{i,j}x^iy^j, \quad g^-(x,y) = \sum_{i+j=0}^{n} d_{i,j}x^iy^j,$$

with any real coefficients $a_{i,j}, b_{i,j}, c_{i,j}$ and $d_{i,j}$, and $|\varepsilon| \neq 0$ is a small parameter. By using the first order averaging theory for discontinuous systems and complex method, we study the maximum number, denoted by $H(n)$, of limit cycles of system (1.2) bifurcating from the period annulus around the center of system (1.1). The main results are summarized as follows.

Theorem 1.1. *For system (1.2) with $|\varepsilon| \neq 0$ sufficiently small, we have*

(i) If $|a| > |b| \neq 0$, then $2[\frac{n}{2}] + 2n + 3 \leq H(n) \leq 2[\frac{n}{2}] + 4n + 14$;

(ii) If $|b| > |a| \neq 0$, then $2[\frac{n}{2}] + 2n + 3 \leq H(n) \leq 4[\frac{n}{2}] + 3n + 14$;

(iii) If $|a| = |b| \neq 0$, then $[\frac{n}{2}] + 2n + 3 \leq H(n) \leq 3n + 6$;

(iv) If $b = 0, a \neq 0$, then $H(n) = 2[\frac{n}{2}] + n + 1$,

where $[\cdot]$ is the integer function, and $H(n)$ denotes the maximum number of limit cycles of system (1.2) bifurcating from the period annulus of the unperturbed system (1.1) at first order in ε.

Remark 1.2. It is noted that the limit cycle bifurcation from the unperturbed system (1.1) with $a, b \in R\backslash\{0\}$ under arbitrarily small smooth polynomial perturbations of degree n is studied in [4], which shows that $3[(n-1)/2] + 4$ if $a \neq b$ and, respectively, $2[(n-1)/2] + 2$ if $a = b$, up to first order in ε, are upper bounds for the number of the limit cycles bifurcating from the period annulus of the cubic center (1.1). Comparing Theorem 1.1 with the results in [4], we obtained that at first order in ε, the lower bound of the maximum number of limit cycles produced by piecewise smooth perturbations is almost twice the upper bound of the maximum number of limit cycles produced by smooth perturbations. Hence, for one differential system, piecewise smooth perturbations generally produce more limit cycles than smooth ones.

The organization of this paper is as follows. In Section 2, we present some preliminary results, including the first order averaging theory for discontinuous systems and the method estimating the number of zeros of some functions. The explicit expression and properties of the averaged function are derived in Section 3. Sections 4-6 are dedicated to the investigation of the lower and upper bounds for the maximum number of the zeros of the averaged function, respectively. Finally we prove Theorem 1.1 in Section 7.

2 Preliminary results

In this section, we briefly introduce the first order averaging theory for discontinuous systems and the method concerning the estimate of the number of zeros of some functions, which will be used in the proof of our main results.

Lemma 2.1 ([34]). *Consider the following discontinuous differential systems*

$$\frac{dr}{d\theta} = \varepsilon F(\theta, r) + \varepsilon^2 R(\theta, r, \varepsilon), \tag{2.1}$$

with

$$F(\theta, r) = F_1(\theta, r) + \text{sign}(h(\theta, r))F_2(\theta, r),$$
$$R(\theta, r, \varepsilon) = R_1(\theta, r, \varepsilon) + \text{sign}(h(\theta, r))R_2(\theta, r, \varepsilon),$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ and $h : \mathbb{R} \times D \to \mathbb{R}$ are continuous functions, T-periodic in the first variable θ and D is an open subset of $\mathbb{R}^n$. We also suppose that h is a C^1 function having zero as a regular value, and the sign function $\text{sign}(u)$ is given by

$$\text{sign}(u) = \begin{cases} 1, & u > 0, \\ 0, & u = 0, \\ -1, & u < 0. \end{cases}$$

Define the averaged function $f : D \to \mathbb{R}^n$ *as*

$$f(r) = \int_0^T F(\theta, r)\, d\theta. \tag{2.2}$$

Assume that the following hypotheses (i), (ii) *and* (iii) *hold.*

(i) F_1, F_2, R_1, R_2 *and* h *are locally Lipschitz with respect to* r.

(ii) There exists an open bounded subset $C \subset D$ *such that for the sufficiently small* $|\varepsilon| > 0$, *every orbit starting in* $\overline{C}$ *reaches the set of discontinuity only at its crossing regions.*

(iii) For $a \in C$ *with* $f(a) = 0$, *there exists a neighborhood* V *of* a *such that* $f(z) \neq 0$ *for all* $z \in \overline{V} \setminus \{a\}$ *and the Brouwer degree function* $d_B(f, V, a) \neq 0$.

Then, for the sufficiently small $|\varepsilon| > 0$ *there exists a T-periodic solution* $r(\theta, \varepsilon)$ *of system* (2.1) *such that* $r(0, \varepsilon) \to a$ *as* $\varepsilon \to 0$.

The result from [3] is often used to replace the condition (iii) in Lemma 2.1, which is stated as follows.

Remark 2.2. $(\widetilde{iii})$ Let $f : D \to \mathbb{R}$ be a C^1 function with $f(a) = 0$, where D is an open subset of $\mathbb{R}$ and $a \in D$. Whenever the Jacobian determinant $J_f(a) \neq 0$, there exists a neighborhood V of a such that $f(r) \neq 0$ for all $r \in \overline{V} \backslash \{a\}$. Then $d_B(f, V, 0) \neq 0$.

To estimate the number of zeros of some functions, we recall an important result from [13].

Lemma 2.3. *Consider* $p+1$ *linearly independent analytical functions* $f_i : U \to \mathbb{R}, i = 0, 1, \ldots, p$, *where* $U \subset \mathbb{R}$ *is an interval. Suppose that there exists* $j \in \{0, 1, \ldots, p\}$ *such that* f_j *has constant sign. Then there exists* $p+1$ *constants* $C_i, i = 0, 1, \ldots, p$ *such that* $f(x) = \sum_{i=0}^{p} C_i f_i(x)$ *has at least* p *simple zeros in* U.

3 Explicit expression of averaged function

This section is devoted to the derivation and simplification of the expression for the averaged function.

After making the polar coordinate transformations $x = r\cos\theta$ and $y = r\sin\theta$, system (1.2) becomes the following

$$\frac{dr}{d\theta} = \begin{cases} \varepsilon X^+(\theta, r) + \varepsilon^2 Y^+(\theta, r, \varepsilon), & \cos\theta > 0, \\ \varepsilon X^-(\theta, r) + \varepsilon^2 Y^-(\theta, r, \varepsilon), & \cos\theta < 0, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} X^+(\theta, r) &= \frac{P^+(\theta, r)}{(r\cos\theta + a)(r\sin\theta + b)}, \qquad X^-(\theta, r) = \frac{P^-(\theta, r)}{(r\cos\theta + a)(r\sin\theta + b)}, \\ Y^+(\theta, r, \varepsilon) &= -\frac{X^+(\theta, r)Q^+(\theta, r)}{r(r\cos\theta + a)(r\sin\theta + b) + \varepsilon Q^+(\theta, r)}, \\ Y^-(\theta, r, \varepsilon) &= -\frac{X^-(\theta, r)Q^-(\theta, r)}{r(r\cos\theta + a)(r\sin\theta + b) + \varepsilon Q^-(\theta, r)}, \end{aligned}$$

with

$$P^{\pm}(\theta, r) = \cos\theta f^{\pm}(r\cos\theta, r\sin\theta) + \sin\theta g^{\pm}(r\cos\theta, r\sin\theta),$$

$$Q^{\pm}(\theta, r) = \cos\theta g^{\pm}(r\cos\theta, r\sin\theta) - \sin\theta f^{\pm}(r\cos\theta, r\sin\theta).$$

Denote

$$r_1 = \begin{cases} -a, & a < 0, \\ +\infty, & a > 0, \end{cases} \qquad r_2 = \begin{cases} a, & a > 0, \\ +\infty, & a < 0, \end{cases} \qquad r_3 = \begin{cases} b, & b > 0, \\ -b, & b < 0, \end{cases}$$

then the functions $X^+(\theta, r)$ and $Y^+(\theta, r, \varepsilon)$ ($X^-(\theta, r)$ and $Y^-(\theta, r, \varepsilon)$, resp.) are well defined in $(0, r_1) \cap (0, r_3)$ ($(0, r_2) \cap (0, r_3)$, resp.) for $ab \neq 0$, while $X^+(\theta, r)$ and $Y^+(\theta, r, \varepsilon)$ ($X^-(\theta, r)$ and $Y^-(\theta, r, \varepsilon)$, resp.) are well defined in $(0, r_1)$ ($(0, r_2)$, resp.) for $b = 0,\ a \neq 0$.

We rewrite system (3.1) as the form

$$\frac{dr}{d\theta} = \varepsilon F(\theta, r) + \varepsilon^2 R(\theta, r, \varepsilon), \tag{3.2}$$

where

$$F(\theta, r) = \begin{cases} X^+(\theta, r), & \cos\theta > 0, \\ X^-(\theta, r), & \cos\theta < 0, \end{cases}$$
$$R(\theta, r, \varepsilon) = \begin{cases} Y^+(\theta, r, \varepsilon), & \cos\theta > 0, \\ Y^-(\theta, r, \varepsilon), & \cos\theta < 0. \end{cases}$$

By Lemma 2.1, the averaged function of system (3.2) can be expressed as

$$\begin{aligned} f(r) &= \int_0^{2\pi} F(\theta, r)\, d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(\theta, r)\, d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} F(\theta, r)\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} X^+(\theta, r)\, d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} X^-(\theta, r)\, d\theta \\ &= \sum_{i+j=1}^{n+1} \omega_{i,j} r^{i+j-1} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\sin\theta + b)} d\theta \\ &\quad + \sum_{i+j=1}^{n+1} \tau_{i,j} r^{i+j-1} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\sin\theta + b)} d\theta, \end{aligned} \tag{3.3}$$

where $\omega_{i,j} = a_{i-1,j} + b_{i,j-1}$, $\tau_{i,j} = c_{i-1,j} + d_{i,j-1}$, and $\omega_{0,0} = \tau_{0,0} = 0$ provided that $a_{-1,j} = b_{i,-1} = c_{-1,j} = d_{i,-1} = 0$.

Define

$$\begin{aligned} I_{i,j}(r) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\sin\theta + b)} d\theta, \\ J_{i,j}(r) &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\sin\theta + b)} d\theta \end{aligned} \tag{3.4}$$

for $i, j \geq 0$.

Remark 3.1. Note that in the interval $\cap_{i=1}^{3}(0, r_i) = (0, \min\{|a|, |b|\})$ ($(0, |a|)$, resp.) for $ab \neq 0$ ($b = 0, a \neq 0$, resp.), the zeros of the function $f(r)$ coincide with the non-zero zeros of $F(r) = rf(r)$. To make the calculation easier, we investigate the zeros of the function $F(r)$ instead of $f(r)$ in the subsequent sections.

Lemma 3.2. *The function $F(r) = rf(r)$ defined above can be expressed as*

$$F(r) = rf(r) = F_1(r) + F_2(r), \tag{3.5}$$

where

$$F_1(r) = \sum_{i=0}^{n+1} r^i I_{i,0}(r) \sum_{j=0}^{[\frac{n+1-i}{2}]} P_{i,j} r^{2j} + \sum_{i=0}^{n} r^{i+1} I_{i,1}(r) \sum_{j=0}^{[\frac{n-i}{2}]} Q_{i,j} r^{2j},$$
$$F_2(r) = \sum_{i=0}^{n+1} r^i J_{i,0}(r) \sum_{j=0}^{[\frac{n+1-i}{2}]} \widetilde{P}_{i,j} r^{2j} + \sum_{i=0}^{n} r^{i+1} J_{i,1}(r) \sum_{j=0}^{[\frac{n-i}{2}]} \widetilde{Q}_{i,j} r^{2j},$$

with

$$P_{i,j} = \sum_{k=0}^{[\frac{i}{2}]} (-1)^k C_{k+j}^k \omega_{i-2k,2k+2j}, \qquad Q_{i,j} = \sum_{k=0}^{[\frac{i}{2}]} (-1)^k C_{k+j}^k \omega_{i-2k,2k+2j+1},$$
$$\widetilde{P}_{i,j} = \sum_{k=0}^{[\frac{i}{2}]} (-1)^k C_{k+j}^k \tau_{i-2k,2k+2j}, \qquad \widetilde{Q}_{i,j} = \sum_{k=0}^{[\frac{i}{2}]} (-1)^k C_{k+j}^k \tau_{i-2k,2k+2j+1}.$$

Moreover, $P_{0,0} = \widetilde{P}_{0,0} = 0$, C_{k+j}^k is the combinatorial number, and the other coefficients $P_{i,j}, Q_{i,j}, \widetilde{P}_{i,j}$ and $\widetilde{Q}_{i,j}$ are independent.

Proof. From (3.3) and (3.4), we have

$$\begin{aligned}
F(r) &= \sum_{i+j=1}^{n+1} \omega_{i,j} r^{i+j} I_{i,j}(r) + \sum_{i+j=1}^{n+1} \tau_{i,j} r^{i+j} J_{i,j}(r) \\
&= \sum_{i=1}^{n+1} r^i \sum_{j=0}^{i} \omega_{i-j,j} I_{i-j,j}(r) + \sum_{i=1}^{n+1} r^i \sum_{j=0}^{i} \tau_{i-j,j} J_{i-j,j}(r) \\
&= \sum_{i=1}^{n+1} r^i \sum_{j=0}^{[\frac{i}{2}]} \omega_{i-2j,2j} I_{i-2j,2j}(r) + \sum_{i=1}^{n+1} r^i \sum_{j=0}^{[\frac{i-1}{2}]} \omega_{i-2j-1,2j+1} I_{i-2j-1,2j+1}(r) \\
&\quad + \sum_{i=1}^{n+1} r^i \sum_{j=0}^{[\frac{i}{2}]} \tau_{i-2j,2j} J_{i-2j,2j}(r) + \sum_{i=1}^{n+1} r^i \sum_{j=0}^{[\frac{i-1}{2}]} \tau_{i-2j-1,2j+1} J_{i-2j-1,2j+1}(r).
\end{aligned} \tag{3.6}$$

On the other hand, some computations shows that

$$\begin{aligned}
I_{i,2j}(r) &= \sum_{k=0}^{j} (-1)^k C_j^k I_{i+2k,0}(r), \qquad I_{i,2j+1}(r) = \sum_{k=0}^{j} (-1)^k C_j^k I_{i+2k,1}(r), \\
J_{i,2j}(r) &= \sum_{k=0}^{j} (-1)^k C_j^k J_{i+2k,0}(r), \qquad J_{i,2j+1}(r) = \sum_{k=0}^{j} (-1)^k C_j^k J_{i+2k,1}(r).
\end{aligned} \tag{3.7}$$

Putting (3.7) into (3.6), we can get (3.5). The independence of $P_{i,j}, Q_{i,j}, \widetilde{P}_{i,j}$ and $\widetilde{Q}_{i,j}$ follows from their definitions.

This completes the proof of Lemma 3.2. □

Define

$$\begin{aligned}
Y_{i,j}(r) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{a + r\cos\theta} d\theta, \qquad Z_{i,j}(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{b + r\sin\theta} d\theta, \\
\widetilde{Y}_{i,j}(r) &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{a + r\cos\theta} d\theta, \qquad \widetilde{Z}_{i,j}(r) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos^i\theta \sin^j\theta}{b + r\sin\theta} d\theta,
\end{aligned} \tag{3.8}$$

for $i, j \geq 0$, then a straightforward computation yields Lemma 3.3 below.

Lemma 3.3. *For* (3.4) *and* (3.8)*, the following equalities hold.*

(i) $I_{i,0}(r) = \frac{1}{r}[Z_{i-1,0}(r) - aI_{i-1,0}(r)]$ *for* $i \geq 1$.

(ii) $Z_{i,0}(r) = -\frac{1}{r^2}[(b^2 - r^2)Z_{i-2,0}(r) - b^* m_{i-2}]$ *for* $i \geq 2$.

(iii) $I_{i,1}(r) = \frac{1}{r}[Y_{i,0}(r) - bI_{i,0}(r)]$ *for* $i \geq 0$.

(iv) $Y_{i,0}(r) = \frac{1}{r}[c^* m_{i-1} - aY_{i-1,0}(r)]$ *for* $i \geq 1$.

(v) $J_{i,0}(r) = \frac{1}{r}[\widetilde{Z}_{i-1,0}(r) - aJ_{i-1,0}(r)]$ *for* $i \geq 1$.

(vi) $\widetilde{Z}_{i,0}(r) = -\frac{1}{r^2}[(b^2 - r^2)\widetilde{Z}_{i-2,0}(r) - \widetilde{b}^* m_{i-2}]$ *for* $i \geq 2$.

(vii) $J_{i,1}(r) = \frac{1}{r}[\widetilde{Y}_{i,0}(r) - bJ_{i,0}(r)]$ *for* $i \geq 0$.

(viii) $\widetilde{Y}_{i,0}(r) = \frac{1}{r}[\widetilde{c}^* m_{i-1} - a\widetilde{Y}_{i-1,0}(r)]$ *for* $i \geq 1$,

where $m_i = \frac{(i-1)!!}{i!!}$, $m_0 = m_1 = 1$, *and*

$$b^* = \begin{cases} \pi b, & i \text{ is even}, \\ 2b, & i \text{ is odd}, \end{cases} \qquad c^* = \begin{cases} \pi, & i \text{ is odd}, \\ 2, & i \text{ is even}, \end{cases}$$

$$\widetilde{b}^* = \begin{cases} \pi b, & i \text{ is even}, \\ -2b, & i \text{ is odd}, \end{cases} \qquad \widetilde{c}^* = \begin{cases} \pi, & i \text{ is odd}, \\ -2, & i \text{ is even}. \end{cases}$$

Moreover, we have

$$\Big(-a^2 - b^2 + r^2\Big) I_{0,0}(r) = -bY_{0,0}(r) + rZ_{1,0}(r) - aZ_{0,0}(r),$$
$$\Big(-a^2 - b^2 + r^2\Big) J_{0,0}(r) = -b\widetilde{Y}_{0,0}(r) - rZ_{1,0}(r) - aZ_{0,0}(r).$$

Now, we start with simplifying $F(r)$. Firstly, substituting Lemma 3.3 into (3.5), we get

$$\begin{aligned} F_1(r) = &\sum_{i=0}^{[\frac{n+1}{2}]} W_{0,i} r^{2i} I_{0,0}(r) + \sum_{i=0}^{[\frac{n}{2}]} Z_{2i,0}(r) r^{2i} \sum_{j=0}^{[\frac{n}{2}]-i} W_{2i+1,j} r^{2j} + \sum_{i=0}^{[\frac{n-1}{2}]} Z_{2i+1,0}(r) r^{2i+1} \sum_{j=0}^{[\frac{n-1}{2}]-i} W_{2i+2,j} r^{2j} \\ &- b \sum_{i=0}^{[\frac{n}{2}]} T_{0,i} r^{2i} I_{0,0}(r) - b \sum_{i=0}^{[\frac{n-1}{2}]} Z_{2i,0}(r) r^{2i} \sum_{j=0}^{[\frac{n-1}{2}]-i} T_{2i+1,j} r^{2j} \\ &- b \sum_{i=0}^{[\frac{n}{2}]-1} Z_{2i+1,0}(r) r^{2i+1} \sum_{j=0}^{[\frac{n}{2}]-1-i} T_{2i+2,j} r^{2j} + \sum_{i=0}^{[\frac{n}{2}]} T_{0,i} r^{2i} Y_{0,0}(r) + \sum_{i=0}^{[\frac{n-1}{2}]} m_{2i} c^* r^{2i} \sum_{j=0}^{[\frac{n-1}{2}]-i} T_{2i+1,j} r^{2j} \\ &+ \sum_{i=0}^{[\frac{n}{2}]-1} m_{2i+1} c^* r^{2i+1} \sum_{j=0}^{[\frac{n}{2}]-1-i} T_{2i+2,j} r^{2j}, \end{aligned} \tag{3.9}$$

where

$$W_{i,j} = \sum_{k=i}^{n+1-2j} P_{k,j}(-a)^{k-i}, \qquad T_{i,j} = \sum_{k=i}^{n-2j} Q_{k,j}(-a)^{k-i}.$$

And $F_2(r)$ can be similarly expressed as

$$\begin{aligned}
F_2(r) = & \sum_{i=0}^{[\frac{n+1}{2}]} \widetilde{W}_{0,i} r^{2i} J_{0,0}(r) + \sum_{i=0}^{[\frac{n}{2}]} \widetilde{Z}_{2i,0}(r) r^{2i} \sum_{j=0}^{[\frac{n}{2}]-i} \widetilde{W}_{2i+1,j} r^{2j} + \sum_{i=0}^{[\frac{n-1}{2}]} \widetilde{Z}_{2i+1,0}(r) r^{2i+1} \sum_{j=0}^{[\frac{n-1}{2}]-i} \widetilde{W}_{2i+2,j} r^{2j} \\
& - b \sum_{i=0}^{[\frac{n}{2}]} \widetilde{T}_{0,i} r^{2i} J_{0,0}(r) - b \sum_{i=0}^{[\frac{n-1}{2}]} \widetilde{Z}_{2i,0}(r) r^{2i} \sum_{j=0}^{[\frac{n-1}{2}]-i} \widetilde{T}_{2i+1,j} r^{2j} \\
& - b \sum_{i=0}^{[\frac{n}{2}]-1} \widetilde{Z}_{2i+1,0}(r) r^{2i+1} \sum_{j=0}^{[\frac{n}{2}]-1-i} \widetilde{T}_{2i+2,j} r^{2j} + \sum_{i=0}^{[\frac{n}{2}]} \widetilde{T}_{0,i} r^{2i} \widetilde{Y}_{0,0}(r) + \sum_{i=0}^{[\frac{n-1}{2}]} m_{2i} \widetilde{c}^* r^{2i} \sum_{j=0}^{[\frac{n-1}{2}]-i} \widetilde{T}_{2i+1,j} r^{2j} \\
& + \sum_{i=0}^{[\frac{n}{2}]-1} m_{2i+1} \widetilde{c}^* r^{2i+1} \sum_{j=0}^{[\frac{n}{2}]-1-i} \widetilde{T}_{2i+2,j} r^{2j},
\end{aligned}$$

where

$$\widetilde{W}_{i,j} = \sum_{k=i}^{n+1-2j} \widetilde{P}_{k,j} (-a)^{k-i}, \qquad \widetilde{T}_{i,j} = \sum_{k=i}^{n-2j} \widetilde{Q}_{k,j} (-a)^{k-i}.$$

Obviously, $W_{i,j}, T_{i,j}, \widetilde{W}_{i,j}$ and $\widetilde{T}_{i,j}$ are independent.

From Lemma 3.3, we have when $k \geq 1$

$$\begin{aligned}
r^{2k} I_{0,0}(r) = & \left(a^2+b^2\right)^k I_{0,0}(r) - b \sum_{i=0}^{k-1} \left(a^2+b^2\right)^{k-1-i} r^{2i} Y_{0,0}(r) \\
& + \sum_{i=0}^{k-1} \left(a^2+b^2\right)^{k-1-i} r^{2i+1} Z_{1,0}(r) - a \sum_{i=0}^{k-1} \left(a^2+b^2\right)^{k-1-i} r^{2i} Z_{0,0}(r).
\end{aligned} \tag{3.10}$$

Noting $W_{0,0} = -aW_{1,0}$ and substituting (3.10) into (3.9), we have

$$\begin{aligned}
F_1(r) = & \left[\sum_{i=0}^{[\frac{n-1}{2}]} T_{2i+1,0} (-b^2)^i - \frac{1}{b} \sum_{i=1}^{[\frac{n}{2}]} W_{2i+1,0} (-b^2)^i \right] (\pi - b Z_{0,0}(r)) \\
& + \left[T_{0,0} - b \sum_{j=1}^{[\frac{n+1}{2}]} (a^2+b^2)^{j-1} W_{0,j} + b^2 \sum_{j=1}^{[\frac{n}{2}]} T_{0,j} (a^2+b^2)^{j-1} \right] (Y_{0,0}(r) - b I_{0,0}(r)) \\
& + \left[W_{1,0} - a \sum_{j=1}^{[\frac{n+1}{2}]} (a^2+b^2)^{j-1} W_{0,j} + ab \sum_{j=1}^{[\frac{n}{2}]} T_{0,j} (a^2+b^2)^{j-1} \right] (Z_{0,0}(r) - a I_{0,0}(r)) \\
& + \left[\sum_{i=1}^{[\frac{n}{2}]} T_{0,i} - b \sum_{i=1}^{[\frac{n+1}{2}]-1} \sum_{j=i+1}^{[\frac{n+1}{2}]} (a^2+b^2)^{j-i-1} W_{0,j} + b^2 \sum_{i=1}^{[\frac{n}{2}]-1} \sum_{j=i+1}^{[\frac{n}{2}]} (a^2+b^2)^{j-i-1} T_{0,j} \right] r^{2i} Y_{0,0}(r) \\
& + \Bigg[\sum_{i=1}^{[\frac{n}{2}]} \sum_{t=0}^{[\frac{n}{2}]-i} \sum_{k+j=i} W_{2k+2t+1,j} C_{k+t}^t (-b^2)^t - b \sum_{i=1}^{[\frac{n-1}{2}]} \sum_{t=0}^{[\frac{n-1}{2}]-i} \sum_{k+j=i} T_{2k+2t+1,j} C_{k+t}^t (-b^2)^t \\
& - a \sum_{i=1}^{[\frac{n+1}{2}]-1} \sum_{j=i+1}^{[\frac{n+1}{2}]} (a^2+b^2)^{j-i-1} W_{0,j} + ab \sum_{i=1}^{[\frac{n}{2}]-1} \sum_{j=i+1}^{[\frac{n}{2}]} (a^2+b^2)^{j-i-1} T_{0,j} \Bigg] r^{2i} Z_{0,0}(r)
\end{aligned}$$

$$+\Bigg[\sum_{i=0}^{[\frac{n-1}{2}]}\sum_{t=0}^{[\frac{n-1}{2}]-i}\sum_{k+j=i}W_{2k+2t+2,j}C_{k+t}^{t}(-b^2)^t-b\sum_{i=0}^{[\frac{n}{2}]-1}\sum_{t=0}^{[\frac{n}{2}]-1-i}\sum_{k+j=i}T_{2k+2t+2,j}C_{k+t}^{t}(-b^2)^t$$
$$+\sum_{i=0}^{[\frac{n+1}{2}]-1}\sum_{j=i+1}^{[\frac{n+1}{2}]}(a^2+b^2)^{j-i-1}W_{0,j}-b\sum_{i=0}^{[\frac{n}{2}]-1}\sum_{j=i+1}^{[\frac{n}{2}]}(a^2+b^2)^{j-i-1}T_{0,j}\Bigg]r^{2i+1}Z_{1,0}(r)$$
$$+\Bigg[\sum_{i=1}^{[\frac{n-1}{2}]}\sum_{k+j=i}T_{2k+1,j}c^*m_{2k}+\sum_{i=1}^{[\frac{n}{2}]-1}\sum_{h=1}^{[\frac{n}{2}]-i}\sum_{k+j=i}W_{2k+2h+1,j}(-b^2)^{h-1}b^*\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t-1)}$$
$$-b\sum_{i=1}^{[\frac{n-1}{2}]-1}\sum_{h=1}^{[\frac{n-1}{2}]-i}\sum_{k+j=i}T_{2k+2h+1,j}(-b^2)^{h-1}b^*\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t-1)}\Bigg]r^{2i}$$
$$+\Bigg[\sum_{i=0}^{[\frac{n}{2}]-1}\sum_{k+j=i}T_{2k+2,j}c^*m_{2k+1}$$
$$+\sum_{i=0}^{[\frac{n-1}{2}]-1}\sum_{h=1}^{[\frac{n-1}{2}]-i}\sum_{k+j=i}W_{2k+2h+2,j}(-b^2)^{h-1}b^*\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t)-1}$$
$$-b\sum_{i=0}^{[\frac{n}{2}]-2}\sum_{h=1}^{[\frac{n}{2}]-1-i}\sum_{k+j=i}T_{2k+2h+2,j}(-b^2)^{h-1}b^*\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t)-1}\Bigg]r^{2i+1}.$$

Moreover, $F_1(r)$ can be expressed as

$$F_1(r)=A_1^{(1)}\Big(\pi-bZ_{0,0}(r)\Big)+A_2\Big(Z_{0,0}(r)-aI_{0,0}(r)\Big)+A_3\Big(Y_{0,0}(r)-bI_{0,0}(r)\Big)+\sum_{i=1}^{[\frac{n}{2}]}B_ir^{2i}Y_{0,0}(r)$$
$$+\sum_{i=1}^{[\frac{n}{2}]}D_i^{(1)}r^{2i}Z_{0,0}(r)+\sum_{i=0}^{[\frac{n-1}{2}]}E_i^{(1)}r^{2i+1}Z_{1,0}(r)+\sum_{i=1}^{[\frac{n-1}{2}]}F_i^{(1)}r^{2i}+\sum_{i=0}^{[\frac{n}{2}]-1}G_i^{(1)}r^{2i+1}.$$

Similarly, due to $\widetilde{W}_{0,0}=-a\widetilde{W}_{1,0}$, $F_2(r)$ takes the form

$$F_2(r)=A_1^{(2)}\Big(\pi-b\widetilde{Z}_{0,0}(r)\Big)+A_4\Big(\widetilde{Z}_{0,0}(r)-aJ_{0,0}(r)\Big)+A_5\Big(\widetilde{Y}_{0,0}(r)-bJ_{0,0}(r)\Big)+\sum_{i=1}^{[\frac{n}{2}]}C_ir^{2i}\widetilde{Y}_{0,0}(r)$$
$$+\sum_{i=1}^{[\frac{n}{2}]}D_i^{(2)}r^{2i}\widetilde{Z}_{0,0}(r)+\sum_{i=0}^{[\frac{n-1}{2}]}E_i^{(2)}r^{2i+1}\widetilde{Z}_{1,0}(r)+\sum_{i=1}^{[\frac{n-1}{2}]}F_i^{(2)}r^{2i}+\sum_{i=0}^{[\frac{n}{2}]-1}G_i^{(2)}r^{2i+1}.$$

Recalling that $Z_{0,0}=\widetilde{Z}_{0,0}, Z_{1,0}=-\widetilde{Z}_{1,0}$, we get

$$\begin{aligned}F(r)=&\ A_1\Big(\pi-bZ_{0,0}(r)\Big)+A_2\Big(Z_{0,0}(r)-aI_{0,0}(r)\Big)\\&+A_3\Big(Y_{0,0}(r)-bI_{0,0}(r)\Big)+A_4\Big(Z_{0,0}(r)-aJ_{0,0}(r)\Big)\\&+A_5\Big(\widetilde{Y}_{0,0}(r)-bJ_{0,0}(r)\Big)+\sum_{i=1}^{[\frac{n}{2}]}B_ir^{2i}Y_{0,0}(r)+\sum_{i=1}^{[\frac{n}{2}]}C_ir^{2i}\widetilde{Y}_{0,0}(r)+\sum_{i=1}^{\left[\frac{n}{2}\right]}D_ir^{2i}Z_{0,0}(r)\\&+\sum_{i=0}^{[\frac{n-1}{2}]}E_ir^{2i+1}Z_{1,0}(r)+\sum_{i=1}^{[\frac{n-1}{2}]}F_ir^{2i}+\sum_{i=0}^{[\frac{n}{2}]-1}G_ir^{2i+1}.\end{aligned}\tag{3.11}$$

where the coefficients $A_i(i=1,2\ldots,5)$, $B_i, C_i, D_i\left(i=1,\ldots,[\frac{n}{2}]\right)$, $E_i\left(i=0,1,\ldots,[\frac{n-1}{2}]\right)$, $F_i\left(i=1,\ldots,[\frac{n-1}{2}]\right)$ and $G_i\left(i=0,1,\ldots,[\frac{n}{2}]-1\right)$ are listed in Appendix.

In particular, for $b=0, a\neq 0$, recall $I_{i,0}(r)=J_{i,0}(r)=0$, $I_{i,1}(r)=\frac{1}{r}Y_{i,0}(r)$, and $J_{i,1}(r)=\frac{1}{r}\widetilde{Y}_{i,0}(r)$ for $i\geq 0$, then $F(r)$ takes the form

$$F(r)=\sum_{i=0}^{[\frac{n}{2}]} B_i r^{2i} Y_{0,0}(r)+\sum_{i=0}^{[\frac{n}{2}]} C_i r^{2i}\widetilde{Y}_{0,0}(r)+\sum_{i=0}^{[\frac{n-1}{2}]} F_i r^{2i}+\sum_{i=0}^{[\frac{n}{2}]-1} G_i r^{2i+1}, \tag{3.12}$$

where B_i, C_i, F_i and G_i are given by taking $b=0$ in the corresponding coefficient formulas of (3.11).

Choosing the appropriate parameter variables from (3.11) and (3.12), respectively, we can get the following Jacobian determinants

$$\left|\frac{\partial\left(A_4,A_3,B_1,\ldots,B_{[\frac{n}{2}]},A_5,C_1,\ldots,C_{[\frac{n}{2}]},A_2,D_1,\ldots,D_{[\frac{n}{2}]},E_0,\ldots,E_{\left[\frac{n-1}{2}\right]},A_1,F_1,\ldots,F_{\left[\frac{n-1}{2}\right]},G_0,\ldots,G_{[\frac{n}{2}]-1}\right)}{\partial\left(\widetilde{W}_{1,0},T_{0,0},T_{0,1}\ldots,T_{0,[\frac{n}{2}]},\widetilde{T}_{0,0},,\widetilde{T}_{0,1}\ldots,\widetilde{T}_{0,[\frac{n}{2}]},W_{1,0},W_{1,1},\ldots,W_{1,[\frac{n}{2}]},W_{2,0},\ldots,W_{2,\left[\frac{n-1}{2}\right]},T_{1,0},T_{1,1},\ldots,T_{1,\left[\frac{n-1}{2}\right]},T_{2,0},\ldots,T_{2,[\frac{n}{2}]-1}\right)}\right|$$
$$=(c^*)^{n-1}\neq 0,$$

and

$$\left|\frac{\partial\left(B_0,B_1,\ldots,B_{[\frac{n}{2}]},C_0,C_1,\ldots,C_{[\frac{n}{2}]},F_0,F_1,\ldots,F_{\left[\frac{n-1}{2}\right]},G_0,G_1,\ldots,G_{[\frac{n}{2}]-1}\right)}{\partial\left(T_{0,0},T_{0,1}\ldots,T_{0,[\frac{n}{2}]},\widetilde{T}_{0,0},,\widetilde{T}_{0,1}\ldots,\widetilde{T}_{0,[\frac{n}{2}]},T_{1,0},T_{1,1},\ldots,T_{1,\left[\frac{n-1}{2}\right]},T_{2,0},\ldots,T_{2,[\frac{n}{2}]-1}\right)}\right|=(c^*)^n\neq 0,$$

which imply Lemma 3.4 below.

Lemma 3.4. *For the functions $F(r)$ in (3.11) and (3.12), their coefficients are independent, respectively.*

4 Properties of some integrals

In this section, we study the properties of some integrals defined in Section 3, which play the important roles in the proof of Theorem 1.1.

A straightforward computation yields Lemma 4.1 below.

Lemma 4.1. *For the integrals $I_{0,0}(r), J_{0,0}(r), Y_{0,0}(r)$ and $\widetilde{Y}_{0,0}(r)$ with $ab\neq 0$, the following equalities hold.*

$$I_{0,0}(r)=\begin{cases}\dfrac{-4b\sqrt{b^2-r^2}\arctan\sqrt{\frac{a-r}{a+r}}+\sqrt{a^2-r^2}\sqrt{b^2-r^2}\ln\frac{b+r}{b-r}-a\pi\sqrt{a^2-r^2}}{(-a^2-b^2+r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2}}, & a>0, r\in(0,a),\\[2ex] \dfrac{-2b\sqrt{b^2-r^2}\ln\frac{r+\sqrt{r^2-a^2}}{a}+\sqrt{r^2-a^2}\sqrt{b^2-r^2}\ln\frac{b+r}{b-r}-a\pi\sqrt{r^2-a^2}}{(-a^2-b^2+r^2)\sqrt{r^2-a^2}\sqrt{b^2-r^2}}, & a>0, r\in(a,+\infty),\\[2ex] \frac{2}{ab}-\frac{1}{b^2}\ln\frac{a+b}{b-a}+\frac{a\pi}{b^2\sqrt{b^2-a^2}}, & a>0, r=a,\\[2ex] \dfrac{4b\sqrt{b^2-r^2}\arctan\sqrt{\frac{a-r}{a+r}}+\sqrt{a^2-r^2}\sqrt{b^2-r^2}\ln\frac{b+r}{b-r}-a\pi\sqrt{a^2-r^2}}{(-a^2-b^2+r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2}}, & a<0, r\in(0,-a).\end{cases}$$

$$J_{0,0}(r)=\begin{cases}\dfrac{-4b\sqrt{b^2-r^2}\arctan\sqrt{\frac{a+r}{a-r}}+\sqrt{a^2-r^2}\sqrt{b^2-r^2}\ln\frac{b-r}{b+r}-a\pi\sqrt{a^2-r^2}}{(-a^2-b^2+r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2}}, & a>0, r\in(0,a),\\[2ex] \dfrac{2b\sqrt{b^2-r^2}\ln\frac{r+\sqrt{r^2-a^2}}{-a}+\sqrt{r^2-a^2}\sqrt{b^2-r^2}\ln\frac{b-r}{b+r}-a\pi\sqrt{r^2-a^2}}{(-a^2-b^2+r^2)\sqrt{r^2-a^2}\sqrt{b^2-r^2}}, & a<0, r\in(-a,+\infty),\\[2ex] \frac{2}{ab}-\frac{1}{b^2}\ln\frac{a+b}{b-a}+\frac{a\pi}{b^2\sqrt{b^2-a^2}}, & a<0, r=-a,\\[2ex] \dfrac{4b\sqrt{b^2-r^2}\arctan\sqrt{\frac{a+r}{a-r}}+\sqrt{a^2-r^2}\sqrt{b^2-r^2}\ln\frac{b-r}{b+r}-a\pi\sqrt{a^2-r^2}}{(-a^2-b^2+r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2}}, & a<0, r\in(0,-a).\end{cases}$$

$$Y_{0,0}(r)=\begin{cases}\frac{-4}{\sqrt{a^2-r^2}}\arctan\sqrt{\frac{a-r}{a+r}}, & a<0, r\in(0,-a),\\ \frac{2}{\sqrt{r^2-a^2}}\ln\left(\frac{r+\sqrt{r^2-a^2}}{a}\right), & a>0, r\in(a,+\infty),\\ \frac{2}{a}, & a>0, r=a,\\ \frac{4}{\sqrt{a^2-r^2}}\arctan\sqrt{\frac{a-r}{a+r}}, & a>0, r\in(0,a).\end{cases}$$

$$\widetilde{Y}_{0,0}(r)=\begin{cases}\frac{4}{\sqrt{a^2-r^2}}\arctan\sqrt{\frac{a+r}{a-r}}, & a>0, r\in(0,a),\\ \frac{-2}{\sqrt{r^2-a^2}}\ln\left(\frac{r+\sqrt{r^2-a^2}}{-a}\right), & a<0, r\in(-a,+\infty),\\ \frac{2}{a}, & a<0, r=-a,\\ \frac{-4}{\sqrt{a^2-r^2}}\arctan\sqrt{\frac{a+r}{a-r}}, & a<0, r\in(0,-a).\end{cases}$$

$$Z_{0,0}(r)=\widetilde{Z}_{0,0}(r)=\frac{\pi}{\sqrt{b^2-r^2}}, \qquad b>0, r\in(0,b) \quad \textit{or} \quad b<0, r\in(0,-b).$$

$$Z_{1,0}(r)=-\widetilde{Z}_{1,0}(r)=\frac{1}{r}\ln\frac{b+r}{b-r}, \qquad b>0, r\in(0,b) \quad \textit{or} \quad b<0, r\in(0,-b).$$

Moreover, we have

Lemma 4.2. *For the integrals $Y_{0,0}(r), \widetilde{Y}_{0,0}(r), Z_{0,0}(r)$ and $Z_{1,0}(r)$, the statements given below are true.*

1. *If $a>0$, then $Y_{0,0}(r)$ can be analytically extended to the interval $(-a,+\infty)$. Furthermore, when $r\to(-a)^+, Y_{0,0}(r)\sim\frac{\sqrt{2}\pi}{\sqrt{a(a+r)}}$; when $r\to+\infty, Y_{0,0}(r)\sim\frac{2\ln(r)}{r}$.*
2. *If $a>0$, then $\widetilde{Y}_{0,0}(r)$ can be analytically extended to the interval $(-\infty,a)$. Furthermore, when $r\to a^-, \widetilde{Y}_{0,0}(r)\sim\frac{\sqrt{2}\pi}{\sqrt{a(a-r)}}$; when $r\to-\infty, \widetilde{Y}_{0,0}(r)\sim-\frac{2\ln(-r)}{r}$.*
3. *If $b>0$, then $Z_{0,0}(r)$ and $Z_{1,0}(r)$ can be analytically extended to the interval $(-b,b)$. Furthermore, when $r\to b^-$, $Z_{0,0}(r)\sim\frac{\pi}{\sqrt{2b(b-r)}}$, $Z_{1,0}(r)\sim\frac{1}{b}\ln\frac{2b}{b-r}$; when $r\to(-b)^+, Z_{0,0}(r)\sim\frac{\pi}{\sqrt{2b(b+r)}}$, $Z_{1,0}(r)\sim\frac{1}{b}\ln\frac{2b}{b+r}$.*

And other cases can be discussed similarly.

Proof. For $a>0$, when $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, the function $\frac{1}{a+r\cos\theta}$ is an analytic function of r in $(-a,+\infty)$, so $Y_{0,0}(r)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{a+r\cos\theta}d\theta$ is also analytic.

For $r\in(-a,0), Y_{0,0}(r)$ takes the same form as given for $r\in(0,a)$. Thus we have that when $r\to(-a)^+$,

$$Y_{0,0}(r)\sim\frac{\sqrt{2}\pi}{\sqrt{a(a+r)}},$$

when $r\to+\infty$,

$$Y_{0,0}(r)\sim\frac{2\ln r}{r}.$$

Similarly, we can prove other results. □

Lemma 4.3.

1. *If* $a > 0$, *then* $Y_{0,0}(r)$ *can be analytically extended to the complex domain* $D_1 = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \leq -a\}$.
2. *If* $a > 0$, *then* $\widetilde{Y}_{0,0}(r)$ *can be analytically extended to the complex domain* $D_2 = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \geq a\}$.
3. *If* $b > 0$, *then* $Z_{0,0}(r)$ *and* $Z_{1,0}(r)$ *can be analytically extended to the complex domain* $D_3 = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \leq -b, r \geq b\}$.

And other cases can be discussed similarly.

Proof. It is not difficult to know that $Y_{0,0}(r)$ takes the form

$$Y_{0,0}(r) = \frac{1}{\sqrt{a^2 - r^2}} \left(\pi - \int_0^r \frac{2}{\sqrt{a^2 - z^2}} dz \right).$$

The function $\sqrt{a^2 - r^2}$ is analytic in the domain $D^* = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r^2 - a^2 \geq 0\}$, so $Y_{0,0}(r)$ is analytic in the domain D^*. Together with Lemma 4.2, $Y_{0,0}(r)$ is analytic in $D_1 = \mathbb{C} \setminus \{r \in \mathbb{R} \mid r \leq -a\}$.

The other results can be obtained in a similar way. □

For $r \leq -a$ with $a > 0$, denote by $Y_{0,0}^+(r)$ ($Y_{0,0}^-(r)$, resp.) the analytic continuation of $Y_{0,0}(\bar{r})$ along an arc such that $Im(\bar{r}) > 0$ ($Im(\bar{r}) < 0$, resp.) for $\bar{r} \in D_1$. Similarly, we can define $\widetilde{Y}_{0,0}^{\pm}(r), Z_{0,0}^{\pm}(r)$ and $Z_{1,0}^{\pm}(r)$ for $r \in \mathbb{C} \setminus D_2$ and $r \in \mathbb{C} \setminus D_3$, respectively.

Lemma 4.4.

1. *If* $a > 0$, *then the functions* $Y_{0,0}^{\pm}(r)$ *defined in* $(-\infty, -a)$ *satisfy*

$$Y_{0,0}^+(r) - Y_{0,0}^-(r) = \frac{c_1 i}{\sqrt{r^2 - a^2}},$$

2. *If* $a > 0$, *then the functions* $\widetilde{Y}_{0,0}^{\pm}(r)$ *defined in* $(a, +\infty)$ *satisfy*

$$\widetilde{Y}_{0,0}^+(r) - \widetilde{Y}_{0,0}^-(r) = \frac{c_2 i}{\sqrt{r^2 - a^2}},$$

3. *If* $b > 0$, *then the functions* $Z_{0,0}^{\pm}(r)$ *defined in* $(-\infty, -b) \cup (b, +\infty)$ *satisfies*

$$Z_{0,0}^+(r) - Z_{0,0}^-(r) = \frac{2i\pi}{\sqrt{r^2 - b^2}}, \qquad r \in (b, +\infty),$$
$$Z_{0,0}^+(r) - Z_{0,0}^-(r) = \frac{-2i\pi}{\sqrt{r^2 - b^2}}, \qquad r \in (-\infty, -b).$$

4. *If* $b > 0$, *then the functions* $Z_{1,0}^{\pm}(r)$ *defined in* $(-\infty, -b) \cup (b, +\infty)$ *satisfy*

$$Z_{1,0}^+(r) - Z_{1,0}^-(r) = \frac{2i\pi}{r},$$

where c_1 *and* c_2 *are all nonzero real numbers and* $i^2 = -1$.

Proof. 1. A straightforward computation shows

$$\left(a^2 - r^2\right) Y'_{0,0}(r) = rY_{0,0}(r) - 2.$$

Noting that $Y^{\pm}_{0,0}(r)$ are the analytic continuation of $Y_{0,0}(r)$ for $r \in (-\infty, -a)$, we have

$$\left(a^2 - r^2\right) Y^{\pm\,\prime}_{0,0}(r) = rY^{\pm}_{0,0}(r) - 2,$$

which leads to

$$\left(a^2 - r^2\right)\left(Y^+_{0,0}(r) - Y^-_{0,0}(r)\right)' = r\left(Y^+_{0,0}(r) - Y^-_{0,0}(r)\right).$$

Solving the equation, we get that

$$Y^+_{0,0}(r) - Y^-_{0,0}(r) = \frac{c_1^*}{\sqrt{r^2 - a^2}},$$

where $c_1^* = c_1 i, c_1$ is a nonzero real number. Otherwise, $-a$ is the analytic point or the pole of $Y_{0,0}(r)$, which contradicts with the fact

$$Y_{0,0}(r) \sim \frac{\sqrt{2}\pi}{\sqrt{a(a+r)}}$$

as $r \to (-a)^+$.

The second result can be proved in a similar way.

2. Note that $Z^{\pm}_{0,0}(r)$ are both analytic continuation of $Z_{0,0}(r)$ for $r \in (b, +\infty)$, where b is not an analytic point, so we have

$$\begin{aligned} Z^+_{0,0}(r) - Z^-_{0,0}(r) &= \frac{\pi}{\sqrt{b+r}}|b-r|^{-\frac{1}{2}} e^{i\frac{\pi}{2}} - \frac{\pi}{\sqrt{b+r}}|b-r|^{-\frac{1}{2}} e^{-i\frac{\pi}{2}} \\ &= \frac{2i\pi}{\sqrt{r^2 - b^2}}. \end{aligned}$$

In a similar way, we can get the other results.

Hence we complete the proof of Lemma 4.4. □

5 Lower bound for the maximum number of zeros of averaged function

In this section we firstly prove the linearly independence of the generating functions of $F(r)$, then give the estimate on the lower bound for the maximum number of zeros of the averaged function $f(r)$.

Lemma 5.1. *For the function $F(r)$, we have*

1. *If $ab \neq 0$ and $|a| \neq |b|$, then the generating functions of the function $F(r)$ in (3.11) are the following $4[\frac{n}{2}] + 2[\frac{n-1}{2}] + 6$ linearly independent functions:*

$$\begin{aligned} &\pi - bZ_{0,0}(r),\ Z_{0,0}(r) - aI_{0,0}(r),\ Z_{0,0}(r) - aJ_{0,0}(r),\ Y_{0,0}(r) - bI_{0,0}(r),\ \widetilde{Y}_{0,0}(r) - bJ_{0,0}(r), \\ &r^2 Y_{0,0}(r),\ r^4 Y_{0,0}(r), \ldots, r^{2[\frac{n}{2}]} Y_{0,0}(r),\ r^2 \widetilde{Y}_{0,0}(r),\ r^4 \widetilde{Y}_{0,0}(r), \ldots, r^{2[\frac{n}{2}]} \widetilde{Y}_{0,0}(r), \\ &r^2 Z_{0,0}(r),\ r^4 Z_{0,0}(r), \ldots, r^{2[\frac{n}{2}]} Z_{0,0}(r),\ rZ_{1,0}(r),\ r^3 Z_{1,0}(r),\ r^5 Z_{1,0}(r), \ldots,\ r^{2[\frac{n-1}{2}]+1} Z_{1,0}(r), \\ &r^2, \ldots, r^{2[\frac{n-1}{2}]},\ r, r^3, \ldots, r^{2[\frac{n}{2}]-1}. \end{aligned} \tag{5.1}$$

2. *If $|a| = |b| \neq 0$, then the generating functions of the function $F(r)$ in (3.11) are the following $3[\frac{n}{2}] + 2[\frac{n-1}{2}] + 6$ linearly independent functions:*

$$\pi - bZ_{0,0}(r),\ Z_{0,0}(r) - aI_{0,0}(r),\ Z_{0,0}(r) - aJ_{0,0}(r),\ Y_{0,0}(r) - bI_{0,0}(r),\ \widetilde{Y}_{0,0}(r) - bJ_{0,0}(r),$$
$$r^2Y_{0,0}(r),\ r^4Y_{0,0}(r), \ldots, r^{2[\frac{n}{2}]}Y_{0,0}(r),\ r^2Z_{0,0}(r),\ r^4Z_{0,0}(r), \ldots, r^{2[\frac{n}{2}]}Z_{0,0}(r),$$
$$rZ_{1,0}(r),\ r^3Z_{1,0}(r),\ r^5Z_{1,0}(r), \ldots, r^{2[\frac{n-1}{2}]+1}Z_{1,0}(r),$$
$$r^2, \ldots, r^{2[\frac{n-1}{2}]},\ r, r^3, \ldots, r^{2[\frac{n}{2}]-1}.$$

3. *If $b = 0, a \neq 0$, then the generating functions of the function $F(r)$ in (3.12) are the following $3[\frac{n}{2}] + [\frac{n-1}{2}] + 3$ linearly independent functions:*

$$Y_{0,0}(r),\ r^2Y_{0,0}(r),\ r^4Y_{0,0}(r), \ldots, r^{2[\frac{n}{2}]}Y_{0,0}(r),$$
$$\widetilde{Y}_{0,0}(r),\ r^2\widetilde{Y}_{0,0}(r),\ r^4\widetilde{Y}_{0,0}(r), \ldots, r^{2[\frac{n}{2}]}\widetilde{Y}_{0,0}(r),$$
$$1, r^2, \ldots, r^{2[\frac{n-1}{2}]},\ r,\ r^3, \ldots, r^{2[\frac{n}{2}]-1}.$$

where $Y_{0,0}(r), \widetilde{Y}_{0,0}(r), Z_{0,0}(r)$ and $Z_{1,0}(r)$ are given by (3.8).

Proof. We only prove the first result, the other ones are similar.

From (3.11), we can analytically extend the domain of $F(r)$ to the complex plane $\mathbb{C} \setminus \{r \in \mathbb{R} \mid |r| \geq \min\{|a|, |b|\}\}$ for $ab \neq 0$, and suppose that there exist some coefficients such that $F(r) \equiv 0$, that is

$$\begin{aligned} F(r) = {} & a_1\Big(\pi - bZ_{0,0}(r)\Big) + a_2\Big(Z_{0,0}(r) - aI_{0,0}(r)\Big) \\ & + a_3\Big(Y_{0,0}(r) - bI_{0,0}(r)\Big) + a_4\Big(Z_{0,0}(r) - aJ_{0,0}(r)\Big) \\ & + a_5\Big(\widetilde{Y}_{0,0}(r) - bJ_{0,0}(r)\Big) + \sum_{i=1}^{[\frac{n}{2}]} b_i r^{2i} Y_{0,0}(r) + \sum_{i=1}^{[\frac{n}{2}]} c_i r^{2i} \widetilde{Y}_{0,0}(r) + \sum_{i=1}^{[\frac{n}{2}]} d_i r^{2i} Z_{0,0}(r) \\ & + \sum_{i=0}^{[\frac{n-1}{2}]} e_i r^{2i+1} Z_{1,0}(r) + \sum_{i=1}^{[\frac{n-1}{2}]} f_i r^{2i} + \sum_{i=0}^{[\frac{n}{2}]-1} g_i r^{2i+1} \equiv 0, \end{aligned} \tag{5.2}$$

we just need to prove $a_i = 0\ \ (i = 1, 2\ldots, 5), b_i = c_i = d_i = 0\ \ \big(i = 1, \ldots, [\frac{n}{2}]\big)$, $e_i = 0\ \ \big(i = 0, 1, \ldots, [\frac{n-1}{2}]\big)$, $f_i = 0\ \ \big(i = 1, \ldots, [\frac{n-1}{2}]\big)$, and $g_i = 0\ \ \big(i = 0, 1, \ldots, [\frac{n}{2}] - 1\big)$.

Obviously, $F(r) \equiv 0$ is equivalent to

$$V_*(r) := (-a^2 - b^2 + r^2)F(r) \equiv 0$$

for $r \in \mathbb{C} \setminus \{r \in \mathbb{R} | |r| \geq min\{|a|, |b|\}\}$ and $ab \neq 0$.

From (5.2), we have

$$\begin{aligned} V_*(r) = {} & \sum_{i=0}^{[\frac{n+2}{2}]} \overline{A}_i r^{2i} Y_{0,0}(r) + \sum_{i=0}^{[\frac{n+2}{2}]} \overline{B}_i r^{2i} \widetilde{Y}_{0,0}(r) + \sum_{i=0}^{[\frac{n+2}{2}]} \overline{C}_i r^{2i} Z_{0,0}(r) \\ & + \sum_{i=0}^{[\frac{n+1}{2}]} \overline{D}_i r^{2i+1} Z_{1,0}(r) + \sum_{i=0}^{[\frac{n-1}{2}]+1} \overline{E}_i r^{2i} + \sum_{i=0}^{[\frac{n}{2}]} \overline{F}_i r^{2i+1}, \end{aligned}$$

where

$$\begin{aligned}
&\overline{A}_0 = aba_2 - a^2a_3, \quad \overline{A}_1 = a_3 - (a^2+b^2)b_1, \quad \overline{A}_i = b_{i-1} - (a^2+b^2)b_i, \quad \overline{A}_{[\frac{n+2}{2}]} = b_{[\frac{n}{2}]},\\
&\overline{B}_0 = aba_4 - a^2a_5, \quad \overline{B}_1 = a_5 - (a^2+b^2)c_1, \quad \overline{B}_i = c_{i-1} - (a^2+b^2)c_i, \quad \overline{B}_{[\frac{n+2}{2}]} = c_{[\frac{n}{2}]},\\
&\overline{C}_0 = b(a^2+b^2)a_1 - b^2(a_2+a_4) + ab(a_3+a_5), \quad \overline{C}_1 = -ba_1 + a_2 + a_4 - (a^2+b^2)d_1,\\
&\overline{C}_i = d_{i-1} - (a^2+b^2)d_i, \quad \overline{C}_{[\frac{n+2}{2}]} = d_{[\frac{n}{2}]},\\
&\overline{D}_0 = a(a_4-a_2) + b(a_5-a_3) - (a^2+b^2)e_0, \quad \overline{D}_i = e_{i-1} - (a^2+b^2)e_i, \quad \overline{D}_{[\frac{n+1}{2}]} = e_{[\frac{n-1}{2}]},\\
&\overline{E}_0 = (-a^2-b^2)a_1\pi, \quad \overline{E}_1 = a_1\pi - (a^2+b^2)f_1, \quad \overline{E}_i = f_{i-1} - (a^2+b^2)f_i, \quad \overline{E}_{[\frac{n-1}{2}]+1} = f_{[\frac{n-1}{2}]},\\
&\overline{F}_0 = (-a^2-b^2)g_0, \qquad \overline{F}_i = g_{i-1} - (a^2+b^2)g_i, \qquad \overline{F}_{[\frac{n}{2}]} = g_{[\frac{n}{2}]-1}.
\end{aligned} \tag{5.3}$$

If $a > b > 0$, then $V_*(r)$ are analytic in $D = D_1 \cap D_2 \cap D_3 = \mathbb{C} \setminus \{r \in \mathbb{R} | r \geq b,\ r \leq -b\}$.

Similarly, we can define the functions $V_*^{\pm}(r)$ as the analytic continuation of $V_*(r)$ to $(-\infty, -b) \cup (b, +\infty)$ from the upper and lower half planes, respectively.

By Lemma 4.4, when $r \in (b, a)$, we have

$$V_*^+(r) - V_*^-(r) = \frac{2i\pi}{\sqrt{r^2-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} \overline{C}_i r^{2i} + \frac{2i\pi}{r} \sum_{i=0}^{[\frac{n+1}{2}]} \overline{D}_i r^{2i+1}.$$

Thus $V_*^+(r) - V_*^-(r) \equiv 0$ yields

$$\frac{1}{\sqrt{r^2-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} \overline{C}_i r^{2i} + \sum_{i=0}^{[\frac{n+1}{2}]} \overline{D}_i r^{2i} \equiv 0,$$

which implies that $\overline{C}_i = 0\ \left(i = 0,1,2,\ldots,[\frac{n+2}{2}]\right), \overline{D}_i = 0\ \left(i = 0,1,2,\ldots,[\frac{n+1}{2}]\right)$.

Similarly, when $r \in (a, +\infty)$,

$$V_*^+(r) - V_*^-(r) = \frac{c_2^*}{\sqrt{r^2-a^2}} \sum_{i=0}^{[\frac{n+2}{2}]} \overline{B}_i r^{2i} \equiv 0$$

if and only if $\overline{B}_i = 0\ \left(i = 0,1,2,\ldots,[\frac{n+2}{2}]\right)$.

And when $r \in (-\infty, -a)$,

$$V_*^+(r) - V_*^-(r) = \frac{c_1^*}{\sqrt{r^2-a^2}} \sum_{i=0}^{[\frac{n+2}{2}]} \overline{A}_i r^{2i} \equiv 0$$

if and only if $\overline{A}_i = 0\ \left(i = 0,1,2,\ldots,[\frac{n+2}{2}]\right)$.

Consequently, $V_*(r) \equiv 0$ becomes

$$\sum_{i=0}^{[\frac{n-1}{2}]+1} \overline{E}_i r^{2i} + \sum_{i=0}^{[\frac{n}{2}]} \overline{F}_i r^{2i+1} \equiv 0,$$

so we get $\overline{E}_i = 0\ \left(i = 0,1,2,\ldots,[\frac{n-1}{2}]+1\right)$ and $\overline{F}_i = 0\ \left(i = 0,1,2,\ldots,[\frac{n}{2}]\right)$.

The above results lead to $a_i = 0\ (i = 1,2,\ldots,5), b_i = c_i = d_i = 0\ \left(i = 1,\ldots,[\frac{n}{2}]\right),\ e_i = 0\ \left(i = 0,1,\ldots,[\frac{n-1}{2}]\right),\ f_i = 0\ \left(i = 1,\ldots,[\frac{n-1}{2}]\right)$, and $g_i = 0\ \left(i = 0,1,\ldots,[\frac{n}{2}]-1\right)$.

Hence, for the case $a > b > 0$, the functions listed in (5.1) are $4[\frac{n}{2}] + 2[\frac{n-1}{2}] + 6$ linearly independent generating ones of $F(r)$.

The other cases for $ab \neq 0, |a| \neq |b|$ can be investigated in the similar way.

This completes the proof of the first result in Lemma 5.1. □

Recall that $[\frac{n}{2}]+[\frac{n-1}{2}]=n-1$, then Lemma 5.2 follows from Remark 3.1, Lemmas 2.3 and 5.1.

Lemma 5.2. *Denoting by $N(f)$ the maximum number of simple zeros of the averaged function $f(r)$ in $r\in(0,min\{|a|,|b|\})$ for $ab\neq 0$ or $r\in(0,|a|)$ for $b=0,a\neq 0$, we have*

1. *when $ab\neq 0$ and $|a|\neq|b|$, $N(f)\geq 2\left[\frac{n}{2}\right]+2n+3$;*
2. *when $|a|=|b|\neq 0$, $N(f)\geq\left[\frac{n}{2}\right]+2n+3$;*
3. *when $b=0,a\neq 0,N(f)\geq 2\left[\frac{n}{2}\right]+n+1$,*

where $[\cdot]$ denotes the integer function.

6 Upper bound for the number of zeros of averaged function

In this section, we extend the variable r to the complex plane to investigate the upper bound for the number of zeros of the function $F(r)$, which is closely related to that of the averaged function $f(r)$.

Here, we look r as the complex number.

Lemma 6.1. *For the complex variable r, we have*

1. *If $a>0$, then when $r\to(-a)^+,Y_{0,0}(r)\sim\frac{\sqrt{2}\pi}{\sqrt{a(a+r)}}$; when $r\to\infty,Y_{0,0}(r)\sim\frac{2\ln(r)}{r}$.*
2. *If $a>0$, then when $r\to a^-,\widetilde{Y}_{0,0}(r)\sim\frac{\sqrt{2}\pi}{\sqrt{a(a-r)}}$; when $r\to\infty,\widetilde{Y}_{0,0}(r)\sim-\frac{2\ln(-r)}{r}$.*
3. *If $b>0$, then when $r\to b^-,Z_{0,0}(r)\sim\frac{\pi}{\sqrt{2b(b-r)}}$, $Z_{1,0}(r)\sim\frac{1}{b}\ln\frac{2b}{b-r}$; when $r\to(-b)^+$, $Z_{0,0}(r)\sim\frac{\pi}{\sqrt{2b(b+r)}}$, $Z_{1,0}(r)\sim\frac{1}{b}\ln\frac{2b}{b+r}$.*

And other cases can be discussed similarly.

Proof. For $Y_{0,0}(r)$, we have

$$\begin{aligned}\left(a^2-r^2\right)Y'_{0,0}(r)&=rY_{0,0}(r)-2,\\ \left(a^2-r^2\right)Y''_{0,0}(r)-3rY'_{0,0}(r)-Y_{0,0}(r)&=0.\end{aligned}\tag{6.1}$$

It is easy to check that ∞ and $-a$ are the regular singular points (see [27]) of the second equation in (6.1), so when the complex variable $r\to-a$ and ∞, the solution $Y_{0,0}(r)$ has the same properties as for the real number r. This fact together with Lemma 4.2 yields the first results.

Similarly, the other results can be proved. □

The results concerning the number of zeros of $F(r)$ are given as follows.

Lemma 6.2. *When $|a|>|b|\neq 0$, we have*

$$N^*(F)\leq 2\left[\frac{n}{2}\right]+4n+14,$$

where $N^*(F)$ denotes the maximum number of non-zero simple zeros of $F(r)$ in $D=D_1\cap D_2\cap D_3=\mathbb{C}\setminus\{r\in\mathbb{R}\mid|r|\geq|b|\}$, and $[\cdot]$ is the integer function.

Proof. We only give the detailed proof for the case $a > b > 0$, other results can be proved in a similar way.

Let $0 < \varepsilon \ll 1 \ll R$ and $D_{\varepsilon,R}$ be the domain obtained by removing four small discs

$$C_{b,\varepsilon} = \{r \in \mathbb{C} \mid |r-b| \le \varepsilon\}, \qquad C_{a,\varepsilon} = \{r \in \mathbb{C} \mid |r-a| \le \varepsilon\},$$
$$C_{-b,\varepsilon} = \{r \in \mathbb{C} \mid |r+b| \le \varepsilon\}, \qquad C_{-a,\varepsilon} = \{r \in \mathbb{C} \mid |r+a| \le \varepsilon\}$$

and four real intervals

$$L_1 = [b+\varepsilon, a-\varepsilon], \qquad L_2 = [a+\varepsilon, R],$$
$$L_3 = [-a+\varepsilon, -b-\varepsilon], \qquad L_4 = [-R, -a-\varepsilon]$$

from $C_R = \{r \in \mathbb{C} \mid |r| \le R\}$, $L_i^{\pm}$ be the upper and lower bounds of L_i for $i = 1,2,3,4$, see Figure 6.1.

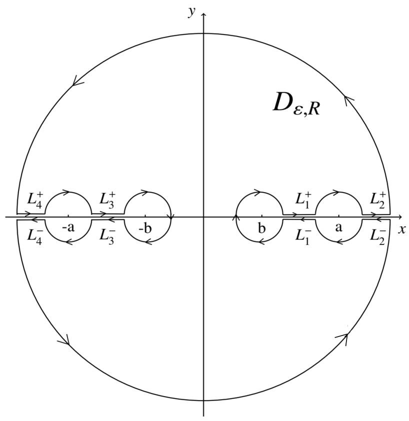

Figure 6.1: The domain $D_{\varepsilon,R}$

Recalling $F(r)$ in (3.11), we define

$$\begin{aligned} V(r) &:= \left(r^2 - a^2 - b^2\right) F(r) \\ &= \sum_{i=0}^{[\frac{n+2}{2}]} \widetilde{A}_i r^{2i} Y_{0,0}(r) + \sum_{i=0}^{[\frac{n+2}{2}]} \widetilde{B}_i r^{2i} \widetilde{Y}_{0,0}(r) + \sum_{i=0}^{[\frac{n+2}{2}]} \widetilde{C}_i r^{2i} Z_{0,0}(r) \\ &\quad + \sum_{i=0}^{[\frac{n+1}{2}]} \widetilde{D}_i r^{2i+1} Z_{1,0}(r) + \sum_{i=0}^{[\frac{n-1}{2}]+1} \widetilde{E}_i r^{2i} + \sum_{i=0}^{[\frac{n}{2}]} \widetilde{F}_i r^{2i+1}, \end{aligned}$$

where $\widetilde{A}_i, \widetilde{B}_i, \widetilde{C}_i, \widetilde{D}_i, \widetilde{E}_i$ and $\widetilde{F}_i$ are similar to the coefficients of (5.3), with a_i $(i = 1,2\ldots,5)$, b_i, c_i, d_i $\left(i = 1,\ldots,[\frac{n}{2}]\right)$, e_i $\left(i = 0,1,\ldots,[\frac{n-1}{2}]\right)$, f_i $\left(i = 1,\ldots,[\frac{n-1}{2}]\right)$ and g_i $\left(i = 0,1,\ldots,[\frac{n}{2}]-1\right)$ being replaced by A_i $(i = 1,2\ldots,5)$, B_i, C_i, D_i $\left(i = 1,\ldots,[\frac{n}{2}]\right)$, E_i $\left(i = 0,1,\ldots,[\frac{n-1}{2}]\right)$, F_i $\left(i = 1,\ldots,[\frac{n-1}{2}]\right)$ and G_i $\left(i = 0,1,\ldots,[\frac{n}{2}]-1\right)$ in (3.11), respectively.

Since the zeros of $F(r)$ coincide with those of $V(r)$ for $r \in D$, we investigate the latter instead of the former. Now consider the change of the argument of $V(r)$ along the boundary of the domain $D_{\varepsilon,R}$.

On $\partial C_{b,\varepsilon}$, $V(r) \sim \frac{C^*}{\sqrt{b-r}}$, where C^* is a constant. This means that the argument of $V(r)$ increases by $\pi + o(1)$. Since $V(b-\varepsilon)$ is a real number, it intersects the real axis only once.

On L_1^+, $V(r)$ is real if and only if $\operatorname{Im} V(r) = 0$, that is

$$0 = V^+(r) - V^-(r) = \frac{2i\pi}{\sqrt{r^2-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} C_i r^{2i} + \frac{2i\pi}{r} \sum_{i=0}^{[\frac{n+1}{2}]} D_i r^{2i+1},$$

where $V^{\pm}(r)$ denote the analytic continuation of $V(r)$ to $(-\infty, -b) \cup (b, +\infty)$ from the upper and lower half planes, respectively. Then letting $u = r^2$, we can get that

$$\sum_{i=0}^{[\frac{n+2}{2}]} C_i u^i + \sqrt{u-b^2} \sum_{i=0}^{[\frac{n+1}{2}]} D_i u^i = 0.$$

Define

$$V_1(u) = \sum_{i=0}^{[\frac{n+2}{2}]} C_i u^i + \sqrt{u-b^2} \sum_{i=0}^{[\frac{n+1}{2}]} D_i u^i = 0,$$

and denote by n_1 the number of the root of the function $V_1(u)$. Then $V(u)$ intersects the real axis exactly n_1 times, which holds for L_1^-. So the argument of $V(r)$ increases by at most $2(n_1+1)\pi + o(1)$ on $\partial C_{b,\varepsilon} \cup L_1^{\pm}$.

Similarly, the argument of $V(r)$ increases by at most $2(n_2+1)\pi + o(1)$ on $\partial C_{-b,\varepsilon} \cup L_3^{\pm}$, where n_2 is the number of zeros of the function $V_2(u)$ defined by

$$V_2(u) = -\sum_{i=0}^{[\frac{n+2}{2}]} C_i u^i + \sqrt{u-b^2} \sum_{i=0}^{[\frac{n+1}{2}]} D_i u^i.$$

On $\partial C_{a,\varepsilon}$, $V(r) \sim \frac{C^{**}}{\sqrt{a-r}}$ for some constant C^{**}, which implies that the argument of $V(r)$ increases by at most $\pi + o(1)$.

On L_2^+, $V(r)$ is real if and only if $\operatorname{Im} V(r) = 0$, that is

$$\begin{aligned} 0 = \frac{V^+(r) - V^-(r)}{i} &= \frac{c_2}{\sqrt{r^2-a^2}} \sum_{i=0}^{[\frac{n+2}{2}]} B_i r^{2i} + \frac{2\pi}{\sqrt{r^2-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} C_i r^{2i} + \frac{2\pi}{r} \sum_{i=0}^{[\frac{n+1}{2}]} D_i r^{2i+1} \\ &= \frac{1}{\sqrt{r^2-a^2}} \sum_{i=0}^{[\frac{n+2}{2}]} B_i^* r^{2i} + \frac{1}{\sqrt{r^2-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} C_i^* r^{2i} + \sum_{i=0}^{[\frac{n+1}{2}]} D_i^* r^{2i}, \end{aligned} \tag{6.2}$$

where $B_i^* = c_2 B_i, C_i^* = 2\pi C_i, D_i^* = 2\pi D_i$. Let $u = r^2$, then the function (6.2) becomes

$$\frac{1}{\sqrt{u-a^2}} \sum_{i=0}^{[\frac{n+2}{2}]} B_i^* u^i + \frac{1}{\sqrt{u-b^2}} \sum_{i=0}^{[\frac{n+2}{2}]} C_i^* u^i + \sum_{i=0}^{[\frac{n+1}{2}]} D_i^* u^i = 0. \tag{6.3}$$

By the Argument Principle, the number of roots of (6.3) is not greater than $n + [\frac{n}{2}] + 4$ on L_2^+. So $V(r)$ intersects the real axis at most $n + [\frac{n}{2}] + 4$ times, which is true for L_2^-. Then the argument of $V(r)$ totally increases by at most $2\left(n + [\frac{n}{2}] + 5\right)\pi + o(1)$ on $C_{a,\varepsilon} \cup L_2^{\pm}$.

In a similar way, we get that the argument of $V(r)$ increases by at most $2\left(n + [\frac{n}{2}] + 5\right)\pi + o(1)$ on $C_{-a,\varepsilon} \cup L_4^{\pm}$.

On C_R, we have

$$r^{2[\frac{n+2}{2}]} Y_{0,0}(r) \sim r^{2[\frac{n+2}{2}]-1} \ln r, \qquad r^{2[\frac{n+2}{2}]} \widetilde{Y}_{0,0}(r) \sim r^{2[\frac{n+2}{2}]-1} \ln(-r),$$
$$r^{2[\frac{n+2}{2}]} Z_{0,0}(r) \sim r^{2[\frac{n+2}{2}]-1}, \qquad r^{2[\frac{n+1}{2}]+1} Z_{1,0}(r) \sim r^{2[\frac{n+1}{2}]},$$

then the corresponding arguments of these terms increase by $2\left(2[\frac{n+2}{2}]-1\right)\pi+o(1)$, $2\left(2[\frac{n+2}{2}]-1\right)\pi+o(1)$, $2\left(2[\frac{n+2}{2}]-1\right)\pi$, $2\left(2[\frac{n+1}{2}]\right)\pi$, respectively. And the argument of r^{n+1} increases by $2(n+1)\pi$. Since $2[\frac{n+2}{2}]-1\leq n+1, 2[\frac{n+1}{2}]\leq n+1$, the argument of $V(r)$ increases by at most $2(n+1)\pi+o(1)$.

So along the boundary of $D_{\varepsilon,R}$, the arguments of $V(r)$ increase by at most

$$\begin{aligned}&2(n_1+1)\pi+2\left(n+\left[\frac{n}{2}\right]+5\right)\pi+2(n_2+1)\pi+2\left(n+\left[\frac{n}{2}\right]+5\right)\pi+2(n+1)\pi+o(1)\\&\quad=2\left(3n+n_1+n_2+2\left[\frac{n}{2}\right]+13\right)\pi+o(1).\end{aligned}$$

On the other hand, a straightforward computation yields

$$\begin{aligned}V_1(u)\cdot V_2(u)&=\left(\sum_{i=0}^{[\frac{n+2}{2}]}C_iu^i+\sqrt{u-b^2}\sum_{i=0}^{[\frac{n+1}{2}]}D_iu^i\right)\cdot\left(-\sum_{i=0}^{[\frac{n+2}{2}]}C_iu^i+\sqrt{u-b^2}\sum_{i=0}^{[\frac{n+1}{2}]}D_iu^i\right)\\&=(u-b^2)\sum_{i=0}^{2[\frac{n+1}{2}]}\widetilde{D}_iu^i-\sum_{i=0}^{2[\frac{n+2}{2}]}\widetilde{C}_iu^i,\end{aligned}$$

which implies that $V_1(u)\cdot V_2(u)=0$ has at most $n+2$ zeros, taking into account the multiplicities. Thus we have $n_1+n_2\leq n+2$. By the Argument Principle, $V(r)$ has at most $4n+2[\frac{n}{2}]+15$ zeros in $D_{\varepsilon,R}$.

Let $\varepsilon\to 0$ and $R\to+\infty$, then we have that $V(r)$ has at most $4n+2\left[\frac{n}{2}\right]+15$ zeros in $D=D_1\cap D_2\cap D_3$ obtained by removing two real intervals $(-\infty,-b]$ and $[b,+\infty)$ from $\mathbb{C}$. Since $V(0)=0$, we get that

$$N^*(V)\leq 2\left[\frac{n}{2}\right]+4n+14,$$

where $N^*(V)$ denotes the maximum number of the non-zero simple zeros of $V(r)$. Thus we also have

$$N^*(F)\leq 2\left[\frac{n}{2}\right]+4n+14.$$

This completes the proof of Lemma 6.2 for the case $a>b>0$. □

Similarly, we have

Lemma 6.3.

1. *When $|b|>|a|\neq 0$, $N^*(F)\leq 4[\frac{n}{2}]+3n+14$.*
2. *When $|a|=|b|\neq 0$, $N^*(F)\leq 3n+6$.*
3. *When $b=0, a\neq 0$, $N^*(F)\leq 2[\frac{n}{2}]+n+1$, where $N^*(F)$ denotes the maximum number of non-zero simple zeros of $F(r)$ in $r\in(0,min\{|a|,|b|\})$ for $ab\neq 0$ or $r\in(0,|a|)$ for $b=0,a\neq 0$.*

Based on Remark 3.1, we get the upper bound of zeros of the averaged function $f(r)$, which is stated as follows.

Lemma 6.4. *Let $N(f)$ be the maximum number of simple zeros of $f(r)$ in $r\in(0,min\{|a|,|b|\})$ for $ab\neq 0$ or $r\in(0,|a|)$ for $b=0,a\neq 0$, then the following statements hold.*

1. *When $ab\neq 0$ and $|a|>|b|$, $N(f)\leq 2[\frac{n}{2}]+4n+14$;*
2. *When $ab\neq 0$ and $|a|<|b|$, $N(f)\leq 4[\frac{n}{2}]+3n+14$;*

3. *When* $|a| = |b| \neq 0,\ N(f) \leq 3n + 6$;
4. *When* $b = 0, a \neq 0,\ N(f) \leq 2[\frac{n}{2}] + n + 1$,

where $[\cdot]$ *is the integer function.*

7 Proof of Theorem 1.1

Following Lemmas 5.2 and 6.4, we obtain Lemma 7.1 below.

Lemma 7.1. *The following statements are true.*

1. *When* $|a| > |b| \neq 0,\ 2[\frac{n}{2}] + 2n + 3 \leq N(f) \leq 2[\frac{n}{2}] + 4n + 14$;
2. *When* $|b| > |a| \neq 0,\ 2[\frac{n}{2}] + 2n + 3 \leq N(f) \leq 4[\frac{n}{2}] + 3n + 14$;
3. *When* $|a| = |b| \neq 0,\ [\frac{n}{2}] + 2n + 3 \leq N(f) \leq 3n + 6$;
4. *When* $b = 0, a \neq 0,\ N(f) = 2[\frac{n}{2}] + n + 1$,

where $N(f)$ *is the same as defined by Lemma 6.4.*

Proof of Theorem 1.1. By the first order averaging theory, the number of non-zero simple zeros of the averaged function $f(r)$ corresponds to that of limit cycles bifurcating from the period annulus around the center of the unperturbed systems (1.1). Then Theorem 1.1 follows from Lemmas 7.1 and 2.1.

This completes the proof of Theorem 1.1. □

Acknowledgements

We would like to thank the referee for the helpful comments and suggestions. This research is supported by the Natural Science Foundation of Beijing, China (No. 1202018), and Intelligent Innovation Fund Project of School of Mathematical Sciences, Beihang University (ZCJJ-2023-01-03).

Appendix. Coefficients of the function $F(r)$ in (3.11)

For the odd number n, the coefficients in (3.11) take the form

$$A_1 = \sum_{i=0}^{\frac{n-1}{2}} \Big(T_{2i+1,0} + \widetilde{T}_{2i+1,0}\Big)\Big(-b^2\Big)^i - \frac{1}{b}\sum_{i=1}^{\frac{n-1}{2}} \Big(W_{2i+1,0} + \widetilde{W}_{2i+1,0}\Big)\Big(-b^2\Big)^i,$$

$$A_2 = W_{1,0} - a\sum_{j=1}^{\frac{n+1}{2}} W_{0,j}\Big(a^2+b^2\Big)^{j-1} + ab\sum_{j=1}^{\frac{n-1}{2}} T_{0,j}\Big(a^2+b^2\Big)^{j-1},$$

$$A_3 = T_{0,0} - b\sum_{j=1}^{\frac{n+1}{2}} W_{0,j}\Big(a^2+b^2\Big)^{j-1} + b^2\sum_{j=1}^{\frac{n-1}{2}} T_{0,j}\Big(a^2+b^2\Big)^{j-1},$$

$$A_4 = \widetilde{W}_{1,0} - a\sum_{j=1}^{\frac{n+1}{2}} \widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-1} + ab\sum_{j=1}^{\frac{n-1}{2}} \widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-1},$$

$$A_5 = \widetilde{T}_{0,0} - b\sum_{j=1}^{\frac{n+1}{2}} \widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-1} + b^2\sum_{j=1}^{\frac{n-1}{2}} \widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-1},$$

$$B_i = T_{0,i} - b\sum_{j=i+1}^{\frac{n+1}{2}} W_{0,j}\Big(a^2+b^2\Big)^{j-i-1} + b^2\sum_{j=i+1}^{\frac{n-1}{2}} T_{0,j}\Big(a^2+b^2\Big)^{j-i-1},$$

$$B_{\left[\frac{n}{2}\right]} = T_{0,\frac{n-1}{2}} - bW_{0,\frac{n+1}{2}},$$

$$C_i = \widetilde{T}_{0,i} - b\sum_{j=i+1}^{\frac{n+1}{2}} \widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-i-1} + b^2\sum_{j=i+1}^{\frac{n-1}{2}} \widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-i-1},$$

$$C_{\left[\frac{n}{2}\right]} = \widetilde{T}_{0,\frac{n-1}{2}} - b\widetilde{W}_{0,\frac{n+1}{2}},$$

$$\begin{aligned} D_i = &\sum_{t=0}^{\frac{n-1}{2}-i}\sum_{k+j=i}\Big(W_{2k+2t+1,j}+\widetilde{W}_{2k+2t+1,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t - a\sum_{j=i+1}^{\frac{n+1}{2}}\Big(W_{0,j}+\widetilde{W}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1} \\ &- b\sum_{t=0}^{\frac{n-1}{2}-i}\sum_{k+j=i}\Big(T_{2k+2t+1,j}+\widetilde{T}_{2k+2t+1,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t + ab\sum_{j=i+1}^{\frac{n-1}{2}}\Big(T_{0,j}+\widetilde{T}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1}, \end{aligned}$$

$$D_{\left[\frac{n}{2}\right]} = \sum_{k+j=\frac{n-1}{2}}\Big(W_{2k+1,j}+\widetilde{W}_{2k+1,j}\Big) - b\sum_{k+j=\frac{n-1}{2}}\Big(T_{2k+1,j}+\widetilde{T}_{2k+1,j}\Big) - a\Big(W_{0,\frac{n+1}{2}}+\widetilde{W}_{0,\frac{n+1}{2}}\Big),$$

$$\begin{aligned} E_i = &\sum_{t=0}^{\frac{n-1}{2}-i}\sum_{k+j=i}\Big(W_{2k+2t+2,j}-\widetilde{W}_{2k+2t+2,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t + \sum_{j=i+1}^{\frac{n+1}{2}}\Big(W_{0,j}-\widetilde{W}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1} \\ &- b\sum_{t=0}^{\frac{n-1}{2}-i-1}\sum_{k+j=i}\Big(T_{2k+2t+2,j}-\widetilde{T}_{2k+2t+2,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t - b\sum_{j=i+1}^{\frac{n-1}{2}}\Big(T_{0,j}-\widetilde{T}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1}, \end{aligned}$$

$$E_{\left[\frac{n-1}{2}\right]} = \sum_{k+j=\frac{n-1}{2}}\Big(W_{2k+2,j}-\widetilde{W}_{2k+2,j}\Big) + \Big(W_{0,\frac{n+1}{2}}-\widetilde{W}_{0,\frac{n+1}{2}}\Big),$$

$$\begin{aligned} F_i = &\sum_{k+j=i}\Big(T_{2k+1,j}c^* + \widetilde{T}_{2k+1,j}\widetilde{c}^*\Big)m_{2k} \\ &+ \sum_{h=1}^{\frac{n-1}{2}-i}\sum_{k+j=i}(-b^2)^{h-1}\Big(W_{2k+2h+1,j}b^* + \widetilde{W}_{2k+2h+1,j}\widetilde{b}^*\Big)\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t-1)} \\ &- b\sum_{h=1}^{\frac{n-1}{2}-i}\sum_{k+j=i}\Big(T_{2k+2h+1,j}b^* + \widetilde{T}_{2k+2h+1,j}\widetilde{b}^*\Big)\Big(-b^2\Big)^{h-1}\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t-1)}, \end{aligned}$$

$$F_{\left[\frac{n-1}{2}\right]} = \sum_{k+j=\frac{n-1}{2}}\Big(T_{2k+1,j}c^* + \widetilde{T}_{2k+1,j}\widetilde{c}^*\Big)m_{2k},$$

$$\begin{aligned} G_i = &\sum_{k+j=i}\Big(T_{2k+2,j}c^* + \widetilde{T}_{2k+2,j}\widetilde{c}^*\Big)m_{2k+1} \\ &+ \sum_{h=1}^{\frac{n-1}{2}-i}\sum_{k+j=i}(-b^2)^{h-1}\Big(W_{2k+2h+2,j}b^* + \widetilde{W}_{2k+2h+2,j}\widetilde{b}^*\Big)\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t)-1} \\ &- b\sum_{h=1}^{\frac{n-3}{2}-i}\sum_{k+j=i}(-b^2)^{h-1}\Big(T_{2k+2h+2,j}b^* + \widetilde{T}_{2k+2h+2,j}\widetilde{b}^*\Big)\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t)-1}, \end{aligned}$$

$$G_{\left[\frac{n}{2}\right]-1}=\sum_{k+j=\frac{n-3}{2}}\Big(T_{2k+2,j}c^*+\widetilde{T}_{2k+2,j}\widetilde{c}^*\Big)m_{2k+1}+\sum_{k+j=\frac{n-3}{2}}\Big(W_{2k+4,j}b^*+\widetilde{W}_{2k+4,j}\widetilde{b}^*\Big)\sum_{t=0}^{k}m_{2(k-t)+1}.$$

For the even number n, the coefficients in (3.11) take the form

$$\begin{aligned}
A_1&=\sum_{i=0}^{\frac{n}{2}-1}\Big(T_{2i+1,0}+\widetilde{T}_{2i+1,0}\Big)\Big(-b^2\Big)^i-\frac{1}{b}\sum_{i=1}^{\frac{n}{2}}\Big(W_{2i+1,0}+\widetilde{W}_{2i+1,0}\Big)\Big(-b^2\Big)^i,\\
A_2&=W_{1,0}-a\sum_{j=1}^{\frac{n}{2}}W_{0,j}\Big(a^2+b^2\Big)^{j-1}+ab\sum_{j=1}^{\frac{n}{2}}T_{0,j}\Big(a^2+b^2\Big)^{j-1},\\
A_3&=T_{0,0}-b\sum_{j=1}^{\frac{n}{2}}W_{0,j}\Big(a^2+b^2\Big)^{j-1}+b^2\sum_{j=1}^{\frac{n}{2}}T_{0,j}\Big(a^2+b^2\Big)^{j-1},\\
A_4&=\widetilde{W}_{1,0}-a\sum_{j=1}^{\frac{n}{2}}\widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-1}+ab\sum_{j=1}^{\frac{n}{2}}\widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-1},\\
A_5&=\widetilde{T}_{0,0}-b\sum_{j=1}^{\frac{n}{2}}\widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-1}+b^2\sum_{j=1}^{\frac{n}{2}}\widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-1},\\
B_i&=T_{0,i}-b\sum_{j=i+1}^{\frac{n}{2}}W_{0,j}\Big(a^2+b^2\Big)^{j-i-1}+b^2\sum_{j=i+1}^{\frac{n}{2}}T_{0,j}\Big(a^2+b^2\Big)^{j-i-1},\\
B_{\left[\frac{n}{2}\right]}&=T_{0,\frac{n}{2}},\\
C_i&=\widetilde{T}_{0,i}-b\sum_{j=i+1}^{\frac{n}{2}}\widetilde{W}_{0,j}\Big(a^2+b^2\Big)^{j-i-1}+b^2\sum_{j=i+1}^{\frac{n}{2}}\widetilde{T}_{0,j}\Big(a^2+b^2\Big)^{j-i-1},\\
C_{\left[\frac{n}{2}\right]}&=\widetilde{T}_{0,\frac{n}{2}},
\end{aligned}$$

$$\begin{aligned}
D_i=&\sum_{t=0}^{\frac{n}{2}-i}\sum_{k+j=i}\Big(W_{2k+2t+1,j}+\widetilde{W}_{2k+2t+1,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t-a\sum_{j=i+1}^{\frac{n}{2}}\Big(W_{0,j}+\widetilde{W}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1}\\
&-b\sum_{t=0}^{\frac{n}{2}-i-1}\sum_{k+j=i}\Big(T_{2k+2t+1,j}+\widetilde{T}_{2k+2t+1,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t+ab\sum_{j=i+1}^{\frac{n}{2}}\Big(T_{0,j}+\widetilde{T}_{0,j}\Big)\Big(a^2+b^2\Big)^{j-i-1},\\
D_{[\frac{n}{2}]}=&\sum_{k+j=\frac{n}{2}}\Big(W_{2k+1,j}+\widetilde{W}_{2k+1,j}\Big),\\
E_i=&\sum_{t=0}^{\frac{n}{2}-1-i}\sum_{k+j=i}\Big(W_{2k+2t+2,j}-\widetilde{W}_{2k+2t+2,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t+\sum_{j=i+1}^{\frac{n}{2}}\Big(a^2+b^2\Big)^{j-i-1}\Big(W_{0,j}-\widetilde{W}_{0,j}\Big)\\
&-b\sum_{t=0}^{\frac{n}{2}-i-1}\sum_{k+j=i}\Big(T_{2k+2t+2,j}-\widetilde{T}_{2k+2t+2,j}\Big)C_{k+t}^t\Big(-b^2\Big)^t-b\sum_{j=i+1}^{\frac{n}{2}}\Big(a^2+b^2\Big)^{j-i-1}\Big(T_{0,j}-\widetilde{T}_{0,j}\Big),\\
E_{\left[\frac{n-1}{2}\right]}=&\sum_{k+j=\frac{n}{2}-1}\Big(W_{2k+2,j}-\widetilde{W}_{2k+2,j}\Big)\\
&-b\sum_{k+j=\frac{n}{2}-1}\Big(T_{2k+2,j}-\widetilde{T}_{2k+2,j}\Big)+\Big(W_{0,\frac{n}{2}}-\widetilde{W}_{0,\frac{n}{2}}\Big)-b\Big(T_{0,\frac{n}{2}}-\widetilde{T}_{0,\frac{n}{2}}\Big),
\end{aligned}$$

$$
\begin{aligned}
F_i = &\sum_{k+j=i}\Big(T_{2k+1,j}c^* + \widetilde{T}_{2k+1,j}\widetilde{c}^*\Big)m_{2k}\\
&+\sum_{h=1}^{\frac{n}{2}-i}\Big(-b^2\Big)^{h-1}\sum_{k+j=i}\Big(W_{2k+2h+1,j}b^* + \widetilde{W}_{2k+2h+1,j}\widetilde{b}^*\Big)\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t-1)}\\
&-b\sum_{h=1}^{\frac{n}{2}-1-i}\sum_{k+j=i}\Big(T_{2k+2h+1,j}b^* + \widetilde{T}_{2k+2h+1,j}\widetilde{b}^*\Big)\Big(-b^2\Big)^{h-1}\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t-1)},
\end{aligned}
$$

$$
F_{\left[\frac{n-1}{2}\right]} = \sum_{k+j=\frac{n}{2}-1}\Big(T_{2k+1,j}c^* + \widetilde{T}_{2k+1,j}\widetilde{c}^*\Big)m_{2k} + \sum_{k+j=\frac{n}{2}-1}\Big(W_{2k+3,j}b^* + \widetilde{T}_{2k+3,j}\widetilde{b}^*\Big)\sum_{t=0}^{k}m_{2(k-t)},
$$

$$
\begin{aligned}
G_i = &\sum_{k+j=i}\Big(T_{2k+2,j}c^* + \widetilde{T}_{2k+2,j}\widetilde{c}^*\Big)m_{2k+1}\\
&+\sum_{h=1}^{\frac{n}{2}-1-i}\Big(-b^2\Big)^{h-1}\sum_{k+j=i}\Big(W_{2k+2h+2,j}b^* + \widetilde{W}_{2k+2h+2,j}\widetilde{b}^*\Big)\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t)-1}\\
&-b\sum_{h=1}^{\frac{n}{2}-1-i}\Big(-b^2\Big)^{h-1}\sum_{k+j=i}\Big(T_{2k+2h+2,j}b^* + \widetilde{T}_{2k+2h+2,j}\widetilde{b}^*\Big)\sum_{t=h-1}^{k+h-1}C_t^{t-h+1}m_{2(k+h-t)-1},
\end{aligned}
$$

$$
G_{\left[\frac{n}{2}\right]-1} = \sum_{k+j=\frac{n}{2}-1}\Big(T_{2k+2,j}c^* + \widetilde{T}_{2k+2,j}\widetilde{c}^*\Big)m_{2k+1}.
$$

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