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Qualitative analysis of a mechanical system of coupled nonlinear oscillators

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Abstract. In this paper we investigate nonlinear systems of second order ODEs describing the dynamics of two coupled nonlinear oscillators of a mechanical system. We obtain, under certain assumptions, some stability results for the null solution. Also, we show that in the presence of a time-dependent external force, every solution starting from sufficiently small initial data and its derivative are bounded or go to zero as the time tends to $+\infty$, provided that suitable conditions are satisfied. Our theoretical results are illustrated with numerical simulations.

Keywords: coupled oscillators, uniform stability, asymptotic stability, uniform asymptotic stability.

2020 Mathematics Subject Classification: 34C15, 34D20.

1. Introduction

Consider a mechanical system of coupled nonlinear oscillators, as shown in Figure 1.1. Specifically, the block of mass m_1 is anchored to a fixed horizontal wall and the block of mass m_2 by springs and dampers, and the block of mass m_2 is also attached to the wall by a pair of springs and dampers. Suppose that the stiffnesses and the dampings are represented by the functions $k_i : \mathbb{R}_+ \to \mathbb{R}_+$ and $d_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \{1, 2, 3\}$, and $\widehat{g}_i : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2\}$, denote external forces acting on the blocks. One may also consider an external force $\widehat{f}(t)$ acting on the block of mass m_1, but for the moment, we restrict our attention to the case $\widehat{f} \equiv 0$. We assume that when the two blocks are in their equilibrium positions, the springs and the dampers are also in their equilibrium positions. Let $x(t)$ and $y(t)$ be the vertical displacements of the blocks from their equilibrium positions.

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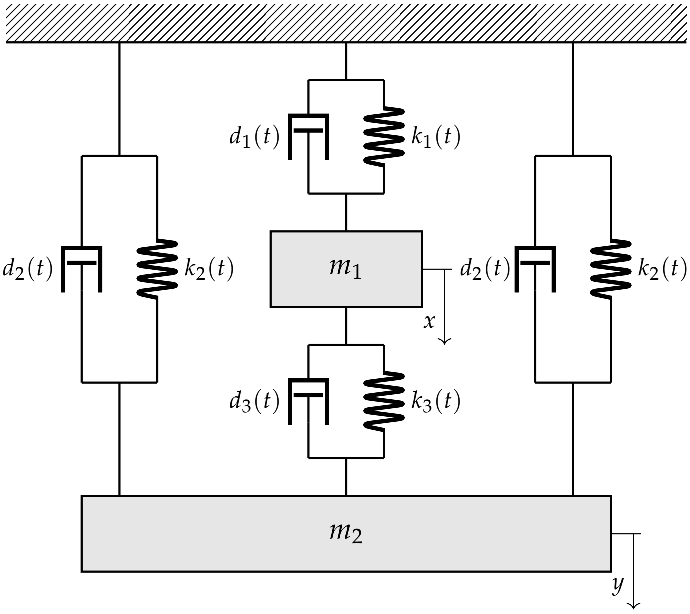


Figure 1.1: A mechanical system of coupled nonlinear oscillators

Then the system of ODEs describing the motion is (see, e.g., [27])

$$\begin{cases} m_1\ddot{x} + k_1(t)x + 2d_1(t)\dot{x} - k_3(t)(y-x) - 2d_3(t)(\dot{y}-\dot{x}) = \widehat{g}_1(t,x,y), \\ m_2\ddot{y} + 2k_2(t)y + 4d_2(t)\dot{y} + k_3(t)(y-x) + 2d_3(t)(\dot{y}-\dot{x}) = \widehat{g}_2(t,x,y), \end{cases}$$

or

$$\begin{cases} \ddot{x} + 2f_1(t)\dot{x} - f_3(t)\dot{y} + \beta(t)x - \gamma_1(t)y + g_1(t,x,y) = 0, \\ \ddot{y} + 2f_2(t)\dot{y} - f_4(t)\dot{x} - \gamma_2(t)x + \delta(t)y + g_2(t,x,y) = 0, \end{cases} \tag{1.1}$$

where

$$\begin{aligned} f_1(t) &:= \frac{1}{m_1}(d_1(t)+d_3(t)), & f_2(t) &:= \frac{1}{m_2}(2d_2(t)+d_3(t)), \\ f_3(t) &:= \frac{2}{m_1}d_3(t), & f_4(t) &:= \frac{2}{m_2}d_3(t), \\ \beta(t) &:= \frac{1}{m_1}(k_1(t)+k_3(t)), & \delta(t) &:= \frac{1}{m_2}(2k_2(t)+k_3(t)), \\ \gamma_1(t) &:= \frac{1}{m_1}k_3(t), & \gamma_2(t) &:= \frac{1}{m_2}k_3(t), \\ g_1(t,x,y) &:= -\frac{1}{m_1}\widehat{g}_1(t,x,y), & g_2(t,x,y) &:= -\frac{1}{m_2}\widehat{g}_2(t,x,y). \end{aligned}$$

The general case of a single 1-D damped nonlinear oscillator is described by the following equation which is well-known in the literature

$$\ddot{x} + 2f^*(t)\dot{x} + \beta^*(t)x + g^*(t,x) = 0, \quad t \in \mathbb{R}_+. \tag{1.2}$$

T. A. Burton and T. Furumochi [2] introduced a new method, based on the Schauder fixed point theorem, to study the stability of the null solution of Eq. (1.2) in the case $\beta^*(t) = 1$. In [14] we reported new stability results for the same equation. Our approach was based on elementary arguments only, involving in particular some Bernoulli type differential inequalities. In [15] we considered Eq. (1.2) under more general assumptions, which required more sophisticated arguments. For other investigations regarding the asymptotic stability of the equilibrium of a single damped nonlinear oscillator, we refer the reader to [7,8,10,11,24], and the references therein.

In the present paper, in Section 2 we will study the stability of the null solution of system (1.1), by two approaches, based on classical differential inequalities and on Lyapunov's method. For other results regarding the asymptotic stability of the equilibria of coupled damped nonlinear oscillators, we refer the reader to [9,16,17,20–23,25], and the references therein. For fundamental concepts and results in stability theory we refer the reader to [1,3,5,6,13,19].

In Section 3 we will consider that the block of mass m_1 is subject to the action of a time dependent external force $\widehat{f} : \mathbb{R}_+ \to \mathbb{R}$. In this case, the system of ODEs describing the dynamics of the mechanical system is

$$\begin{cases} \ddot{x} + 2f_1(t)\dot{x} - f_3(t)\dot{y} + \beta(t)x - \gamma_1(t)y - f(t) + g_1(t,x,y) = 0, \\ \ddot{y} + 2f_2(t)\dot{y} - f_4(t)\dot{x} - \gamma_2(t)x + \delta(t)y + g_2(t,x,y) = 0, \end{cases} \tag{1.3}$$

with the same functions as before, and $f(t) := \frac{1}{m_1}\widehat{f}(t)$, and we will derive certain qualitative properties of the solutions of system (1.3) with initial data small enough.

The model in Figure 1.1 could be used, e.g., to describe the dynamics in vertical direction of vibration reduction systems for horizontal cranes with loadings suspended in two sides [12,28]. For other models of coupled oscillators or for models from electric circuit theory, structural dynamics, described by systems of type (1.1) or (1.3), we refer the reader to the monographs [4,18,26].

2. A stability result for the system (1.1)

In this section we shall use the following hypotheses.

(H1) $f_i \in C^1(\mathbb{R}_+), f_j \in C(\mathbb{R}_+)$, $f_i(t) \geq 0$, $f_j(t) \geq 0$, $\forall t \in \mathbb{R}_+$, and $\int_0^{+\infty} f_j(t)\mathrm{d}t < +\infty$, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$;

(H2) there exist constants h, K_1, $K_2 \geq 0$ such that

$$\left|\dot{f}_i(t) + f_i^2(t)\right| \leq K_i\widetilde{f}(t),\ \forall t \in [h,+\infty),\ \forall i \in \{1,2\},$$

where $\widetilde{f}(t) := \min\{f_1(t), f_2(t)\}$, $\forall t \in \mathbb{R}_+$;

(H3) $\int_0^{+\infty} \widetilde{f}(t)\mathrm{d}t = +\infty$.

(H4) $\beta, \delta \in C^1(\mathbb{R}_+)$, β, δ are decreasing and

$$\beta(t) \geq \beta_0 > 0,\ \delta(t) \geq \delta_0 > 0, \quad \forall t \in \mathbb{R}_+,$$

where β_0, δ_0 are constants such that

$$\frac{K_1}{\sqrt{\beta_0}} + \frac{K_2}{\sqrt{\delta_0}} < 1;$$

(H5) $\gamma_i \in C(\mathbb{R}_+)$, $\gamma_i(t) \geq 0$, $\forall t \in \mathbb{R}_+$, and $\int_0^{+\infty} \gamma_i(t)\mathrm{d}t < +\infty$, $\forall i \in \{1,2\}$;

(H6) $g_i = g_i(t,x,y) \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$, g_i are locally Lipschitzian with respect to x, y, $i \in \{1,2\}$, and fulfill the relations

$$|g_1(t,x,y)| \leq r_1(t)\mathrm{O}(|x|), \quad \forall t \in \mathbb{R}_+,\ \forall y \in \mathbb{R}, \tag{2.1}$$

$$|g_2(t,x,y)| \leq r_2(t)\mathrm{O}(|y|), \quad \forall t \in \mathbb{R}_+,\ \forall x \in \mathbb{R}, \tag{2.2}$$

where $r_i \in C(\mathbb{R}_+)$, $r_i(t) \geq 0$, $\forall t \in \mathbb{R}_+$, $\int_0^{+\infty} r_i(t)\mathrm{d}t < +\infty$, $\forall i \in \{1,2\}$, and $\mathrm{O}(|x|)$ denotes the big-O Landau symbol as $x \to 0$ (similarly for $\mathrm{O}(|y|)$);

(H7) There is a $p > 0$, such that $f_i(t) \geq p$, $\forall t \geq 0$, $\forall i \in \{1,2\}$.

Remark 2.1. If (H1) and (H2) hold, then f_i, $\dot{f}_i$ are bounded, $i \in \{1,2\}$. Indeed, by (H2) we see that

$$(t \geq h,\ f_i(t) > K_i) \Longrightarrow \dot{f}_i(t) < 0.$$

This, combined with (H1), implies

$$f_i(t) \leq M_i := \max\{f_i(h), K_i\}, \quad \forall t \geq h.$$

So, using again (H2), we obtain

$$\left|\dot{f}_i(t)\right| \leq 2M_i^2, \quad \forall t \geq h.$$

This concludes the proof, since, by (H1), f_i, $\dot{f}_i \in C[0,h]$, $i \in \{1,2\}$.

Remark 2.2. Since we are going to discuss the stability of the null solution of system (1.1) and the large-time behavior of the solutions to (1.3) starting from small initial data, we can replace the inequalities (2.1) and (2.2) by

$$|g_1(t,x,y)| \leq r_1(t)|x|, \quad |g_2(t,x,y)| \leq r_2(t)|y|, \quad \forall t \in \mathbb{R}_+,\ \forall x,\ y \in \mathbb{R}, \tag{2.3}$$

possibly with $M_i r_i(t)$ instead of $r_i(t)$, where $M_i > 0$, and some functions $\widetilde{g}_i$ instead of g_i, $\forall i \in \{1,2\}$.

Indeed, from (2.1) there exist M_1, $a_1 > 0$, such that

$$|g_1(t,x,y)| \leq r_1(t)M_1|x|, \quad \text{if } |x| < a_1.$$

If we define the function $\widetilde{g}_1 : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$\widetilde{g}_1(t,x,y) := \begin{cases} g_1(t,a_1,y), \text{ if } x \geq a_1, \\ g_1(t,x,y), \text{ if } |x| < a_1, \\ g_1(t,-a_1,y), \text{ if } x \leq -a_1, \end{cases}$$

for all $t \geq 0$, $y \in \mathbb{R}$, then

$$|\widetilde{g}_1(t,x,y)| \leq r_1(t)M_1|x|, \quad \forall (t,x,y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R},$$

$\widetilde{g}_1 \in C(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$, and $\widetilde{g}_1$ is locally Lipschitzian in x, y. Similar reasonings work for the functions g_2 and r_2, possibly with another constant a_2.

2.1. A stability result via differential inequalities

We can state and prove the following stability result.

Theorem 2.3.

a) *Suppose that the hypotheses (H1), (H2), (H4)–(H6) are satisfied. Then the null solution of the system (1.1) is uniformly stable.*

b) *If the hypotheses (H1)–(H6) are fulfilled, then the null solution of (1.1) is asymptotically stable.*

c) *If the hypotheses (H1), (H2), (H4)–(H7) are fulfilled, then the null solution of (1.1) is uniformly asymptotically stable.*

Proof. By using the following transformation (inspired from [2])

$$\begin{cases} \dot{x} = u - f_1(t)x \\ \dot{u} = \left[\dot{f}_1(t) + f_1^2(t) - \beta(t)\right]x - f_1(t)u + [\gamma_1(t) - f_2(t)f_3(t)]y + f_3(t)v - g_1(t,x,y) \\ \dot{y} = v - f_2(t)y \\ \dot{v} = [\gamma_2(t) - f_1(t)f_4(t)]x + f_4(t)u + \left[\dot{f}_2(t) + f_2^2(t) - \delta(t)\right]y - f_2(t)v - g_2(t,x,y) \end{cases} \tag{2.4}$$

the system (1.1) becomes

$$\dot{z} = A(t)z + B(t)z + F(t,z), \tag{2.5}$$

where

$$z = \begin{pmatrix} x \\ u \\ y \\ v \end{pmatrix}, \qquad A(t) = \begin{pmatrix} -f_1(t) & 1 & 0 & 0 \\ -\beta(t) & -f_1(t) & \gamma_1(t) & 0 \\ 0 & 0 & -f_2(t) & 1 \\ \gamma_2(t) & 0 & -\delta(t) & -f_2(t) \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \dot{f}_1(t) + f_1^2(t) & 0 & -f_2(t)f_3(t) & f_3(t) \\ 0 & 0 & 0 & 0 \\ -f_1(t)f_4(t) & f_4(t) & \dot{f}_2(t) + f_2^2(t) & 0 \end{pmatrix}, \qquad F(t,z) = \begin{pmatrix} 0 \\ -g_1(t,x,y) \\ 0 \\ -g_2(t,x,y) \end{pmatrix}.$$

Using the boundedness of the functions f_i, $\dot{f}_i$, f_j, β, γ_i, δ, r_i, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$, we easily deduce that our stability question of the null solution of the system (1.1) reduces to the stability of the null solution $z(t) = 0$ of the system (2.5) .

Let $t_0 \geq 0$ and

$$Z(t,t_0) = \left(a_{ij}(t,t_0)\right)_{i,j\in\overline{1,4}}, \; t \geq t_0,$$

be the fundamental matrix of the system

$$\dot{z} = A(t)z, \tag{2.6}$$

which equals the identity matrix for $t = t_0$. Then we deduce

$$\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0) + \delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0) \leq \beta(t_0)\mathrm{e}^{\int_{t_0}^{t}\left[-2\widetilde{f}(u) + \frac{\gamma(u)}{\sqrt{\zeta(u)}}\right]\mathrm{d}u}, \tag{2.7}$$

$$\beta(t)a_{12}^2(t,t_0) + a_{22}^2(t,t_0) + \delta(t)a_{32}^2(t,t_0) + a_{42}^2(t,t_0) \leq \mathrm{e}^{\int_{t_0}^{t}\left[-2\widetilde{f}(u) + \frac{\gamma(u)}{\sqrt{\zeta(u)}}\right]\mathrm{d}u}, \tag{2.8}$$

$$\beta(t)a_{13}^2(t,t_0) + a_{23}^2(t,t_0) + \delta(t)a_{33}^2(t,t_0) + a_{43}^2(t,t_0) \leq \delta(t_0)\mathrm{e}^{\int_{t_0}^{t}\left[-2\widetilde{f}(u) + \frac{\gamma(u)}{\sqrt{\zeta(u)}}\right]\mathrm{d}u}, \tag{2.9}$$

$$\beta(t)a_{14}^2(t,t_0) + a_{24}^2(t,t_0) + \delta(t)a_{34}^2(t,t_0) + a_{44}^2(t,t_0) \leq \mathrm{e}^{\int_{t_0}^{t}\left[-2\widetilde{f}(u) + \frac{\gamma(u)}{\sqrt{\zeta(u)}}\right]\mathrm{d}u}, \tag{2.10}$$

for all $t \geq t_0$, where $\gamma(t) := \max\{\gamma_1(t), \gamma_2(t)\}$, $\zeta(t) := \min\{\beta(t), \delta(t)\}$, $\forall t \in \mathbb{R}_+$.

Indeed, from (2.6) we get the following system

$$\begin{cases} \dot{a}_{11}(t,t_0) = -f_1(t)a_{11}(t,t_0) + a_{21}(t,t_0) \\ \dot{a}_{21}(t,t_0) = -\beta(t)a_{11}(t,t_0) - f_1(t)a_{21}(t,t_0) + \gamma_1(t)a_{31}(t,t_0) \\ \dot{a}_{31}(t,t_0) = -f_2(t)a_{31}(t,t_0) + a_{41}(t,t_0) \\ \dot{a}_{41}(t,t_0) = \gamma_2(t)a_{11}(t,t_0) - \delta(t)a_{31}(t,t_0) - f_2(t)a_{41}(t,t_0). \end{cases} \tag{2.11}$$

From the first two equations of (2.11) and hypothesis (H4) we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0)\right] \\ &\qquad \leq -f_1(t)\left[\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0)\right] + \gamma_1(t)a_{21}(t,t_0)a_{31}(t,t_0) \end{aligned} \tag{2.12}$$

and, similarly,

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0)\right] \\ &\qquad \leq -f_2(t)\left[\delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0)\right] + \gamma_2(t)a_{11}(t,t_0)a_{41}(t,t_0). \end{aligned} \tag{2.13}$$

By relations (2.12) and (2.13) we obtain successively

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0) + \delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0)\right] \\ &\quad \leq -f_1(t)\left[\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0)\right] - f_2(t)\left[\delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0)\right] \\ &\qquad + \gamma_1(t)a_{21}(t,t_0)a_{31}(t,t_0) + \gamma_2(t)a_{11}(t,t_0)a_{41}(t,t_0) \\ &\quad \leq \left[-\widetilde{f}(t) + \frac{\gamma(t)}{2\sqrt{\zeta(t)}}\right]\left[\beta(t)a_{11}^2(t,t_0) + a_{21}^2(t,t_0) + \delta(t)a_{31}^2(t,t_0) + a_{41}^2(t,t_0)\right], \end{aligned}$$

for all $t \geq t_0$, and (2.7) follows immediately. The inequalities (2.8)–(2.10) can be derived in the same way.

Let $\|\cdot\|_0$ be the norm in $\mathbb{R}^4$ defined by

$$\|z\|_0 = \left(\beta_0 x^2 + u^2 + \delta_0 y^2 + v^2\right)^{1/2}, \quad \text{for } z = (x,u,y,v)^\top, \tag{2.14}$$

which is equivalent to the Euclidean norm.

For $z_0 = (x_0,u_0,y_0,v_0)^\top \in \mathbb{R}^4$, from (2.7)–(2.10) and (H4), we deduce

$$\|Z(t,t_0)z_0\|_0 \leq \lambda\|z_0\|_0\sqrt{\beta(t_0) + \delta(t_0) + 2}\,\mathrm{e}^{\int_{t_0}^t\left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}, \quad \forall t \geq t_0, \tag{2.15}$$

where $\lambda := \max\{1, 1/\sqrt{\beta_0}, 1/\sqrt{\delta_0}\}$,

$$\begin{aligned} \left\|Z(t,t_0)Z(s,t_0)^{-1}e_2\right\|_0 &\leq \mathrm{e}^{\int_s^t\left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}, \\ \left\|Z(t,t_0)Z(s,t_0)^{-1}e_4\right\|_0 &\leq \mathrm{e}^{\int_s^t\left[-\widetilde{f}(u) + \frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}, \end{aligned} \tag{2.16}$$

for all $t \geq s \geq t_0 \geq 0$, where $e_2 = (0,1,0,0)^\top$, $e_4 = (0,0,0,1)^\top$.

Proof of a). Let $z_0 \neq 0$ with $\|z_0\|_0$ small enough, $t_0 \geq 0$, and $z(t,t_0,z_0) = (x(t,t_0,z_0), u(t,t_0,z_0), y(t,t_0,z_0),\ v(t,t_0,z_0))^\top$ be the unique solution of (2.5) which equals z_0 for $t = t_0$.

From the continuity and the boundedness of the functions f_i, $\dot{f}_i$, f_j, β, γ_i, δ, r_i, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$, there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous and bounded function, such that

$$\|A(t)z + B(t)z + F(t,z)\|_0 \leq \psi(t)\|z\|_0, \quad \forall (t,z) \in \mathbb{R}_+ \times \mathbb{R}^4.$$

By applying a classical result of global existence in the future to system (2.5) (see, e.g., [3, Corollary, p. 53]) it follows that $z(t,t_0,z_0)$ exists on the whole interval $[t_0, +\infty)$.

We have

$$z(t,t_0,z_0) = Z(t,t_0)z_0 + \int_{t_0}^{t} Z(t,t_0)Z(s,t_0)^{-1}[B(s)z(s,t_0,z_0) + F(s,z(s,t_0,z_0))]\mathrm{d}s, \tag{2.17}$$

for all $t \geq t_0$.

From the relations (2.15)–(2.17) we get

$$\begin{aligned}\|z(t,t_0,z_0)\|_0 \leq{}& \lambda\|z_0\|_0\sqrt{\beta(t_0)+\delta(t_0)+2}\mathrm{e}^{\int_{t_0}^{t}\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u} + \int_{t_0}^{t}\mathrm{e}^{\int_{s}^{t}\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u} \\ &\times\Big[\big|\dot{f}_1(s)+f_1^2(s)\big||x(s,t_0,z_0)| + \big|\dot{f}_2(s)+f_2^2(s)\big||y(s,t_0,z_0)| \\ &+ f_1(s)f_4(s)|x(s,t_0,z_0)| + f_2(s)f_3(s)|y(s,t_0,z_0)| \\ &+ f_3(s)|v(s,t_0,z_0)| + f_4(s)|u(s,t_0,z_0)| \\ &+ |g_1(s,x(s,t_0,z_0),y(s,t_0,z_0))| \\ &+ |g_2(s,x(s,t_0,z_0),y(s,t_0,z_0))|\Big]\mathrm{d}s,\end{aligned} \tag{2.18}$$

for all $t \geq t_0$.

In what follows we consider two cases.

Case 1: $0 \leq t_0 < h$. Since $f_i \in C^1[t_0,h]$, $f_j, \beta, \gamma_i, \delta, \in C[t_0,h]$, $g_i \in C([t_0,h]\times\mathbb{R}\times\mathbb{R})$, $\forall i \in \{1,2\}$, $\forall j \in \{3,4\}$, from (2.18) it results that

$$\|z(t,t_0,z_0)\|_0 \leq \lambda D_1\sqrt{\beta(t_0)+\delta(t_0)+2}\|z_0\|_0 + D\int_{t_0}^{t}\|z(s,t_0,z_0)\|_0\mathrm{d}s, \qquad \forall t \in [t_0,h],$$

with D, D_1 positive constants. Using the Gronwall lemma we get

$$\|z(t,t_0,z_0)\|_0 \leq \lambda D_1\sqrt{\beta(t_0)+\delta(t_0)+2}\|z_0\|_0\mathrm{e}^{Dh}, \qquad \forall t \in [t_0,h]. \tag{2.19}$$

For all $t \geq h$, from the relation (2.18) and the hypothesis (H2) we deduce

$$\begin{aligned}\|z(t,t_0,z_0)\|_0 \leq{}& \lambda\sqrt{\beta(h)+\delta(h)+2}\|z(h,t_0,z_0)\|_0\mathrm{e}^{\int_{h}^{t}\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u} \\ &+ \int_{h}^{t}\mathrm{e}^{\int_{s}^{t}\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\Big[K_1\widetilde{f}(s)|x(s,t_0,z_0)| + K_2\widetilde{f}(s)|y(s,t_0,z_0)| \\ &+ f_1(s)f_4(s)|x(s,t_0,z_0)| + f_2(s)f_3(s)|y(s,t_0,z_0)| \\ &+ f_3(s)|v(s,t_0,z_0)| + f_4(s)|u(s,t_0,z_0)| \\ &+ |g_1(s,x(s,t_0,z_0),y(s,t_0,z_0))| \\ &+ |g_2(s,x(s,t_0,z_0),y(s,t_0,z_0))|\Big]\mathrm{d}s.\end{aligned} \tag{2.20}$$

By (2.3) and (2.20) we obtain

$$\begin{aligned}\|z(t,t_0,z_0)\|_0 \leq{}& \lambda\sqrt{\beta(h)+\delta(h)+2}\|z(h,t_0,z_0)\|_0 \mathrm{e}^{\int_h^t \left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u} \\ &+\int_h^t \mathrm{e}^{\int_s^t \left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u}\left[\left(\frac{K_1}{\sqrt{\beta_0}}+\frac{K_2}{\sqrt{\delta_0}}\right)\widetilde{f}(s)\right. \\ &+\frac{f_1(s)f_4(s)}{\sqrt{\beta_0}}+\frac{f_2(s)f_3(s)}{\sqrt{\delta_0}}+f_3(s)+f_4(s) \\ &\left.+\frac{r_1(s)}{\sqrt{\beta_0}}+\frac{r_2(s)}{\sqrt{\delta_0}}\right]\|z(s,t_0,z_0)\|_0\mathrm{d}s \\ =:{}& \sigma(t), \qquad \forall t\geq h. \end{aligned} \tag{2.21}$$

Straightforward calculations lead us to

$$\dot{\sigma}(t)\leq\omega(t)\sigma(t), \qquad \forall t\geq h, \tag{2.22}$$

$$\sigma(h)=\lambda\sqrt{\beta(h)+\delta(h)+2}\|z(h,t_0,z_0)\|_0.$$

where

$$\omega(t):=-K\widetilde{f}(t)+\varphi(t), \qquad \forall t\geq 0,\ K=1-\frac{K_1}{\sqrt{\beta_0}}-\frac{K_2}{\sqrt{\delta_0}},$$
$$\varphi(t):=\frac{\gamma(t)}{2\sqrt{\zeta(t)}}+\frac{f_1(t)f_4(t)}{\sqrt{\beta_0}}+\frac{f_2(t)f_3(t)}{\sqrt{\delta_0}}+f_3(t)+f_4(t)+\frac{r_1(t)}{\sqrt{\beta_0}}+\frac{r_2(t)}{\sqrt{\delta_0}}, \qquad \forall t\geq 0.$$

From (2.21) and (2.22) using classical differential inequalities, we obtain

$$\|z(t,t_0,z_0)\|_0\leq\lambda\sqrt{\beta(h)+\delta(h)+2}\|z(h,t_0,z_0)\|_0\mathrm{e}^{-K\int_h^t\widetilde{f}(s)\mathrm{d}s}\mathrm{e}^{\int_h^t\varphi(s)\mathrm{d}s}, \qquad \forall t\geq h. \tag{2.23}$$

It is readily seen from the hypotheses (H1), (H5), (H6), and Remark 2.1, that

$$\int_h^{+\infty}\varphi(s)\mathrm{d}s<+\infty.$$

Let $\varepsilon>0$ be arbitrary and

$$\eta=\eta(\varepsilon):=\frac{\varepsilon\mathrm{e}^{-\int_h^{+\infty}\varphi(s)\mathrm{d}s}\mathrm{e}^{-Dh}}{\lambda^2D_1\sqrt{\beta(0)+\delta(0)+2}\sqrt{\beta(h)+\delta(h)+2}}.$$

Then, if $\|z_0\|_0<\eta$, by (2.19) and the hypothesis (H4) it results

$$\|z(t)\|_0\leq\frac{\varepsilon\mathrm{e}^{-\int_h^{+\infty}\varphi(s)\mathrm{d}s}}{\lambda\sqrt{\beta(h)+\delta(h)+2}}, \qquad \forall t\in[t_0,h]. \tag{2.24}$$

From the relations (2.23), (2.24), and the hypothesis (H4), it follows that $\|z(t,t_0,z_0)\|_0<\varepsilon$, $\forall t\geq h$.

Case 2: $t_0\geq h$. We similarly get

$$\|z(t,t_0,z_0)\|_0\leq\lambda\sqrt{\beta(t_0)+\delta(t_0)+2}\|z_0\|_0\mathrm{e}^{-K\int_{t_0}^t\widetilde{f}(s)\mathrm{d}s}\mathrm{e}^{\int_{t_0}^t\varphi(s)\mathrm{d}s}, \tag{2.25}$$

for all $t \geq t_0$. With the same η as before, if $\|z_0\|_0 < \eta$, then $\|z(t, t_0, z_0)\|_0 < \varepsilon$, $\forall t \geq t_0$.

Therefore, the null solution of (1.1) is uniformly stable.

Proof of b). If, in addition (H3) holds, then from (2.25) we can easily obtain that the null solution of (1.1) is asymptotically stable.

Proof of c). We know from a) that the null solution of (1.1) is uniformly stable. It remains to prove that there exists $\xi > 0$, such that for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$, such that $\|z_0\|_0 < \xi$ implies $\|z(t, t_0, z_0)\|_0 < \varepsilon$, for all $t_0 \geq 0$ and $t \geq t_0 + T$.

Indeed, if (H7) also holds, then $\int_{t_0}^{t} \widetilde{f}(s)\mathrm{d}s \geq p(t - t_0)$, $\forall t \geq t_0 \geq 0$. From (2.25) we obtain for all $t \geq t_0 \geq 0$, that

$$\|z(t, t_0, z_0)\|_0 \leq \lambda \sqrt{\beta(0) + \delta(0) + 2}\|z_0\|_0 \mathrm{e}^{-Kp(t-t_0)} N, \tag{2.26}$$

where $N := \mathrm{e}^{\int_0^{+\infty} \varphi(s)\mathrm{d}s}$. Let $\xi := \frac{1}{\lambda\sqrt{\beta(0)+\delta(0)+2}}$, $\varepsilon > 0$, and

$$T = T(\varepsilon) := \begin{cases} \frac{1}{Kp} \ln \frac{N}{\varepsilon}, \text{ if } \varepsilon < N, \\ 0, \text{ if } \varepsilon \geq N. \end{cases}$$

Consider $z_0 \in \mathbb{R}^4$, $z_0 \neq 0$, with $\|z_0\|_0 < \xi$ and let $t_0 \geq 0$. Then for all $t \geq t_0 + T$, by (2.26) we successively deduce

$$\|z(t, t_0, z_0)\|_0 < \lambda \sqrt{\beta(0) + \delta(0) + 2}\xi \mathrm{e}^{-Kp(t-t_0)} N = N\mathrm{e}^{-Kp(t-t_0)} \leq \varepsilon.$$

Therefore the null solution of (1.1) is uniformly asymptotically stable. □

Example 2.4. An example of functions f_i, f_j, β, δ, γ_i, g_i, $i \in \{1, 2\}$, $j \in \{3, 4\}$, is

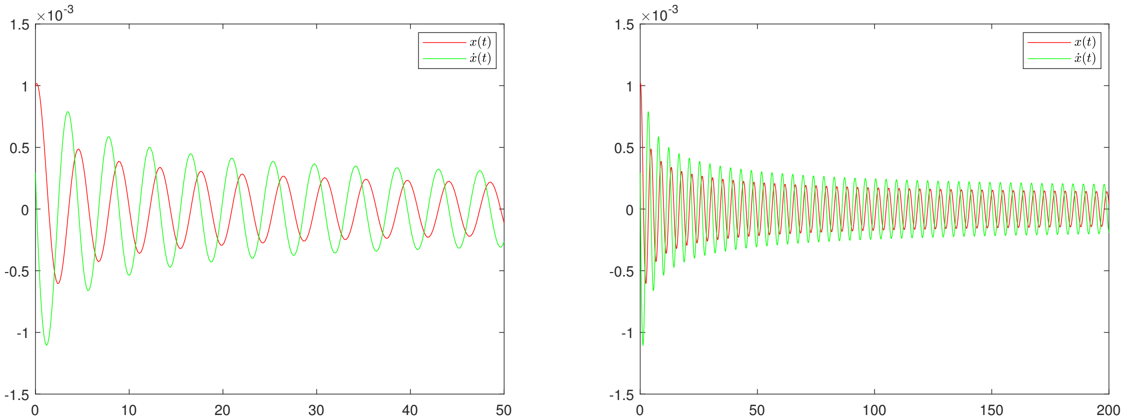
$$f_1(t) = \frac{1}{2t + \sqrt{t^2 + 2}}, \quad f_2(t) = \frac{1}{t + \sqrt{t^2 + 1}}, \quad f_3(t) = \frac{1}{(t + 1)^4}, \quad f_4(t) = \frac{2}{(t + 1)^3}, \quad \forall t \geq 0,$$

$$\beta(t) = \frac{2t + 3}{t + 1}, \quad \delta(t) = \frac{2t^3 + 5}{t^3 + 2}, \quad \gamma_1(t) = \frac{1}{t\sqrt{t^2 + 1} + 1}, \quad \gamma_2(t) = \mathrm{e}^{-t/2}, \quad \forall t \geq 0,$$

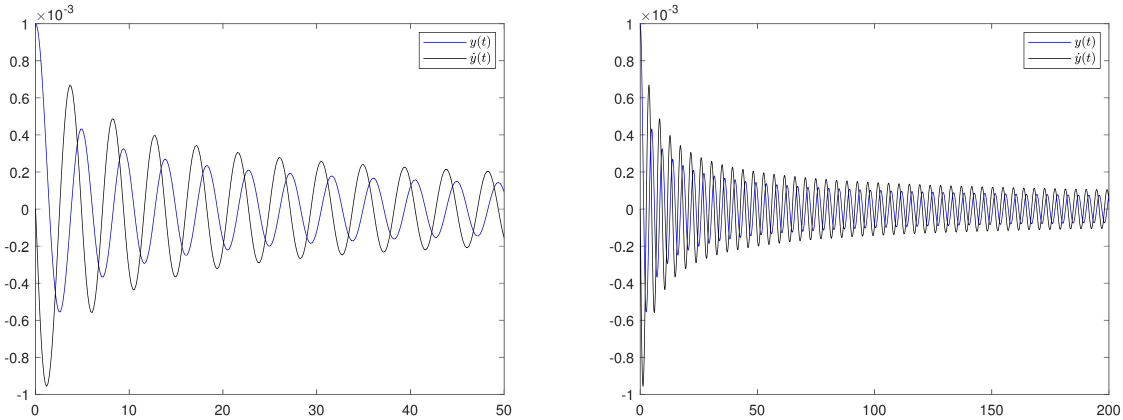
$$g_1(t, x, y) = \mathrm{e}^{-t^2/2} x^3, \quad g_2(t, x, y) = \frac{3}{t^2\sqrt{t + 1}} y^4, \quad \forall t \geq 0, \forall x, y \in \mathbb{R}.$$

These functions satisfy the hypotheses (H1)–(H6), with $\beta_0 = 2$, $\delta_0 = 2$, $K_1 = 1/\sqrt{2}$, $K_2 = (2 + \sqrt{3}) \times (3 - 2\sqrt{2})$, $h = 1$, $r_1(t) = \mathrm{e}^{-t^2/2}$, $r_2(t) = \frac{3}{t^2\sqrt{t+1}}$, $\forall t \geq 0$. In Figure 2.1 the solution of (1.1) and its derivative are plotted on two time intervals, for small initial data. The solution in the planes $(x, \dot{x})$ and $(y, \dot{y})$ on the same time intervals can be observed in Figure 2.2.

Example 2.5. If in Example 2.4 one changes only f_1, f_2 to $f_1(t) = \frac{1}{10} + \frac{1}{t+1}$, respectively $f_2(t) = \frac{1}{5} + \frac{2}{t+1}$, $\forall t \geq 0$, then the hypotheses (H1), (H2), (H4)–(H7) are verified with $K_1 = 1/5$, $K_2 = 4/5$, $h = 7$, $p = \frac{1}{10}$, and the same β_0, δ_0, $r_1(t)$, $r_2(t)$ and we obtain the solution of (1.1) and its derivative plotted in Figure 2.3 on the same time intervals and for the same initial data. In Figure 2.4 the solution is generated in the planes $(x, \dot{x})$ and $(y, \dot{y})$.



$t \in [0, 50]$ $t \in [0, 200]$



$t \in [0, 50]$ $t \in [0, 200]$

Figure 2.1: The solution of system (1.1) and its derivative, with the initial data $z_0 = [0.01, 0.01, 0.01, 0.01]$ and the functions f_1, f_2, f_3, f_4, β, δ, γ_1, γ_2, g_1, g_2 given in Example 2.4.

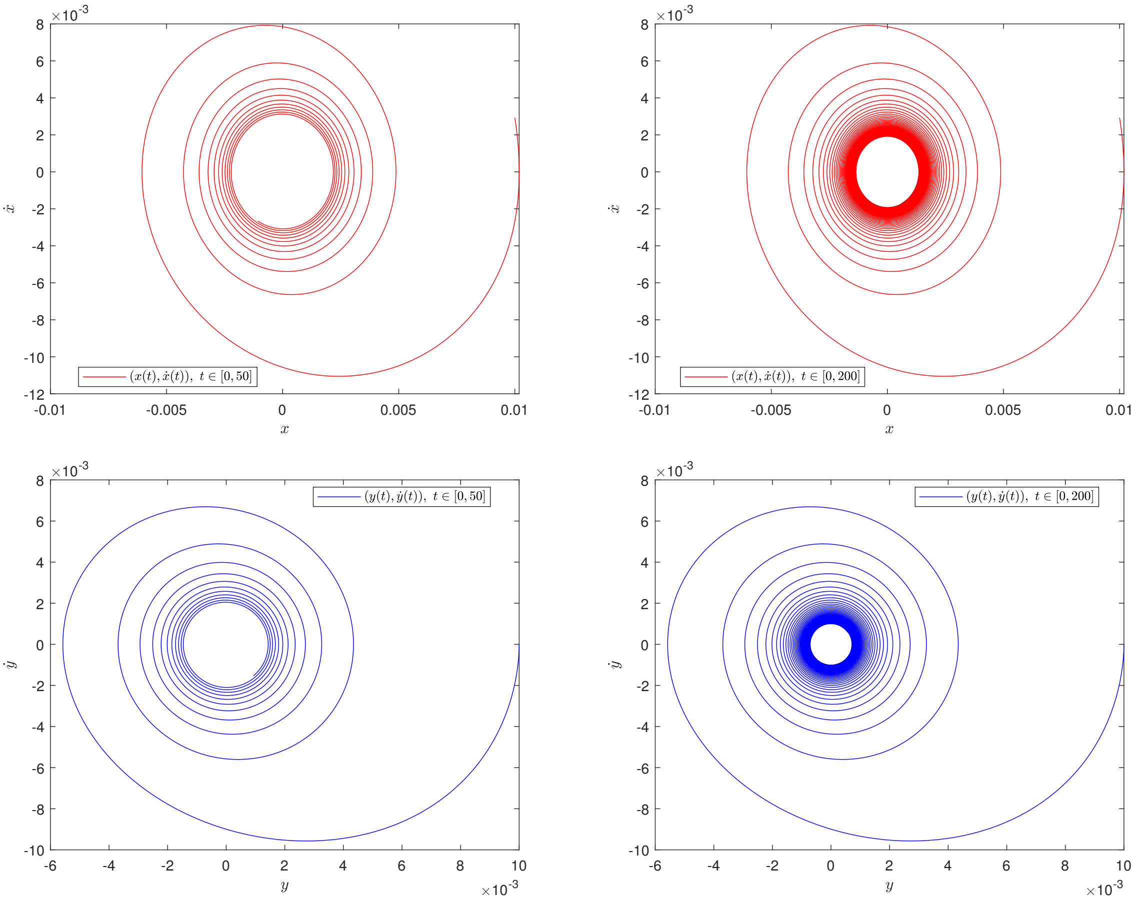
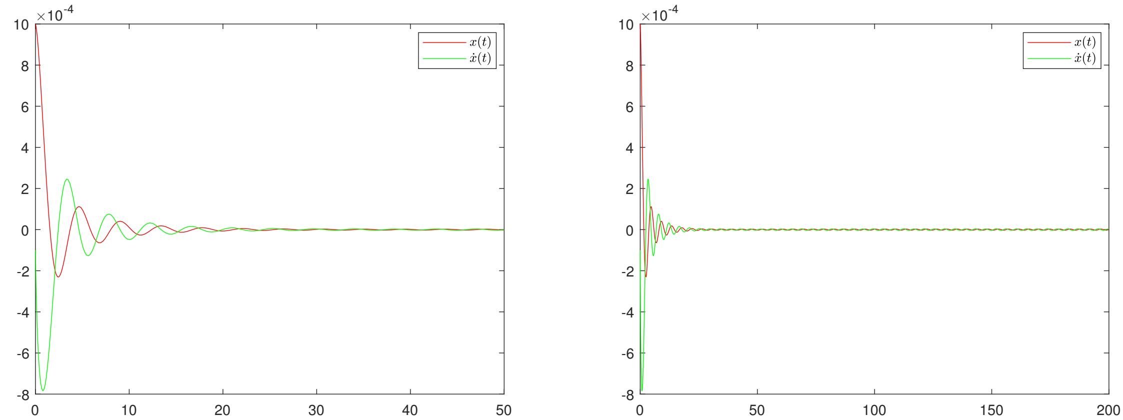
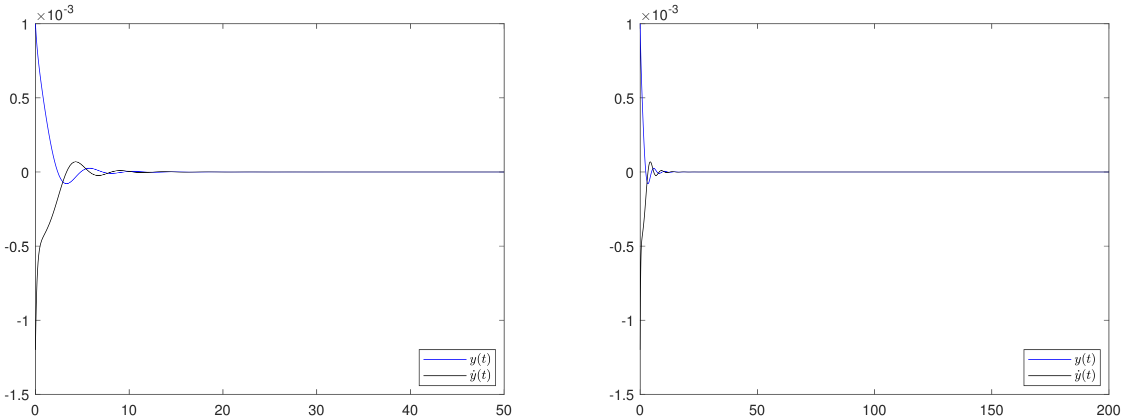


Figure 2.2: The solution of (1.1) in the planes $(x, \dot{x})$ and $(y, \dot{y})$, with the data from Example 2.4.



$t \in [0, 50]$ $t \in [0, 200]$



$t \in [0, 50]$ $t \in [0, 200]$

Figure 2.3: The solution of system (1.1) and its derivative, with the initial data $z_0 = [0.01, 0.01, 0.01, 0.01]$ and the functions f_1, f_2, f_3, f_4, β, δ, γ_1, γ_2, g_1, g_2 given in Example 2.5.

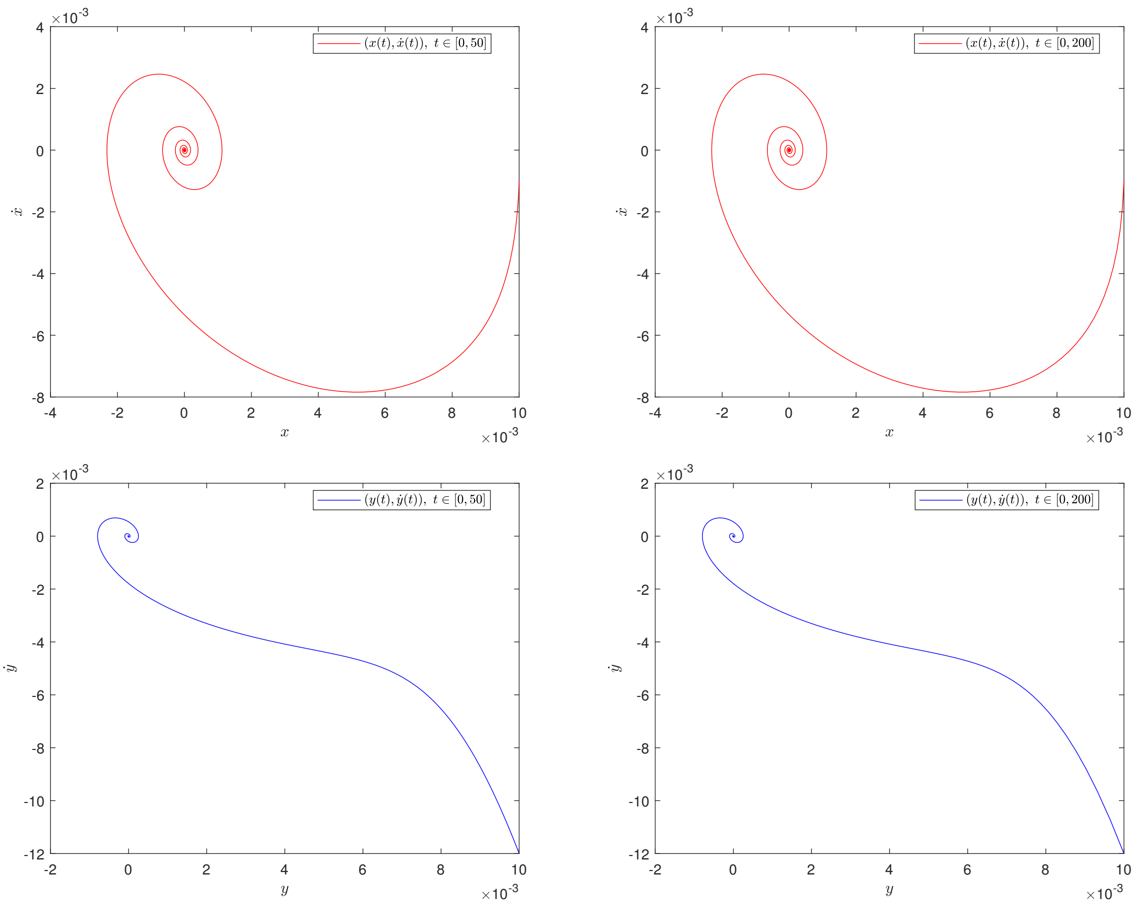


Figure 2.4: The solution of (1.1) in the planes $(x, \dot{x})$ and $(y, \dot{y})$, with the data from Example 2.5.

2.2. A stability result via Lyapunov's method

We are going to use the following additional assumptions.

(H1*) $f_i \in C(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$, $f_j \in C(\mathbb{R}_+)$, $f_i(t) \geq 0$, $f_j(t) \geq 0$, $\forall t \in \mathbb{R}_+$, and $\int_0^{+\infty} f_j(t)\mathrm{d}t < +\infty$, $\forall i \in \{1, 2\}$, $\forall j \in \{3, 4\}$;

(H3*) $\int_0^{+\infty} \widetilde{f}(t)\mathrm{d}t < +\infty$;

(H4*) β, $\delta \in C^1(\mathbb{R}_+)$, β, δ are decreasing and

$$\beta(t) \geq \beta_0 > 0,\ \delta(t) \geq \delta_0 > 0, \quad \forall t \in \mathbb{R}_+.$$

Let us state and prove the following result.

Theorem 2.6. *Suppose that the hypotheses (H1*), (H3*), (H4*), (H5), (H6) are fulfilled. Then the null solution of the system* (1.1) *is uniformly stable.*

Proof. Let us remark that using the classical change of variables $x = x$, $u = \dot{x}$, $y = y$, $v = \dot{y}$, the system (1.1) becomes

$$\dot{z} = F(t, z), \tag{2.27}$$

where

$$z = \begin{pmatrix} x \\ u \\ y \\ v \end{pmatrix}, \qquad F(t,z) = \begin{pmatrix} u \\ -\beta(t)x - 2f_1(t)u + \gamma_1(t)y + f_3(t)v - g_1(t,x,y) \\ v \\ \gamma_2(t)x + f_4(t)u - \delta(t)y - 2f_2(t)v - g_2(t,x,y) \end{pmatrix}$$

and our stability question reduces to the stability of the null solution $z(t) = 0$ of the system (2.27). Let us remark that the global existence in the future of the solutions of (2.27) follows as in the proof of Theorem 2.3, this time the boundedness of the functions f_1, f_2 being ensured by the hypothesis (H1*).

We are going to use again the norm $\|\cdot\|_0$ defined by (2.14). Consider the function $V : \mathbb{R}_+ \times \Delta \to \mathbb{R}$,

$$V(t,z) = \frac{1}{2}\left[\beta(t)x^2 + u^2 + \delta(t)y^2 + v^2\right]\mathrm{e}^{-\int_0^t \left[\tilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s},$$

for $z = (x,u,y,v)^\top \in \Delta$, where $\Delta \subset \mathbb{R}^4$ is a neighborhood of the origin of $\mathbb{R}^4$,

$$\Delta = \left\{z \in \mathbb{R}^4,\ \|z\|_0 < a\right\},$$

where $a = \min\{a_1\sqrt{\beta_0},\ a_2\sqrt{\delta_0}\}$, $a_1 > 0$, $a_2 > 0$ are as in Remark 2.2, $\gamma(t) := \max\{\gamma_1(t), \gamma_2(t)\}$, $\zeta(t) := \min\{\beta(t), \delta(t)\}$, $\forall t \in \mathbb{R}_+$, and $r(t) := \max\{r_1(t), r_2(t)\}$, $\forall t \geq 0$.

Obviously,

$$\begin{aligned} V(t,z) &\geq \frac{1}{2}\left(\beta_0 x^2 + u^2 + \delta_0 y^2 + v^2\right)\mathrm{e}^{-\int_0^t \left[\tilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ &= \frac{1}{2}\|z\|_0^2 \mathrm{e}^{-\int_0^t \left[\tilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s}, \end{aligned}$$

for all $(t,z) \in \mathbb{R}_+ \times \Delta$.

By using hypotheses (H1*), (H3*), (H4*), (H5), (H6), we deduce

$$V(t,z) \geq \frac{1}{2}\|z\|_0^2 \mathrm{e}^{-\left[\int_0^{+\infty}\tilde{f}(s)\mathrm{d}s + \int_0^{+\infty} f_3(s)\mathrm{d}s + \int_0^{+\infty} f_4(s)\mathrm{d}s + \int_0^{+\infty}\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\mathrm{d}s\right]}, \quad \forall (t,z) \in \mathbb{R}_+ \times \Delta$$

and so the function V is positive definite.

The function V is also decrescent. Indeed,

$$\begin{aligned} V(t,z) &\leq \frac{1}{2}\left[\beta(0)x^2 + u^2 + \delta(0)y^2 + v^2\right]\mathrm{e}^{-\int_0^t \left[\tilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ &\leq \frac{1}{2}\max\left\{\frac{\beta(0)}{\beta_0}, \frac{\delta(0)}{\delta_0}\right\}\|z\|_0^2, \quad \forall (t,z) \in \mathbb{R}_+ \times \Delta. \end{aligned}$$

We prove that the time derivative of V along the solutions of the system (2.27) is less than or equal to 0. Indeed, for every $(t,z) \in \mathbb{R}_+ \times \Delta$,

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) = \frac{1}{2}&\big[\dot{\beta}(t)x^2 + 2\beta(t)x\dot{x} + 2u\dot{u} + \dot{\delta}(t)y^2 + 2\delta(t)y\dot{y} + 2v\dot{v}\big] \\
&\times \mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\
&- \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right] V(t,z) \\
\leq &\ \{\gamma(t)(|u||y| + |x||v|) + [f_3(t)+f_4(t)]|u||v| + |u||g_1(t,x,y)| + |v||g_2(t,x,y)|\} \\
&\times \mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\
&- 2\big[f_1(t)u^2 + f_2(t)v^2\big]\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\
&- \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right] V(t,z).
\end{aligned} \tag{2.28}$$

From (2.28) and (2.3) for all $(t,z) \in \mathbb{R}_+ \times \Delta$ we successively obtain

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) \leq &\Big\{\gamma(t)(|u||y| + |x||v|) + [f_3(t)+f_4(t)]|u||v| + [r_1(t)|x||u| + r_2(t)|y||v|] \\
&- 2\big[f_1(t)u^2 + f_2(t)v^2\big]\Big\}\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\
&- \left[\widetilde{f}(t) + f_3(t) + f_4(t) + 2\frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right] V(t,z) \\
\leq &\left[f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right] V - 2\big[f_1(t)u^2 + f_2(t)v^2\big] \\
&\times \mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\
&- \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right] V(t,z) \\
= &-\widetilde{f}(t)V - 2\big[f_1(t)u^2 + f_2(t)v^2\big]\mathrm{e}^{-\int_0^t \left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s}.
\end{aligned} \tag{2.29}$$

Then, from (2.29) we easily get

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) \leq 0, \quad \forall (t,z) \in \mathbb{R}_+ \times \Delta.$$

From Persidski's Theorem (see, e.g., [3, second Corollary, p. 101], [17, Theorem 2.1]), it follows that the null solution of (1.1) is uniformly stable. □

Remark 2.7. Let us remark that by using the transformation (2.4) we obtained the uniform, the asymptotic, and the uniform asymptotic stability, while by using the classical transformation ($x = x$, $u = \dot{x}$, $y = y$, $v = \dot{y}$) and the Lyapunov's method we were only able to achieve the uniform stability of the null solution of (1.1). Hence the first method, based on the transformation (2.4), is more effective.

Remark 2.8. Note that the null solution of the system (1.1) can be uniformly stable and not asymptotically stable. Indeed, this can be seen by considering the following functions

$$f_1(t) = \frac{e^{-t}}{t+1}, \quad f_2(t) = \frac{|\cos^3 t|}{t^2+4}, \quad \forall t \geq 0, \quad f_3(t) = \frac{|\sin t^2|}{t+2}, \quad f_4(t) = \frac{e^{-t^2}}{t+1}, \quad \forall t \geq 0,$$

$$\beta(t) = 0.3 + \frac{1}{t^2+1}, \quad \delta(t) = 0.2 + \frac{1}{\sqrt{t^2+2}}, \quad \gamma_1(t) = \frac{t}{t+2}e^{-t^2}, \quad \gamma_2(t) = \frac{3|\cos t|}{(t+1)^2}, \quad \forall t \geq 0,$$

$$g_1(t,x,y) = \frac{3x^3}{(t^2+2)^2}, \quad g_2(t,x,y) = \frac{2y^2}{(t+1)^3}, \quad \forall t \geq 0, \quad \forall x,y \in \mathbb{R}.$$

These functions satisfy the hypotheses (H1*), (H3*), (H4*), (H5), (H6), with $\beta_0 = 0.3$, $\delta_0 = 0.2$, $r_1(t) = \frac{3}{(t^2+2)^2}$, $r_2(t) = \frac{2}{(t+1)^3}$, $\forall t \geq 0$. For small initial data, the solution of (1.1) and its derivative can be observed in Figure 2.5 on some time intervals. The plottings of the solution in the planes $(x, \dot{x})$, $(y, \dot{y})$ are given in Figure 2.6.

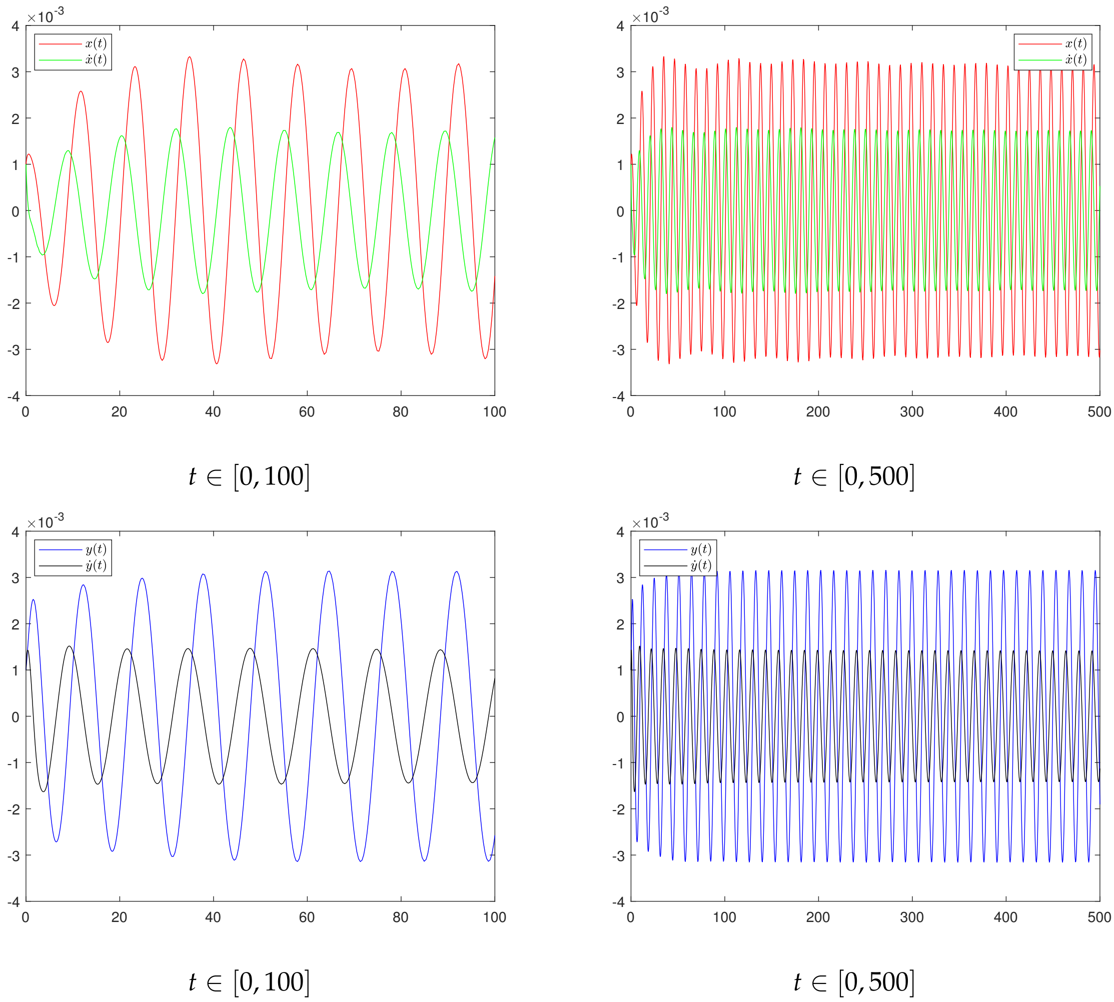


Figure 2.5: The solution of (1.1) and its derivative, with the initial data $z_0 = [0.001, 0.001, 0.001, 0.001]$ and the functions f_1, f_2, f_3, f_4, β, δ, γ_1, γ_2, g_1, g_2 given in Remark 2.8.

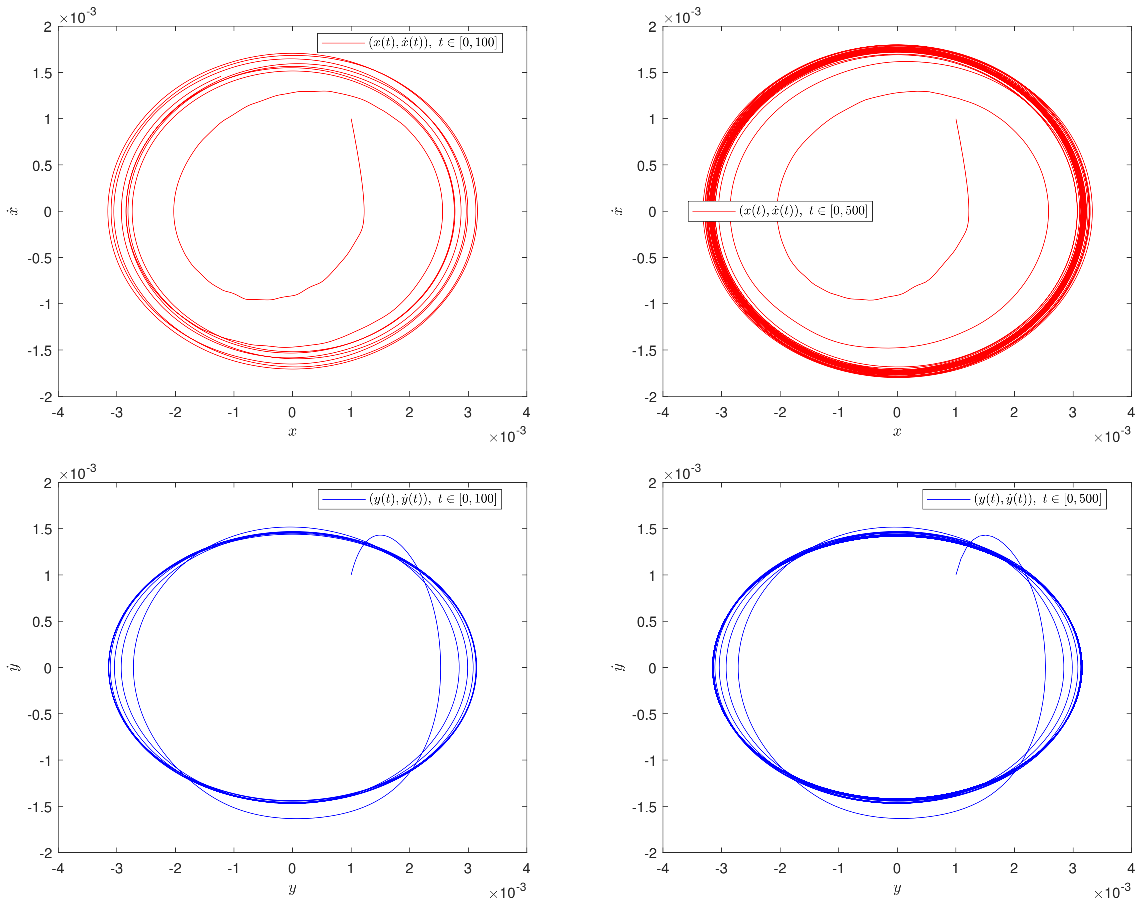


Figure 2.6: The solution of (1.1) in the planes $(x, \dot{x})$ and $(y, \dot{y})$, with the data from Remark 2.8.

3. Analysis of the inhomogeneous system (1.3)

Suppose that the block of mass m_1 is subject to the action of a time-dependent external force $\widehat{f} : \mathbb{R}_+ \to \mathbb{R}$. In this case, we obtain the inhomogeneous system (1.3).

We are going to use the following hypotheses.

(H8) $f \in C(\mathbb{R}_+)$ and $f \in L^1(\mathbb{R}_+)$;

(H9) $f \in C(\mathbb{R}_+)$ and $\lim_{t\to+\infty} f(t) = 0$.

3.1. Qualitative properties of solutions via differential inequalities

Theorem 3.1.

a) Suppose that the hypotheses (H1), (H2), (H4)–(H6), (H8) are fulfilled. Then every solution of the system (1.3) *starting from sufficiently small initial data and its derivative are bounded.*

b) If the hypotheses (H1), (H2), (H4)–(H6), (H7) with p big enough, and (H9) are satisfied, then for every solution (x, y) of (1.3) *starting from small initial data, we have* $\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} \dot{x}(t) = \lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \dot{y}(t) = 0$.

Proof. This time we use the following transformation (of the same type as the one from [2])

$$\begin{cases} \dot{x} = u - f_1(t)x \\ \dot{u} = [\dot{f}_1(t) + f_1^2(t) - \beta(t)]x - f_1(t)u + [\gamma_1(t) - f_2(t)f_3(t)]y + f_3(t)v + f(t) - g_1(t,x,y) \\ \dot{y} = v - f_2(t)y \\ \dot{v} = [\gamma_2(t) - f_1(t)f_4(t)]x + f_4(t)u + \left[\dot{f}_2(t) + f_2^2(t) - \delta(t)\right]y - f_2(t)v - g_2(t,x,y) \end{cases} \tag{3.1}$$

and the system (1.3) becomes

$$\dot{z} = A(t)z + B(t)z + G(t,z), \tag{3.2}$$

where

$$G(t,z) = \begin{pmatrix} 0 \\ f(t) - g_1(t,x,y) \\ 0 \\ -g_2(t,x,y) \end{pmatrix}$$

and $A(t)$ and $B(t)$ are the same as in the proof of Theorem 2.3.

Let $z_0 \in \mathbb{R}^4 \backslash \{0\}$ with $\|z_0\|_0$ small enough, $t_0 \geq 0$, and

$$z(t,t_0,z_0) = (x(t,t_0,z_0), u(t,t_0,z_0), y(t,t_0,z_0), v(t,t_0,z_0))^\top$$

be the unique solution of (3.2) which is equal to z_0 for $t = t_0$.

Similarly (by applying, e.g., [3, Corollary, p. 53]) we conclude that $z(t,t_0,z_0)$ exists on $[t_0, +\infty)$, this time having

$$\|A(t)z + B(t)z + G(t,z)\|_0 \leq \psi(t)\|z\|_0 + |f(t)|, \quad \forall (t,z) \in \mathbb{R}_+ \times \mathbb{R}^4.$$

As before we deduce

$$
\begin{aligned}
\|z(t,t_0,z_0)\|_0 \le \lambda \|z_0\|_0 \sqrt{\beta(t_0)+\delta(t_0)+2}\,\mathrm{e}^{\int_{t_0}^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u} &+ \int_{t_0}^t \mathrm{e}^{\int_s^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u} \\
\times \Big[\big|\dot{f}_1(s)+f_1^2(s)\big||x(s,t_0,z_0)| &+ \big|\dot{f}_2(s)+f_2^2(s)\big||y(s,t_0,z_0)| \\
&+ f_1(s)f_4(s)|x(s,t_0,z_0)| + f_2(s)f_3(s)|y(s,t_0,z_0)| \\
&+ f_3(s)|v(s,t_0,z_0)| + f_4(s)|u(s,t_0,z_0)| + |f(s)| \\
&+ |g_1(s,x(s,t_0,z_0),y(s,t_0,z_0))| \\
&+ |g_2(s,x(s,t_0,z_0),y(s,t_0,z_0))| \Big]\mathrm{d}s, \qquad (3.3)
\end{aligned}
$$

for all $t \ge t_0$.

We distinguish two cases again.

Case 1: $0 \le t_0 < h$. As in the proof of Theorem 2.3, we obtain the relation (2.19), with D, $D_1 > 0$.

From (3.3) and using Remark 2.2, we deduce for all $t \ge h$

$$
\begin{aligned}
\|z(t,t_0,z_0)\|_0 &\le \lambda\sqrt{\beta(h)+\delta(h)+2}\,\|z(h,t_0,z_0)\|_0 \mathrm{e}^{\int_h^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u} \\
&\quad + \int_h^t \mathrm{e}^{\int_s^t\left[-\widetilde{f}(u)+\frac{\gamma(u)}{2\sqrt{\zeta(u)}}\right]\mathrm{d}u} \Bigg\{ \Bigg[\left(\frac{K_1}{\sqrt{\beta_0}}+\frac{K_2}{\sqrt{\delta_0}}\right)\widetilde{f}(s) \\
&\quad + \frac{f_1(s)f_4(s)}{\sqrt{\beta_0}} + \frac{f_2(s)f_3(s)}{\sqrt{\delta_0}} + f_3(s) + f_4(s) \\
&\quad + \frac{r_1(s)}{\sqrt{\beta_0}} + \frac{r_2(s)}{\sqrt{\delta_0}} \Bigg] \|z(s,t_0,z_0)\|_0 + |f(s)| \Bigg\}\mathrm{d}s \\
&=: \rho(t), \qquad \forall t \ge h.
\end{aligned}
$$

Straightforward calculations lead us to

$$
\begin{cases}
\dot{\rho}(t) \le \omega(t)\rho(t) + |f(t)|, & \forall t \ge h, \\
\rho(h) = \lambda\sqrt{\beta(h)+\delta(h)+2}\,\|z(h,t_0,z_0)\|_0,
\end{cases}
$$

with $\omega(t)$, $t \ge 0$, as in the proof of Theorem 2.3.

We easily deduce

$$
\begin{aligned}
\|z(t,t_0,z_0)\|_0 &\le \left(\rho(h) + \int_h^t \mathrm{e}^{-\int_h^s[-K\widetilde{f}(u)+\varphi(u)]\mathrm{d}u}|f(s)|\mathrm{d}s\right)\mathrm{e}^{\int_h^t[-K\widetilde{f}(s)+\varphi(s)]\mathrm{d}s} \\
&=: \mu(t), \quad \forall t \ge h. \qquad (3.4)
\end{aligned}
$$

Proof of a). By using the hypotheses (H1), (H5), (H6), and Remark 2.1, it is readily seen that $\overline{\varphi} := \int_0^{+\infty}\varphi(t)\mathrm{d}t < +\infty$. From (3.4) and the hypothesis (H8) we derive that

$$
\begin{aligned}
\|z(t,t_0,z_0)\|_0 &\le \rho(h)\mathrm{e}^{\int_h^t\varphi(u)\mathrm{d}u} + \int_h^t \mathrm{e}^{\int_s^t\varphi(u)\mathrm{d}u}|f(s)|\mathrm{d}s \\
&\le \mathrm{e}^{\overline{\varphi}}\left(\rho(h) + \int_h^t |f(s)|\mathrm{d}s\right) \\
&\le \mathrm{e}^{\overline{\varphi}}\left(\rho(h) + \|f\|_{L^1[0,+\infty)}\right) < +\infty, \quad \forall t \ge h
\end{aligned}
$$

and so every solution of (1.3) with initial data small enough is bounded. The boundedness of $\dot{z}(t, t_0, z_0)$ follows immediately.

Proof of b). Let us estimate the limit of μ at $+\infty$. We have

$$\lim_{t\to+\infty} \mu(t) = \lim_{t\to+\infty} \frac{\rho(h) + \int_h^t e^{-\int_h^s [-K\widetilde{f}(u)+\varphi(u)]du} |f(s)| ds}{e^{-\int_h^t [-K\widetilde{f}(s)+\varphi(s)]ds}}. \tag{3.5}$$

If $\int_h^{+\infty} e^{-\int_h^s [-K\widetilde{f}(u)+\varphi(u)]du} |f(s)| ds < +\infty$, then, from (3.5) and the hypothesis (H7), we easily obtain

$$\lim_{t\to+\infty} \mu(t) = 0.$$

If $\int_h^{+\infty} e^{-\int_h^s [-K\widetilde{f}(u)+\varphi(u)]du} |f(s)| ds = +\infty$, then we estimate

$$\lim_{t\to+\infty} \frac{\frac{d}{dt}\left(\rho(h) + \int_h^t e^{-\int_h^s [-K\widetilde{f}(u)+\varphi(u)]du} |f(s)| ds\right)}{\frac{d}{dt}\left(e^{-\int_h^t [-K\widetilde{f}(s)+\varphi(s)]ds}\right)} = \lim_{t\to+\infty} \frac{|f(t)|}{K\widetilde{f}(t) - \varphi(t)}. \tag{3.6}$$

Using the hypotheses (H1), (H5)–(H7), and Remark 2.1,

$$K\widetilde{f}(t) - \varphi(t) \geq Kp - \varphi_0, \quad \forall t \geq 0,$$

where $\varphi_0 = \sup_{t\geq 0}\{\varphi(t)\}$. Hence, if $p > \frac{\varphi_0}{K}$, then $K\widetilde{f}(t) - \varphi(t) > 0$, $\forall t \geq 0$, and, from (3.6), the hypothesis (H9), and L'Hospital's rule, we obtain $\lim_{t\to+\infty} \mu(t) = 0$. Hence, by (3.4) it follows that $\lim_{t\to+\infty} \|z(t, t_0, z_0)\|_0 = 0$ and we also infer $\lim_{t\to+\infty} \|\dot{z}(t, t_0, z_0)\|_0 = 0$.

Case 2: $t_0 \geq h$. The proofs of a) and b) follow as in *Case 1*, this time by using the inequality

$$\begin{aligned} \|z(t, t_0, z_0)\|_0 \leq &\left(\lambda\sqrt{\beta(t_0) + \delta(t_0) + 2}\|z_0\|_0 + \int_{t_0}^t e^{-\int_{t_0}^s [-K\widetilde{f}(u)+\varphi(u)]du} |f(s)| ds\right) \\ &\times e^{\int_{t_0}^t [-K\widetilde{f}(s)+\varphi(s)]ds}, \qquad \forall t \geq t_0. \end{aligned}$$

□

Example 3.2. If we consider the functions

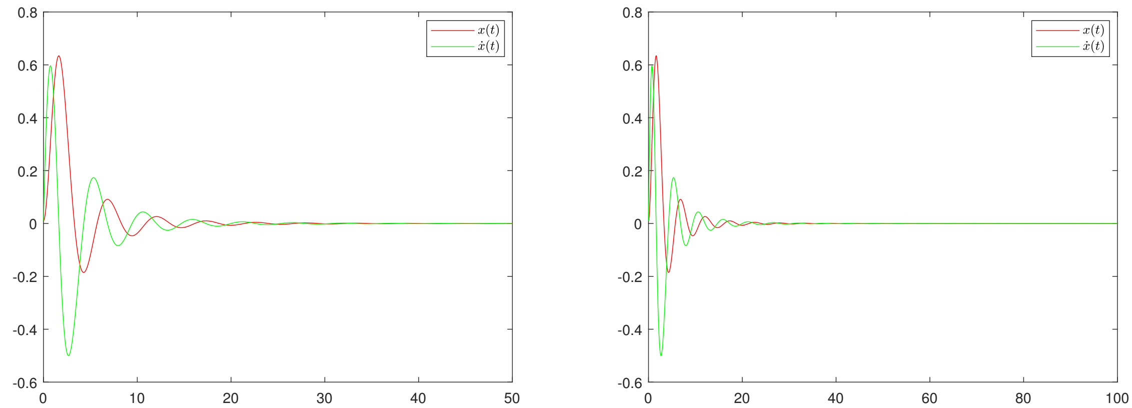
$$f_1(t) = \begin{cases} \frac{\ln t}{t}, & t \geq e \\ \frac{t}{e^3}(2e - t), & t \in [0, e) \end{cases}, \quad f_2(t) = \begin{cases} \frac{\ln t}{t-1}, & t \geq e \\ \frac{t}{e(e-1)^2}(2e - 1 - t), & t \in [0, e) \end{cases},$$

$$f_3(t) = \frac{\arctan t}{(t+1)^2}, \quad f_4(t) = \frac{\sqrt{t}}{(t+2)^2}, \quad f(t) = \frac{2t+3}{t+2}e^{-t}, \quad \forall t \geq 0,$$

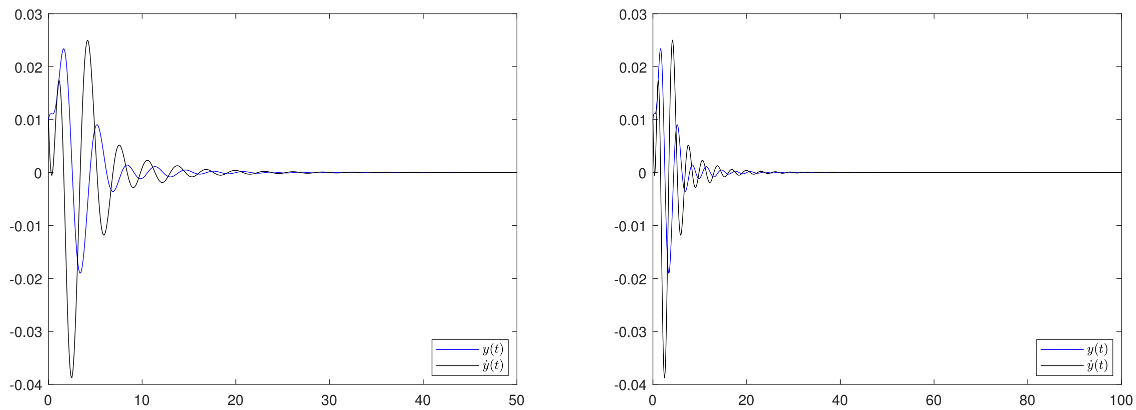
$$\beta(t) = \frac{9}{e^2} + \frac{1}{\sqrt{t+2}}, \ \delta(t) = \frac{49}{4(e-1)^2} + e^{-2t}, \ \gamma_1(t) = \frac{e^{-3t}}{t^2+1}, \ \gamma_2(t) = \frac{\sin^2 t}{(t+1)^3}, \ \forall t \geq 0,$$

$$g_1(t, x, y) = \frac{2|\sin t| x^3}{t\sqrt{t+1}}, \quad g_2(t, x, y) = \frac{3|\cos t| y^2}{(t+1)\sqrt{t+1}}, \quad \forall t \geq 0, \ \forall x, y \in \mathbb{R},$$

then the hypotheses (H1), (H2), (H4)–(H6), (H8) are fulfilled with $\beta_0 = \frac{9}{e^2}$, $\delta_0 = \frac{49}{4(e-1)^2}$, $K_1 = 2/e$, $K_2 = 1/(e-1)$, $h = e$, $r_1(t) = \frac{2|\sin t|}{t\sqrt{t+1}}$, $r_2(t) = \frac{3|\cos t|}{(t+1)\sqrt{t+1}}$, $\forall t \geq 0$. In Figure 3.1 one can observe the solution of (1.3) and its derivative, for small initial data on two time intervals and in Figure 3.2 the solution is plotted in the planes $(x, \dot{x})$, $(y, \dot{y})$ on the same time intervals.



$t \in [0, 50]$ $t \in [0, 50]$



$t \in [0, 100]$ $t \in [0, 100]$

Figure 3.1: The solution of (1.3) and its derivative, with the initial data $z_0 = [0.01, 0.01, 0.01, 0.01]$ and the functions f_1, f_2, f_3, f_4, f, β, δ, γ_1, γ_2, g_1, g_2 given in Example 3.2.

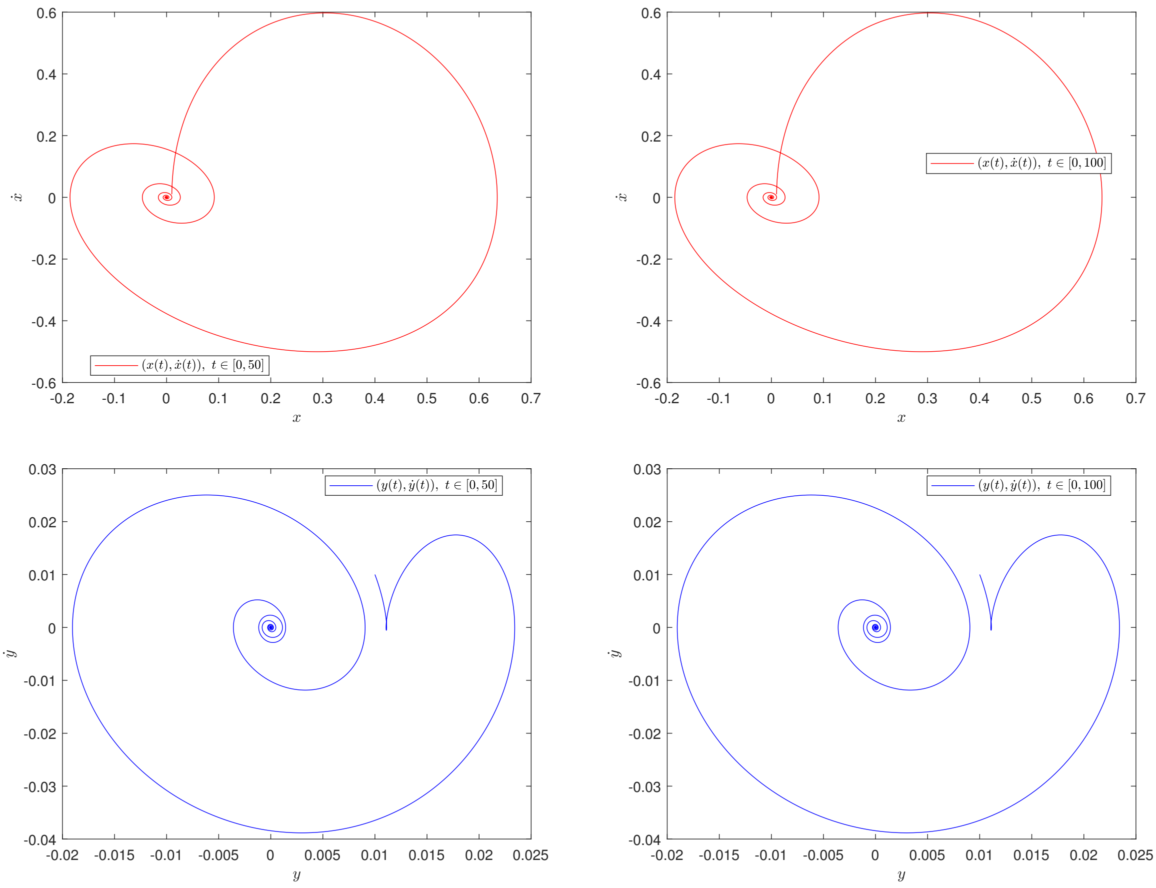


Figure 3.2: The solution of (1.3) in the planes $(x, \dot{x})$ and $(y, \dot{y})$, with the data from Example 3.2.

Remark 3.3. Let us remark the difference between the graphs of the first and second components of the solution near the origin. Due to the action of the external force $\widehat{f}(t)$ on the first block m_1, at least near the origin, the absolute values of $x = x(t)$ are much bigger than the ones of $y = y(t)$.

3.2. Boundedness of solutions

Theorem 3.4. *Suppose that the hypotheses (H1*), (H4*), (H5), (H6), (H8) are fulfilled. Then every solution of the system* (1.3) *with sufficiently small initial data is bounded.*

Proof. Let us remark that using the classical change of variables $x = x$, $u = \dot{x}$, $y = y$, $v = \dot{y}$, the system (1.3) becomes

$$\dot{z} = F(t, z), \tag{3.7}$$

where

$$z = \begin{pmatrix} x \\ u \\ y \\ v \end{pmatrix} \qquad F(t, z) = \begin{pmatrix} u \\ -\beta(t)x - 2f_1(t)u + \gamma_1(t)y + f_3(t)v + f(t) - g_1(t, x, y) \\ v \\ \gamma_2(t)x + f_4(t)u - \delta(t)y - 2f_2(t)v - g_2(t, x, y) \end{pmatrix}.$$

We will use again the norm $\|\cdot\|_0$ defined by (2.14) and the function $V : \mathbb{R}_+ \times \Delta \to \mathbb{R}$,

$$V(t,z) = \frac{1}{2}\left[\beta(t)x^2 + u^2 + \delta(t)y^2 + v^2\right]\mathrm{e}^{-\int_0^t\left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s},$$

for $z = (x,u,y,v)^\top \in \Delta$, where $\Delta \subset \mathbb{R}^4$ is as in the proof of Theorem 2.6, $\gamma(t) := \max\{\gamma_1(t), \gamma_2(t)\}$, $\zeta(t) := \min\{\beta(t), \delta(t)\}$, $\forall t \in \mathbb{R}_+$, and $r(t) := \max\{r_1(t), r_2(t)\}$, $\forall t \geq 0$.

Let us calculate the time derivative of V along the solutions of the system (3.7), whose global existence in the future is deduced as in the proof of Theorem 2.6. For every $(t,z) \in \mathbb{R} \times \Delta$ we have

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) = {} & \frac{1}{2}\left[\dot\beta(t)x^2 + 2\beta(t)x\dot x + 2u\dot u + \dot\delta(t)y^2 + 2\delta(t)y\dot y + 2v\dot v\right] \\ & \times \mathrm{e}^{-\int_0^t\left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ & - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right]V(t,z).\end{aligned}$$

By hypothesis (H4*) we get for every $(t,z) \in \mathbb{R}_+ \times \Delta$,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) \leq {} & \{\gamma(t)(|u||y| + |x||v|) + [f_3(t) + f_4(t)]|u||v| + |u||g_1(t,x,y)| + |v||g_2(t,x,y)| \\ & + |f(t)||u|\} \times \mathrm{e}^{-\int_0^t\left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ & - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right]V(t,z) \\ & - 2\left[f_1(t)u^2 + f_2(t)v^2\right]\mathrm{e}^{-\int_0^t\left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s}.\end{aligned} \tag{3.8}$$

From relations (3.8) and Remark 2.2, we successively deduce

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) \leq {} & \Big\{\gamma(t)(|u||y| + |x||v|) + [f_3(t) + f_4(t)]|u||v| + [r_1(t)|x||u| + r_2(t)|y||v| \\ & + |f(t)||u|] - 2\left[f_1(t)u^2 + f_2(t)v^2\right]\Big\}\mathrm{e}^{-\int_0^t\left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ & - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right]V(t,z) \\ \leq {} & \left[f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right]V(t,z) + |f(t)||u|\mathrm{e}^{-\int_0^t\left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ & - 2\left[f_1(t)u^2 + f_2(t)v^2\right]\mathrm{e}^{-\int_0^t\left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ & - \left[\widetilde{f}(t) + f_3(t) + f_4(t) + \frac{\gamma(t)+r(t)}{\sqrt{\zeta(t)}}\right]V(t,z) \\ \leq {} & -\widetilde{f}(t)V(t,z) + |f(t)||u|\mathrm{e}^{-\int_0^t\left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s},\end{aligned} \tag{3.9}$$

for all $(t,z) \in \mathbb{R}_+ \times \Delta$. Then, from (3.9) we easily obtain $\forall (t,z) \in \mathbb{R}_+ \times \Delta$

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(t,z) \leq & -\widetilde{f}(t)V(t,z) + |f(t)|\sqrt{\beta(t)x^2+u^2+\delta(t)y+v^2} \\ & \times \mathrm{e}^{-\int_0^t\left[\widetilde{f}(s)+f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s} \\ \leq & -\widetilde{f}(t)V(t,z) + |f(t)|\sqrt{2V(t,z)}\mathrm{e}^{-\frac{1}{2}\int_0^t \widetilde{f}(s)\mathrm{d}s},\end{aligned}$$

which actually represents an inequality of Bernoulli type.

Let $z_0 \in \Delta$, $t_0 \geq 0$, and $z(t,t_0,z_0)$ be the unique solution of (3.7) which is equal to z_0 for $t = t_0$. Using classical differential estimates, we find

$$V(t,z(t,t_0,z_0)) \leq \mathrm{e}^{-\int_{t_0}^t \widetilde{f}(s)\mathrm{d}s}\left[\sqrt{V(t_0,z_0)} + \frac{\sqrt{2}}{2}\int_{t_0}^t |f(s)|\mathrm{e}^{-\frac{1}{2}\int_0^{t_0}\widetilde{f}(u)\mathrm{d}u}\mathrm{d}s\right]^2, \quad \forall t \geq t_0.$$

Therefore, by using the hypotheses (H1*), (H5), (H6), it follows that

$$\|z(t,t_0,z_0)\|_0 \leq M\left[\sqrt{V(t_0,z_0)} + \frac{\sqrt{2}}{2}\int_{t_0}^t |f(s)|\mathrm{e}^{-\frac{1}{2}\int_0^{t_0}\widetilde{f}(u)\mathrm{d}u}\mathrm{d}s\right], \quad \forall t \geq t_0,$$

where $M := \sqrt{2}\mathrm{e}^{\frac{1}{2}\int_0^{t_0}\widetilde{f}(s)\mathrm{d}s+\frac{1}{2}\int_0^{+\infty}\left[f_3(s)+f_4(s)+\frac{\gamma(s)+r(s)}{\sqrt{\zeta(s)}}\right]\mathrm{d}s}$. If the hypothesis (H8) comes into play, then

$$\|z(t,t_0,z_0)\|_0 \leq M\left[\sqrt{V(t_0,z_0)} + \frac{\sqrt{2}}{2}\|f\|_{L^1[0,+\infty)}\mathrm{e}^{-\frac{1}{2}\int_0^{t_0}\widetilde{f}(s)\mathrm{d}s}\right], \quad \forall t \geq t_0. \qquad \square$$

Remark 3.5. Note that by using the classical transformation $(x = x,\ u = \dot{x},\ y = y,\ v = \dot{y})$, we could only deduce the boundedness of the solutions of (1.3) for initial data small enough. In contrast, the transformation (3.1) allowed us to obtain in addition that the solutions of (1.3), starting from sufficiently small initial data, have the limit zero at $+\infty$.

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